mathematics

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Existence and Stability Results on Hadamard Type Fractional Time-Delay Semilinear Differential Equations

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Received: 6 July 2020; Accepted: 19 July 2020; Published: 30 July 2020

Abstract: A delayed perturbation of the Mittag-Leffler type matrix function with logarithm is proposed. This combines the classic Mittag–Leffler type matrix function with a logarithm and delayed Mittag–Leffler type matrix function. With the help of this introduced delayed perturbation of the Mittag–Leffler type matrix function with a logarithm, we provide an explicit form for solutions to non-homogeneous Hadamard-type fractional time-delay linear differential equations. We also examine the existence, uniqueness, and Ulam–Hyers stability of Hadamard-type fractional time-delay nonlinear equations.

Keywords: mittag-leffler type matrix function; Hadamard type derivative; Ulam–Hyers stability

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Mathematical descriptions of the models described through differential equations with derivatives of non-integer orders have proved to be a very useful instrument for modeling in viscoelasticity, stability theory, controllability theory, and other related fields. Time-delays are often related with physico-chemical processes, electric networks, hydraulic networks, heredity in population growth, the economy and other related industries. In general, a peculiarity of these mathematical models is that the rate of change of these processes depends on past history. Differential systems describing these models are called time-delay differential equations. The qualitative theory of linear time-delay equations is well investigated. Recently, time-delay differential equations have been considered. In [1–9] authors derived the exact expressions for solutions of linear continuous and discrete delay equations by proposing the concept of delayed matrix functions. On the other hand, stability concepts and relative controllability problems of linear time-delay differential equations were investigated in [10–17].

The unification of differential equations with delay and differential equations with fractional derivatives is provided by differential equations including both delay and non-integer derivatives, so called time-delay fractional differential equations. In applications, this unification is useful for creating useful models of some systems with memory. One can notice that works in this field involve Riemann-Liouville and Caputo type fractional derivatives. For the literature on the related field of fractional time-delay equations of Caputo type and Riemann–Liouville type, we refer the researcher to [13–23].

Besides these derivatives, there is another fractional derivative, involving the logarithmic function: the so-called Hadamard type fractional derivative. Details of the Hadamard type fractional integral and derivative can be found in [24]. Recent results on the existence and uniqueness of solution for fractional differential equations in Hadamard sense can be found in [25–32].

In ([24], p. 235) it is shown that a solution of a Hadamard type fractional linear system

$$\left({}^{H}D_{1^+}^{\alpha}y\right)(t) = \lambda y(t) + f(t), \; t \in (1,T], h > 0,$$
$$\left({}^{H}I_{1^+}^{1-\alpha}y\right)(1^+) = a \in \mathbb{R}, \; \lambda \in \mathbb{R}, \; 0 < \alpha < 1,$$

has the form

$$y(t) = a(\ln t)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(\ln t)^{\alpha}\right] + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\left(\ln \frac{t}{s}\right)^{\alpha}\right] f(s) \frac{ds}{s}.$$

However, we find that there exists only one [33] work on the representation of explicit solutions of Hadamard type fractional order delay linear differential equations. In [33] authors studied the Hadamard type fractional linear time-delay system

$$\begin{aligned} \left({}^{H}D_{1^+}^{\alpha}y\right)(t) &= \mathbb{B}y(t-h), \; t \in (1,T], h > 0, \\ y(t) &= \varphi(t), \; 1 \le t \le h, \\ \left({}^{H}I_{1^+}^{1-\alpha}y\right)(1^+) &= a \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $\mathbb{B}$ is a constant $n \times n$ square matrix.

Motivated by the above researches, we investigate a new class of Hadamard-type fractional delay differential equations. We consider an explicit representation of solutions of a Hadamard type fractional time-delay differential equation of the following form by introducing a new delayed M-L type function with logarithm

$$\begin{cases} \left({}^{H}D_{1^+}^{\alpha}y\right)(t) = \mathbb{A}y(t) + \mathbb{B}y\left(\frac{t}{h}\right) + f(t), \; t \in (1,T], h > 0, \\ y(t) = \varphi(t), \; \frac{1}{h} < t \le 1, \\ \left({}^{H}I_{1^+}^{1-\alpha}y\right)(1^+) = a \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $\left({}^{H}D_{1^+}^{\alpha}y\right)(\cdot)$ is the Hadamard derivative of order $\alpha \in (0,1)$, $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{n\times n}$ denote constant matrices, and $\varphi : \left[\frac{1}{h}, 1\right] \to \mathbb{R}^n$ is an arbitrary Hadamard differentiable vector function, $f \in C([1,T], \mathbb{R}^n)$, $T = h^l$ for a fixed natural number l.

The second purpose of this paper is to study the existence and stability of solutions for a Hadamard type fractional delay differential equation

$$\begin{cases} \left({}^{H}D_{1^+}^{\alpha}y\right)(t) = \mathbb{A}y(t) + \mathbb{B}y\left(\frac{t}{h}\right) + f(t, y(t)), \; t \in (1,T], h > 0, \\ y(t) = \varphi(t), \; \frac{1}{h} < t \le 1, \\ \left({}^{H}I_{1^+}^{1-\alpha}y\right)(1^+) = a \in \mathbb{R}^n, \end{cases} \tag{3}$$

At the end of this section, we state the main contribution of the paper as follows:

(i) We propose delayed perturbation of the M-L type functions $Y_{h,\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)$ with logarithms, by means of the matrix Equations (6). We show that for $\mathbb{B} = \Theta$ the function $Y_{h,\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)$ coincides with the M-L type function with two parameters $(\ln t - \ln s)^{\beta-1} E_{\alpha,\beta}\left(\mathbb{A}(\ln t - \ln s)^{\alpha}\right)$. For $\mathbb{A} = \Theta$ the delayed M-L type function $Y_{h,\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)$ coincides with the delayed M-L type matrix function with two parameters $E_{h,\alpha,\beta}^{\mathbb{B}}(\ln t - \ln h)$, introduced in (4).

(ii) We explicitly write the solution of the Hadamard type fractional delay linear system (2) via delayed perturbation of the M-L type function with logarithm. Using this representation we study existence, uniqueness, and Ulam–Hyers stability of the nonlinear Equation (3).

1. Preliminaries

Let $0 < a < b < \infty$ and $C[a,b]$ be the Banach space of all continuous functions $y : [a,b] \to \mathbb{R}^n$ with the norm $\|y\|_C := \max\{\|y(t)\| : t \in [a,b]\}$. For $0 \le \gamma < 1$, we denote the space $C_{\gamma,\ln}(a,b]$ by the weighted Banach space of the continuous function $y : [a,b] \to \mathbb{R}^n$, which is given by

$$C_{\gamma,\ln}(a,b] := \left\{ y(t) : \left(\ln \frac{t}{a}\right)^{\gamma} y(t) \in C[a,b] \right\},$$

endowed with the norm $\|y\|_{\gamma} := \sup\left\{ \left(\ln \frac{t}{a}\right)^{\gamma} \|y(t)\| : t \in (a,b] \right\}$.

The following definitions and lemmas will be used in this paper.

Definition 1. *[24] Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $y(t)$ is defined by*

$$\left({}^{H}I^{\alpha}_{a+}y\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s},\ 1 \le a < t \le b,$$

where Γ is the Gamma function.

Definition 2. *[24] Hadamard fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $y(t)$ is defined by*

$$\left({}^{H}D^{\alpha}_{a+}y\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^{n} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha+1} y(s) \frac{ds}{s},\ 1 \le a < t \le b.$$

Lemma 1. *[24] If $a, \gamma, \beta > 0$ then*

- $\left({}^{H}I^{\gamma}_{a+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}\left(\ln \frac{t}{a}\right)^{\beta+\gamma-1}.$
- $\left({}^{H}D^{\gamma}_{a+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\left(\ln \frac{t}{a}\right)^{\beta-\gamma-1}.$
- *For $0 < \beta < 1$, $\left({}^{H}D^{\beta}_{a+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(t) = 0$.*

Definition 3. *M-L type matrix function with two parameters $e_{\alpha,\beta}(\mathbb{A};t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by*

$$e_{\alpha,\beta}(\mathbb{A};t) := t^{\beta-1}E_{\alpha,\beta}(\mathbb{A};t) := t^{\beta-1}\sum_{k=0}^{\infty} \frac{\mathbb{A}^k t^{\alpha k}}{\Gamma(k\alpha+\beta)},\quad \alpha,\beta > 0, t \in \mathbb{R}.$$

Next, we introduce a definition of delayed M-L type matrix function $E^{\mathbb{B}}_{h,\alpha,\beta}(\ln t) : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ with logarithm generated by $\mathbb{B}$.

Definition 4. *Two parameters delayed M-L type matrix function $E^{\mathbb{B}}_{h,\alpha,\beta}(\ln t) : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ with logarithm generated by $\mathbb{B}$ is defined by*

$$E^{\mathbb{B}}_{h,\alpha,\beta}(\ln t) := \begin{cases} \Theta, & 0 < t \le \frac{1}{h}, \\ I\frac{(\ln t + \ln h)^{\beta-1}}{\Gamma(\beta)}, & \frac{1}{h} < t \le 1, \\ I\frac{(\ln t + \ln h)^{\beta-1}}{\Gamma(\beta)} + \mathbb{B}\frac{(\ln t)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \ldots + \mathbb{B}^p\frac{(\ln t-(p-1)\ln h)^{p\alpha+\beta-1}}{\Gamma(p\alpha+\beta)}, & h^{p-1} < t \le h^p. \end{cases} \tag{4}$$

Our definition of the two-parameter delayed M-L type matrix function with logarithm differs substantially from the definition given in [33].

In order to give a definition of delayed perturbation of the M-L type matrix functions with logarithm, we introduce the following matrices $Y_{\alpha,\beta,k}$, $k = 0, 1, 2, ...$

$$\begin{aligned} Y_{\alpha,\beta,0}(t,s) &= \left(\ln\frac{t}{s}\right)^{\beta-1} E_{\alpha,\beta}\left(\mathbb{A};\ln\frac{t}{s}\right), \\ Y_{\alpha,\beta,1}(t,sh) &= \int_{sh}^{t} e_{\alpha,\alpha}\left(\mathbb{A};\ln\frac{t}{r}\right)\mathbb{B}Y_{\alpha,\beta,0}\left(\frac{r}{h},s\right)\frac{dr}{r}, \\ Y_{\alpha,\beta,k}\left(t,sh^k\right) &= \int_{sh^k}^{t} e_{\alpha,\alpha}\left(\mathbb{A};\ln\frac{t}{r}\right)\mathbb{B}Y_{\alpha,\beta,k-1}\left(\frac{r}{h},sh^{k-1}\right)\frac{dr}{r}. \end{aligned} \tag{5}$$

Definition 5. *Let $\mathbb{A},\mathbb{B} \in \mathbb{R}^{n\times n}$ be fixed matrices and $k \in \mathbb{N}\cup\{0\}$. Delayed perturbation of M-L type function $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(\cdot,\cdot) : \mathbb{R}\times\mathbb{R}\to\mathbb{R}^n$ with logarithm generated by $\mathbb{A},\mathbb{B}$ is defined by*

$$Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) := \sum_{j=0}^{\infty} Y_{\alpha,\beta,j}\left(t,sh^j\right)H\left(t-sh^j\right) \tag{6}$$

$$= \begin{cases} \Theta, & 0 < t < s, \\ I, & t = s, \\ Y_{\alpha,\beta,0}(t,s) + Y_{\alpha,\beta,1}(t,sh) + ... + Y_{\alpha,\beta,k}\left(t,sh^k\right), & sh^k < t \le sh^{k+1}, \end{cases}$$

where $H(t)$ is a Heaviside function: $H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0. \end{cases}$.

Lemma 2. *Let $a, b > -1$. For $sh^k < t \le sh^{k+1}$, $k \in \mathbb{N}\cup\{0\}$, one has*

$$\int_r^t \left(\ln\frac{t}{s}\right)^a \left(\ln\frac{s}{r}\right)^b \frac{ds}{s} = \left(\ln\frac{t}{r}\right)^{a+b+1} \mathcal{B}[a+1,b+1], \tag{7}$$

$$\frac{1}{\Gamma(1-\alpha)}\int_{sh^k}^{t}\left(\ln\frac{t}{r}\right)^{-\alpha} Y_{\alpha,\beta,k}\left(r,sh^k\right)\frac{dr}{r} = \int_{sh^k}^{t} E_{\alpha,1}\left(\mathbb{A};\ln\frac{t}{r}\right)\mathbb{B}Y_{\alpha,\beta,k-1}\left(\frac{r}{h},sh^{k-1}\right)\frac{dr}{r}. \tag{8}$$

Proof. Let $\ln\frac{s}{r} = \tau\ln\frac{t}{r}$. Then $s = r\left(\frac{t}{r}\right)^{\tau}$, $ds = r\left(\frac{t}{r}\right)^{\tau}\ln\frac{t}{r}d\tau$. So we have

$$\ln\frac{t}{s} = \ln\left(\frac{t}{r}\right)^{1-\tau} = (1-\tau)\ln\left(\frac{t}{r}\right),$$

and

$$\begin{aligned} \int_r^t \left(\ln\frac{t}{s}\right)^a \left(\ln\frac{s}{r}\right)^b \frac{ds}{s} &= \int_0^1 \left(\ln\frac{t}{r}\right)^a (1-\tau)^a \left(\ln\frac{t}{r}\right)^b \tau^b \ln\frac{t}{r}d\tau \\ &= \left(\ln\frac{t}{r}\right)^{a+b+1} \mathcal{B}[a+1,b+1]. \end{aligned}$$

To prove (8), firstly using (7) we calculate it for $k = 0$:

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha)}\int_s^t \left(\ln\frac{t}{r}\right)^{-\alpha} Y_{\alpha,\beta,0}(r,s)\frac{dr}{r} &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\beta)}\int_s^t \left(\ln\frac{t}{r}\right)^{-\alpha}\left(\ln\frac{r}{s}\right)^{\beta-1}\frac{dr}{r} \\ &= \frac{(\ln t - \ln s)^{-\alpha+\beta}}{\Gamma(-\alpha+\beta+1)}. \end{aligned} \tag{9}$$

Similarly, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \int_{sh^k}^{t} \left(\ln \frac{t}{r}\right)^{-\alpha} Y_{\alpha,\beta,k}\left(r, sh^k\right) \frac{dr}{r} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{sh^k}^{t} \left(\ln \frac{t}{r}\right)^{-\alpha} \int_{sh^k}^{r} e_{\alpha,\alpha}\left(\mathbb{A}; \ln \frac{r}{r_1}\right) \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{r_1}{h}, sh^{k-1}\right) H\left(r - sh^k\right) \frac{dr_1}{r_1} \frac{dr}{r} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{sh^k}^{t} \int_{r_1}^{t} \left(\ln \frac{t}{r}\right)^{-\alpha} e_{\alpha,\alpha}\left(\mathbb{A}; \ln \frac{r}{r_1}\right) \frac{dr}{r} \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{r_1}{h}, sh^{k-1}\right) H\left(r - sh^k\right) \frac{dr_1}{r_1} \\
&= \int_{sh^k}^{t} E_{\alpha,1}\left(\mathbb{A}; \ln \frac{t}{r_1}\right) \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{r_1}{h}, sh^{k-1}\right) \frac{dr_1}{r_1}.
\end{aligned}$$

□

Lemma 3. *If $\mathbb{A}$ and $\mathbb{B}$ are commutative matrices, then*

$$Y_{\alpha,\beta,k}\left(t, sh^k\right) = \mathbb{B}^k \sum_{n=0}^{\infty} \begin{pmatrix} n+k \\ k \end{pmatrix} \mathbb{A}^n \frac{\left(\ln t - \ln sh^k\right)^{n\alpha+k\alpha+\beta-1}}{\Gamma(n\alpha + k\alpha + \beta)}. \tag{10}$$

Proof. The proof is based on the equality (7). Using Lemma 2, for $k = 1$, we have

$$\begin{aligned}
Y_{\alpha,\beta,1}(t, sh) &= \int_{sh}^{t} e_{\alpha,\alpha}\left(\mathbb{A}; \ln \frac{t}{r}\right) \mathbb{B} Y_{\alpha,\beta,0}\left(\frac{r}{h}, s\right) \frac{dr}{r} \\
&= \mathbb{B} \int_{sh}^{t} \sum_{n=0}^{\infty} \mathbb{A}^n \left(\ln \frac{t}{r}\right)^{\alpha n+\alpha-1} \frac{1}{\Gamma(\alpha n+\alpha)} \sum_{k=0}^{\infty} \mathbb{A}^k \left(\ln \frac{r}{sh}\right)^{\alpha n+\beta-1} \frac{1}{\Gamma(\alpha k+\beta)} \frac{dr}{r} \\
&= \mathbb{B} \int_{sh}^{t} \sum_{n=0}^{\infty} \sum_{k=0}^{k} \mathbb{A}^k \left(\ln \frac{t}{r}\right)^{\alpha k+\alpha-1} \frac{1}{\Gamma(\alpha k+\alpha)} \mathbb{A}^{n-k} \left(\ln \frac{r}{sh}\right)^{\alpha(n-k)+\beta-1} \frac{1}{\Gamma(\alpha(n-k)+\beta)} \frac{dr}{r} \\
&= \mathbb{B} \sum_{n=0}^{\infty} \sum_{k=0}^{k} \mathbb{A}^n \frac{1}{\Gamma(\alpha k+\alpha)\Gamma(\alpha(n-k)+\beta)} \int_{sh}^{t} \left(\ln \frac{t}{r}\right)^{\alpha k+\alpha-1} \left(\ln \frac{r}{sh}\right)^{\alpha(n-k)+\beta-1} \frac{dr}{r} \\
&= \mathbb{B} \sum_{n=0}^{\infty} \sum_{k=0}^{k} \mathbb{A}^n \frac{(\ln t - \ln sh)^{\alpha n+\alpha+\beta-1}}{\Gamma(\alpha k+\alpha)\Gamma(\alpha(n-k)+\beta)} \mathcal{B}(\alpha k+\alpha, \alpha(n-k)+\beta) \\
&= \mathbb{B} \sum_{n=0}^{\infty} \begin{pmatrix} n+1 \\ 1 \end{pmatrix} \mathbb{A}^n \frac{(\ln t - \ln sh)^{\alpha n+\alpha+\beta-1}}{\Gamma(\alpha n+\alpha+\beta)}.
\end{aligned}$$

For $k = 2$, we get

$$\begin{aligned}
Y_{\alpha,\beta,2}\left(t, sh^2\right) &= \int_{sh^2}^{t} e_{\alpha,\alpha}\left(\mathbb{A}; \ln \frac{t}{r}\right) \mathbb{B} Y_{\alpha,\beta,1}\left(\frac{r}{h}, sh\right) dr \\
&= \mathbb{B} \int_{sh^2}^{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbb{A}^k \left(\ln \frac{t}{r}\right)^{k\alpha+\alpha-1} \frac{1}{\Gamma(k\alpha+\alpha)} \mathbb{B} \begin{pmatrix} n+1-k \\ 1 \end{pmatrix} \\
&\quad \times \mathbb{A}^{n-k} \left(\ln \frac{r}{sh^2}\right)^{n\alpha-k\alpha+\alpha+\beta-1} \frac{1}{\Gamma(n\alpha-k\alpha+\alpha+\beta)} \\
&= \mathbb{B}^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{pmatrix} n+1-k \\ 1 \end{pmatrix} \mathbb{A}^n \frac{1}{\Gamma(k\alpha+\beta)\Gamma(n\alpha-k\alpha+\alpha+\beta)} \int_{sh^2}^{t} \left(\ln \frac{t}{r}\right)^{k\alpha+\alpha-1} \left(\ln \frac{r}{sh^2}\right)^{n\alpha-k\alpha+\alpha+\beta-1} dr \\
&= \mathbb{B}^2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{pmatrix} n+1-k \\ 1 \end{pmatrix}\right) \mathbb{A}^n \left(\ln \frac{r}{sh^2}\right)^{n\alpha+2\alpha+\beta-1} \frac{1}{\Gamma(n\alpha+2\alpha+\beta)} \\
&= \mathbb{B}^2 \sum_{n=0}^{\infty} \begin{pmatrix} n+2 \\ 2 \end{pmatrix} \mathbb{A}^n \frac{(\ln t - \ln sh^2)^{n\alpha+2\alpha+\beta-1}}{\Gamma(n\alpha+2\alpha+\beta)}.
\end{aligned}$$

Using the Mathematical Induction in a similar manner we can get (10). □

According to Lemma 3 in the case $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$ delayed M-L type function $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s)$ has a simple form:

$$Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) := \begin{cases} \Theta, & 0<t<s, \\ I, & t=s, \\ \sum_{i=0}^{\infty}\mathbb{A}^i\frac{(\ln t-\ln s)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}+\sum_{i=1}^{\infty}\begin{pmatrix} i \\ 1\end{pmatrix}\mathbb{A}^{i-1}\mathbb{B}\frac{(\ln t-\ln sh)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} & \\ +...+\sum_{i=p}^{\infty}\begin{pmatrix} i \\ p\end{pmatrix}\mathbb{A}^{i-p}\mathbb{B}^p\frac{(\ln t-\ln sh^p)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, & sh^p<t\leq sh^{p+1}. \end{cases} \tag{11}$$

Next lemma shows some special cases of the delayed M-L type function.

Lemma 4. *Let $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s)$ be defined by (6). Then the following holds true:*

(i) *if $\mathbb{A}=\Theta$ then $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,1) = E^{\mathbb{B}}_{h,\alpha,\beta}\left(\ln\frac{t}{h}\right),\ h^{k-1}<\frac{t}{h}\leq h^k$,*

(ii) *if $\mathbb{B}=\Theta$ then $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) = \left(\ln\frac{t}{s}\right)^{\beta-1}E_{\alpha,\beta}\left(\mathbb{A}\left(\ln\frac{t}{s}\right)^{\alpha}\right) = e_{\alpha,\beta}\left(\mathbb{A};\ln\frac{t}{s}\right)$,*

(iii) *if $\alpha=\beta=1$ and $\mathbb{A}\mathbb{B}=\mathbb{B}\mathbb{A}$ then $Y^{\mathbb{A},\mathbb{B}}_{h,1,1}(t,s) = e^{\mathbb{A}(\ln t-\ln s)}e_h^{\mathbb{B}(\ln t-\ln h)}$, $\mathbb{B}_1=\mathbb{B}e^{-\mathbb{A}\ln h}$, $sh^k<t\leq sh^{k+1}$.*

Proof. (i) If $\mathbb{A}=\Theta$, then the formula (5)

$$\begin{aligned} Y_{\alpha,\beta,0}(t,s) &= e_{\alpha,\beta}\left(\Theta,\ln\frac{t}{s}\right) = I\frac{(\ln t-\ln s)^{\beta-1}}{\Gamma(\beta)}, \\ Y_{\alpha,\beta,1}(t,sh) &= \int_{sh}^{t}e_{\alpha,\alpha}\left(\Theta,\ln\frac{t}{r}\right)\mathbb{B}Y_0\left(\frac{r}{h},s\right)\frac{dr}{r} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\mathbb{B}\int_{sh}^{t}\left(\ln\frac{t}{r}\right)^{\alpha-1}\left(\ln\frac{r}{sh}\right)^{\beta-1}\frac{dr}{r} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\mathbb{B}\left(\ln\frac{t}{sh}\right)^{\alpha+\beta-1}\mathcal{B}[\alpha,\beta] = \frac{1}{\Gamma(\alpha+\beta)}\mathbb{B}\left(\ln\frac{t}{sh}\right)^{\alpha+\beta-1}, \\ Y_{\alpha,\beta,2}\left(t,sh^2\right) &= \int_{sh^2}^{t}e_{\alpha,\alpha}\left(\Theta,\ln\frac{t}{r}\right)\mathbb{B}Y_1\left(\frac{r}{h},sh\right)dr = \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\beta)}\mathbb{B}^2\int_{sh^2}^{t}\left(\ln\frac{t}{r}\right)^{\alpha-1}\left(\ln\frac{r}{sh^2}\right)^{\alpha+\beta-1}\frac{dr}{r} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\beta)}\mathbb{B}^2\left(\ln\frac{t}{sh^2}\right)^{2\alpha+\beta-1}\mathcal{B}[\alpha,\alpha+\beta] = \frac{1}{\Gamma(2\alpha+\beta)}\mathbb{B}^2\left(\ln\frac{t}{sh^2}\right)^{2\alpha+\beta-1}, \\ Y_{\alpha,\beta,k}\left(t,sh^k\right) &= \mathbb{B}^k\left(\ln\frac{t}{sh^k}\right)^{k\alpha+\beta-1}\frac{1}{\Gamma(k\alpha+\beta)},\ \ k\geq 0. \end{aligned}$$

So $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,1)$ coincides with $E^{\mathbb{B}}_{h,\alpha,\beta}(\ln t-\ln h)$:

$$\begin{aligned} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,1) &= \sum_{i=0}^{k}\mathbb{B}^i\left(\ln\frac{t}{h^i}\right)^{i\alpha+\beta-1}\frac{1}{\Gamma(i\alpha+\beta)} = I\frac{(\ln t)^{\beta-1}}{\Gamma(\beta)}+\mathbb{B}\frac{(\ln t-\ln h)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+...+\mathbb{B}^k\frac{(\ln t-k\ln h)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} \\ &= E^{\mathbb{B}}_{h,\alpha,\beta}(\ln t-\ln h),\ \ h^{k-1}<\frac{t}{h}\leq h^k. \end{aligned}$$

(ii) Trivially, from definition of $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s)$ we have: if $\mathbb{B}=\Theta$, then

$$Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) = \left(\ln\frac{t}{s}\right)^{\beta-1}E_{\alpha,\beta}\left(\mathbb{A}\left(\ln\frac{t}{s}\right)^{\alpha}\right).$$

(iii) By (11) for the case $\alpha = \beta = 1$ and $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$, we have

$$\begin{aligned} Y^{\mathbb{A},\mathbb{B}}_{h,1,1}(t,s) &= \sum_{i=0}^{\infty} \mathbb{A}^i \frac{(\ln t - \ln s)^i}{\Gamma(i+1)} + \sum_{i=0}^{\infty} \binom{i+1}{1} \mathbb{A}^i \mathbb{B} \frac{(\ln t - \ln sh)^{i+1}}{\Gamma(i+1)} \\ &+ \ldots + \sum_{i=0}^{\infty} \binom{i+k}{k} \mathbb{A}^i \mathbb{B}^k \frac{\left(\ln t - \ln sh^k\right)^{i+k}}{\Gamma(i+k+1)} \\ &= e^{\mathbb{A}(\ln t - \ln s)} + e^{\mathbb{A}(\ln t - \ln sh)} \mathbb{B}(\ln t - \ln sh) + \ldots + e^{\mathbb{A}\left(\ln t - \ln sh^k\right)} \mathbb{B}^k \frac{1}{k!}(\ln t - k \ln h)^k \\ &= e^{\mathbb{A}(\ln t - \ln s)} e_h^{\mathbb{B}_1(\ln t - \ln h)}. \end{aligned}$$

□

Lemma 5. *For any $s \in \mathbb{R}$ the function $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(\cdot, s) : (1,\infty) \to \mathbb{R}^{n\times n}$ is continuous.*

Proof. The proof is similar to that of [34] and is omitted. □

It turns out that $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s)$ is a delayed perturbation of the Cauchy matrix with logarithm of the homogeneous Equation (2) with $f = 0$.

Lemma 6. $Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ *is a solution of*

$$^{H}D^{\alpha}_{1^+} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) = \left(\ln \frac{t}{s}\right)^{-\alpha+\beta-1} \frac{1}{\Gamma(-\alpha+\beta)} + \mathbb{A} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) + \mathbb{B} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}\left(\frac{t}{h}, s\right). \tag{12}$$

Proof. According to (9) we have

$$\begin{aligned} \left({}^{H}D^{\alpha}_{1^+} Y_{\alpha,\beta,0}(t,s)\right)(t) &= \frac{1}{\Gamma(1-\alpha)} \left(t\frac{d}{dt}\right) \int_1^t \left(\ln \frac{t}{r}\right)^{-\alpha} Y_{\alpha,\beta,0}(r,s) \frac{dr}{r} \\ &= \frac{1}{\Gamma(1-\alpha)} \left(t\frac{d}{dt}\right) \int_1^t \left(\ln \frac{t}{r}\right)^{-\alpha} e_{\alpha,\beta}\left(\mathbb{A}; \ln \frac{r}{s}\right) \frac{dr}{r} \\ &= \left(t\frac{d}{dt}\right) e_{\alpha,1-\alpha+\beta}\left(\mathbb{A}; \ln \frac{t}{s}\right) \\ &= \left(\ln \frac{t}{s}\right)^{-\alpha+\beta-1} \frac{1}{\Gamma(-\alpha+\beta)} + \mathbb{A} e_{\alpha,\beta}\left(\mathbb{A}; \ln \frac{t}{s}\right) \\ &= \left(\ln \frac{t}{s}\right)^{-\alpha+\beta-1} \frac{1}{\Gamma(-\alpha+\beta)} + \mathbb{A} Y_{\alpha,\beta,0}(t,s). \end{aligned} \tag{13}$$

On the other hand for any $k \in \mathbb{N}$:

$$\begin{aligned} \left({}^{H}D^{\alpha}_{1^+} Y_{\alpha,\beta,k}\left(t, sh^k\right)\right)(t) &= \frac{1}{\Gamma(1-\alpha)} \left(t\frac{d}{dt}\right) \int_1^t \left(\ln \frac{t}{r}\right)^{-\alpha} Y_{\alpha,\beta,k}(r,s) \frac{dr}{r} \\ &= \left(t\frac{d}{dt}\right) \int_{sh^k}^t E_{\alpha,1}\left(\mathbb{A}; \ln \frac{t}{r}\right) \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{r}{h}, sh^{k-1}\right) \frac{dr}{r} \\ &= \int_{sh^k}^t \left(t\frac{d}{dt}\right) E_{\alpha,1}\left(\mathbb{A}; \ln \frac{t}{r}\right) \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{r}{h}, sh^{k-1}\right) \frac{dr}{r} \\ &+ \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{t}{h}, sh^{k-1}\right) \\ &= \mathbb{A} Y_{\alpha,\beta,k}\left(t, sh^k\right) + \mathbb{B} Y_{\alpha,\beta,k-1}\left(\frac{t}{h}, sh^{k-1}\right). \end{aligned} \tag{14}$$

From (13) and (14) it follows that for $sh^k < t \leq sh^{k+1}$

$$\begin{aligned} {}^{H}D^{\alpha}_{1+}Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) &= {}^{H}D^{\alpha}_{1+}Y_{\alpha,\beta,0}(t,s) + {}^{H}D^{\alpha}_{1+}Y_{\alpha,\beta,1}(t,sh) + \ldots + {}^{H}D^{\alpha}_{1+}Y_{\alpha,\beta,k}\left(t,sh^k\right) \\ &= \left(\ln\frac{t}{s}\right)^{-\alpha+\beta-1}\frac{1}{\Gamma(-\alpha+\beta)} + \mathbb{A}Y_{\alpha,\beta,0}(t,s) + \mathbb{A}Y_{\alpha,\beta,1}(t,sh) + \mathbb{B}Y_{\alpha,\beta,0}\left(\frac{t}{h},s\right) \\ &+ \ldots + \mathbb{A}Y_{\alpha,\beta,k}\left(t,sh^k\right) + \mathbb{B}Y_{\alpha,\beta,k-1}\left(\frac{t}{h},sh^{k-1}\right) \\ &= \left(\ln\frac{t}{s}\right)^{-\alpha+\beta-1}\frac{1}{\Gamma(-\alpha+\beta)} + \mathbb{A}Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}(t,s) + \mathbb{B}Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\beta}\left(\frac{t}{h},s\right). \end{aligned}$$

The proof is complete. □

Theorem 1. *The solution $y(t)$ of (2) with zero initial condition has the form*

$$y(t) = \int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,f(s)\frac{ds}{s}, \quad t \geq 0.$$

Proof. Assume that any solution of a nonhomogeneous system $y(t)$ has the form

$$y(t) = \int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,h(s)\frac{ds}{s}, \quad t \geq 0, \tag{15}$$

where $h(s), 1 \leq s \leq t \leq T$ is an unknown continuous vector function and $y(1) = 0$. Having Hadamard fractional differentiation on both sides of (15) , for $1 < t \leq h$ we have

$$\begin{aligned} \left({}^{H}D^{\alpha}_{1+}y\right)(t) &= \mathbb{A}y(t) + \mathbb{B}y\left(\frac{t}{h}\right) + f(t) \\ &= \mathbb{A}\int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,h(s)\frac{ds}{s} + \mathbb{B}\int_1^{t/h} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}\left(\frac{t}{h},s\right)h(s)\frac{ds}{s} + f(t) \\ &= \mathbb{A}\int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,h(s)\frac{ds}{s} + f(t). \end{aligned}$$

On the other hand, according to Lemma 2, we have

$$\begin{aligned} \left({}^{H}D^{\alpha}_{1+}y\right)(t) &= \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_1^t \left(\ln\frac{t}{r}\right)^{-\alpha}\left(\int_1^{\mathbb{R}} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(r,s)\,h(s)\frac{ds}{s}\right)\frac{dr}{r} \\ &= \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_1^t\int_s^t \left(\ln\frac{t}{r}\right)^{-\alpha} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(r,s)\,h(s)\frac{dr}{r}\frac{ds}{s} \\ &= c(t) + \frac{1}{\Gamma(1-\alpha)}\int_1^t \left(t\frac{d}{dt}\right)\int_s^t \left(\ln\frac{t}{r}\right)^{-\alpha} Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(r,s)\,h(s)\frac{dr}{r}\frac{ds}{s} \\ &= h(t) + \mathbb{A}\int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,h(s)\frac{ds}{s}. \end{aligned}$$

Therefore, $h(t) \equiv f(t)$. The proof is complete. □

Theorem 2. *The solution $y \in C([1,T],\mathbb{R}^n)$ of (2) with $f = 0$ has a form*

$$y(t) = Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,a + \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,\mathbb{B}\varphi\left(\frac{s}{h}\right)\frac{ds}{s}.$$

Proof. We are looking for a solution which depends on an unknown constant c, and a vector function $g(t)$, of the form

$$y(t) = Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,c + \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,g(s)\,\frac{ds}{s},$$

Moreover, $y(t)$ satisfies initial conditions

$$y(t) = Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,c + \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,g(s)\,\frac{ds}{s}, \quad 1 < t \le h,$$
$$\left({}^{H}I^{1-\alpha}_{1^+}y\right)(1^+) = a.$$

We have

$$\begin{aligned} a &= \left({}^{H}I^{1-\alpha}_{1^+}y\right)(1^+) = \lim_{t\to1^+}\left({}^{H}I^{1-\alpha}_{1^+}y\right)(t) \\ &= \lim_{t\to1^+}\left(\frac{1}{\Gamma(1-\alpha)}\int_1^t (\ln t - \ln s)^{-\alpha}\, Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(s,1)\,c\frac{ds}{s}\right) \\ &= \lim_{t\to1^+}\left(\frac{1}{\Gamma(1-\alpha)}\int_1^t (\ln t - \ln s)^{-\alpha}\, e_{\alpha,\alpha}(\mathbb{A},\ln s)\,\frac{ds}{s}c\right) = c. \end{aligned}$$

Thus $c = a$. Since $1 < t \le h$, we obtain that

$$Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s) = \begin{cases} \left(\ln\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbb{A}\left(\ln\frac{t}{s}\right)^{\alpha}\right), & 1 \le s < t \le h, \\ \Theta, & t < s \le h. \end{cases}$$

Consequently, on interval $1 < t \le h$, we can easily derive

$$\begin{aligned} y(t) &= Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,a + \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,g(s)\,\frac{ds}{s} \\ &= Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,a + \int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,g(s)\,\frac{ds}{s} + \int_t^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,g(s)\,\frac{ds}{s} \\ &= (\ln t)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbb{A}(\ln t)^{\alpha}\right) a + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbb{A}\left(\ln\frac{t}{s}\right)^{\alpha}\right) g(s)\,\frac{ds}{s}. \end{aligned} \tag{16}$$

Having differentiated (16) in the Hadamard sense, we obtain

$$\begin{aligned} \left({}^{H}D^{\alpha}_{1^+}y\right)(t) &= \mathbb{A}(\ln t)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbb{A}(\ln t)^{\alpha}\right) a + \mathbb{A}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbb{A}\left(\ln\frac{t}{s}\right)^{\alpha}\right) g(s)\,\frac{ds}{s} + g(t) \\ &= \mathbb{A}y(t) + g(t). \end{aligned}$$

Therefore, $g(t) = \left({}^{H}D^{\alpha}_{1^+}y\right)(t) - \mathbb{A}y(t) = \mathbb{B}\varphi\left(\frac{t}{h}\right)$ and the desired formula holds. □

Combining Theorems 1 and 2 together we get the following result.

Corollary 1. *A solution $y \in C([1,T],\mathbb{R}^n)$ of (2) has a form*

$$\begin{aligned} y(t) = Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,a &+ \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,\mathbb{B}\varphi\left(\frac{s}{h}\right)\frac{ds}{s} \\ &+ \int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,f(s)\,\frac{ds}{s}. \end{aligned}$$

2. Existence Uniqueness and Stability

In this section, we consider the following equivalent integral form of the nonlinear Cauchy problem for fractional time-delay differential equations with Hadamard derivative (3):

$$\begin{aligned} y(t) = Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,1)\,a + \int_1^h Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\,\mathbb{B}\varphi\left(\frac{s}{h}\right)\frac{ds}{s} \\ + \int_1^t Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\,f(s,y(s))\,\frac{ds}{s}. \end{aligned} \tag{17}$$

Let us introduce the conditions under which existence and uniqueness of the integral Equation (17) will be investigated.

(A1) $f : [1,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $f(t,y) \in C_{\gamma,\ln}[1,T]$ with $\gamma < \alpha$ for any $y \in \mathbb{R}^n$;

(A2) There exists a positive constant $L_f > 0$ such that

$$\|f(t,y_1) - f(t,y_2)\| \leq L_f \|y_1 - y_2\|,$$

for each $(t,y_1),(t,y_2) \in [1,T] \times \mathbb{R}^n$.

From (A1) and (A2), it follows that

$$\|f(t,y)\| \leq L_f \|y\| + L_2 \quad \text{for some } L_2 > 0.$$

To prove existence uniqueness and stability of (17) we use the following properties of $Y_{\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)$.

Lemma 7. *We have for $sh^p < t \leq sh^{p+1}$, $p = 0,1,...,$*

$$\left\|Y_{h,\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)(t,s)\right\| \leq Y_{h,\alpha,\beta}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,s) \leq Y_{1,\alpha,\beta}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,1).$$

Proof. Indeed,

$$\begin{aligned} \left\|Y_{h,\alpha,\beta}^{\mathbb{A},\mathbb{B}}(t,s)\right\| \leq Y_{h,\alpha,\beta}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,s) &= \sum_{k=0}^{p}\sum_{n=0}^{\infty}\binom{n+k}{k}\|\mathbb{B}\|^k\|\mathbb{A}\|^n\frac{\left(\ln t - \ln sh^k\right)^{n\alpha+k\alpha+\beta-1}}{\Gamma(n\alpha+k\alpha+\beta)} \\ &\leq \sum_{k=0}^{p}\sum_{n=0}^{\infty}\binom{n+k}{k}\|\mathbb{B}\|^k\|\mathbb{A}\|^n\frac{(\ln t)^{n\alpha+k\alpha+\beta-1}}{\Gamma(n\alpha+k\alpha+\beta)} \\ &= \sum_{k=0}^{p}\sum_{n=0}^{\infty}\binom{n+k}{k}\|\mathbb{B}\|^k\|\mathbb{A}\|^n\frac{(\ln t)^{n\alpha+k\alpha+\beta-1}}{\Gamma(n\alpha+k\alpha+\beta)} \\ &= Y_{1,\alpha,\beta}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,1). \end{aligned}$$

□

Our first result on existence and uniqueness of (17) is based on the Banach contraction principle.

Theorem 3. *Assume that (A1), (A2) hold. If*

$$L_f(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1) < 1,$$

then the Cauchy problem (3) has a unique solution on $[1,T]$.

Proof. We define an operator Θ on $\mathcal{B}_r := \left\{ y \in C_{\gamma,\ln}[1,T] : \|y\|_\gamma \le r \right\}$ as follows

$$\begin{aligned}(\Theta y)(t) = Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,1)\,a + \int_1^h Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,\mathbb{B}\varphi\left(\frac{s}{h}\right)\frac{ds}{s} \\ + \int_1^t Y^{\mathbb{A},\mathbb{B}}_{h,\alpha,\alpha}(t,s)\,f(s,y(s))\,\frac{ds}{s},\end{aligned}$$

where $r \ge \dfrac{M_2}{1-M_1}$,

$$\begin{aligned}M_2 &:= (\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(t,1)\,\|a\| + (\ln h)^{-\gamma+1}(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|_{\gamma,\ln} \\ &\quad + L_2(\ln t)^{\gamma+1}\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(T,1), \\ M_1 &:= L_f(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\,\|y\|_{\gamma,\ln}.\end{aligned}$$

It is obvious that Θ is well-defined due to (A1). Therefore, the existence of a solution of the Cauchy problem (3) is equivalent to that of the operator Θ has a fixed point on $\mathcal{B}_r$. We will use the Banach contraction principle to prove that Θ has a fixed point. The proof is divided into two steps.

Step 1. $\Theta y \in \mathcal{B}_r$ for any $y \in \mathcal{B}_r$.

Indeed, for any $y \in \mathcal{B}_r$ and any $\delta > 0$, by (A3)

$$\begin{aligned}\left\|(\ln t)^\gamma(\Theta y)(t)\right\| \le (\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(t,1)\,\|a\| + (\ln t)^\gamma\int_1^h Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(t,s)\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|\frac{ds}{s} \\ + (\ln t)^\gamma\int_1^t Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(t,s)\,\|f(s,y(s))\|\,\frac{ds}{s}\end{aligned} \tag{18}$$

Firstly, we estimate the first integral:

$$\begin{aligned}&(\ln t)^\gamma\int_1^h Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(t,s)\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|\frac{ds}{s} \\ &\le (\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\int_1^h(\ln s)^{-\gamma}\frac{ds}{s}\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|_{\gamma,\ln} \\ &\le (\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)(\ln h)^{-\gamma+1}\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|_{\gamma,\ln} \\ &= (\ln h)^{-\gamma+1}(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|_{\gamma,\ln}.\end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned}&(\ln t)^\gamma\int_1^t Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(t,s)\,\|f(s,y(s))\|\,\frac{ds}{s} \\ &\le (\ln t)^\gamma\int_1^t Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(t,s)\,(L_1\|y(s)\| + L_2)\,\frac{ds}{s} \\ &\le L_f(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\,\|y\|_{\gamma,\ln} + L_2(\ln t)^{\gamma+1}\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(T,1).\end{aligned} \tag{20}$$

Inserting (19) and (20) into (18) we get

$$\begin{aligned}&\left\|(\ln t)^\gamma(\Theta y)(t)\right\| \\ &\le (\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(t,1)\,\|a\| + (\ln h)^{-\gamma+1}(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\left\|\mathbb{B}\varphi\left(\frac{s}{h}\right)\right\|_{\gamma,\ln} \\ &+ L_f(\ln t)^\gamma\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{h,\alpha,\alpha}(T,1)\,\|y\|_{\gamma,\ln} + L_2(\ln t)^{\gamma+1}\, Y^{\|\mathbb{A}\|,\|\mathbb{B}\|}_{1,\alpha,\alpha}(T,1) \\ &\le M_2 + M_1\|y\|_{\gamma,\ln} \le M_2 + M_1 r \le r.\end{aligned}$$

Step 2. Let $y, z \in C_{\gamma,\ln}[1,T]$. Then similar to the estimation (20) we get

$$\begin{aligned}
&\left\|(\ln t)^{\gamma}\left((\Theta y)(t)-(\Theta z)(t)\right)\right\| \leq (\ln t)^{\gamma}\int_1^t Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,s)\left\|f(s,y(s))-f(s,z(s))\right\|\frac{ds}{s}\\
&\leq L_f(\ln t)^{\gamma} Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\int_1^t (\ln s)^{-\gamma}(\ln s)^{\gamma}\left\|y(s)-z(s)\right\|\frac{ds}{s}\\
&\leq L_f(\ln t)^{\gamma} Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\int_1^t (\ln s)^{-\gamma}\frac{ds}{s}\left\|y-z\right\|_{\gamma,\ln}\\
&\leq L_f \ln t Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\left\|y-z\right\|_{\gamma,\ln}
\end{aligned}$$

which implies that

$$\left\|\Theta y-\Theta z\right\|_{\gamma,\ln} \leq L_f(\ln T)\, Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\left\|y-z\right\|_{\gamma,\ln}. \tag{21}$$

Hence, the operator Θ is contraction on $\mathcal{B}_r$ and the proof is competed by using the Banach fixed point theorem. □

Secondly, we discuss the Ulam–Hyers stability for the problems (3) by means of integral operator given by

$$y(t) = (\Theta y)(t),$$

where Θ is defined by (17).

Define the following nonlinear operator $Q : C_{\gamma,\ln}([1,T],\mathbb{R}^n) \to C_{\gamma,\ln}([1,T],\mathbb{R}^n)$:

$$Q(y)(t) := \left({}^{H}D_{1^+}^{\alpha}y\right)(t) - \mathbb{A}y(t) - \mathbb{B}y\left(\frac{t}{h}\right) - f(t,y(t)).$$

For some $\varepsilon > 0$, we look at the following inequality:

$$\left\|Q(y)\right\|_{\gamma,\ln} \leq \varepsilon. \tag{22}$$

Definition 6. *We say that the Equation (17) is Ulam–Hyers stable, if there exist $V > 0$ such that for every solution $y^* \in C_{\gamma,\ln}([\frac{1}{h},T],\mathbb{R}^n)$ of the inequality (22), there exists a unique solution $y \in C_{\gamma,\ln}([\frac{1}{h},T],\mathbb{R}^n)$ of problem (17) with*

$$\left\|y-y^*\right\|_{\gamma,\ln} \leq V\varepsilon. \tag{23}$$

Theorem 4. *Under the assumptions of Theorem 3, the problem (17) is stable in Ulam–Hyers sense.*

Proof. Let $y \in C_{\gamma,\ln}([\frac{1}{h},T],\mathbb{R}^n)$ be the solution of the problem (17). Let y^* be any solution satisfying (22):

$$\left({}^{H}D_{1^+}^{\alpha}y^*\right)(t) = \mathbb{A}y^*(t) + \mathbb{B}y^*\left(\frac{t}{h}\right) + f(t,y^*(t)) + Q(y^*)(t).$$

So

$$y^*(t) = \Theta(y^*)(t) + \int_1^t Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\,Q(y^*)(s)\frac{ds}{s}.$$

It follows that

$$\begin{aligned}
&(\ln t)^{\gamma}\left\|\Theta(y^*)(t)-y^*(t)\right\| \leq (\ln t)^{\gamma}\int_1^t \left\|Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\right\|\left\|Q(y^*)(s)\right\|\frac{ds}{s}\\
&\leq (\ln T)^{\gamma+1} Y_{1,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\varepsilon.
\end{aligned}$$

Therefore, we deduce by the fixed-point property (21) of the operator Θ, that

$$\begin{aligned}(\ln t)^{\gamma}\|y(t)-y^{*}(t)\| &\leq (\ln t)^{\gamma}\|\Theta(y)(t)-\Theta(y^{*})(t)\|+(\ln t)^{\gamma}\|\Theta(y^{*})(t)-y^{*}(t)\| \\ &\leq L_{f}(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\|y-y^{*}\|_{\gamma,\ln}+(\ln T)^{\gamma+1}\,Y_{1,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\varepsilon, \end{aligned} \tag{24}$$

and

$$\|y-y^{*}\|_{\gamma,\ln}\leq \frac{(\ln T)^{\gamma+1}\,Y_{1,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)}{1-L_{f}(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)}\varepsilon.$$

Thus, the problem (3) is Ulam–Hyers stable with

$$V=\frac{(\ln T)^{\gamma+1}\,Y_{1,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)}{1-L_{f}(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)}.$$

□

3. Existence Result

Our second existence result is based on the well known Schaefer's fixed point theorem. We use the following linear growth condition to replace (A_2):

(A_3) There exists $M_f > 0$ such that

$$\|f(t,y)\|\leq M_{f}\|y\|,\quad \text{for each } t\in[1,T],\ y\in\mathbb{R}^{n},$$
$$M_{f}(\ln T)^{\gamma}\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)<1.$$

Theorem 5. *Assume that (A_1) and (A_3) hold. Then the Cauchy problem (3) has at least one solution on $C_{\gamma,\ln}[1,T]$.*

Proof. Consider the operator $\Theta : C_{\gamma,\ln}[1,T] \to C_{\gamma,\ln}[1,T]$ defined as follows

$$\begin{aligned}(\Theta y)(t) &= Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,1)\,a+\int_{1}^{h}Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\,\mathbb{A}\varphi\left(\frac{s}{h}\right)\frac{ds}{s} \\ &\quad +\int_{1}^{t}Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\,f(s,y(s))\,\frac{ds}{s}.\end{aligned}$$

For the sake of convenience, we will split the proof into several steps.

Step 1. Θ is continuous.

Let $\{y_n\} \subset C_{\gamma,\ln}[1,T]$ be a sequence converging to $y \in C_{\gamma,\ln}[1,T]$. Then for each $t \in [1,T]$, we have

$$\begin{aligned}&\left\|(\ln t)^{\gamma}(\Theta y_{n})(t)-(\Theta y)(t)\right\| \\ &\leq (\ln t)^{\gamma}\int_{1}^{t}\left\|Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t,s)\right\|\|f(s,y_{n}(s))-f(s,y(s))\|\frac{ds}{s} \\ &\leq L_{f}(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\|f(\cdot,y_{n}(\cdot))-f(\cdot,y(\cdot))\|_{\gamma,\ln}.\end{aligned}$$

Since $f \in C_{\gamma,\ln}[1,T]$, we have

$$\begin{aligned}&\|\Theta y_{n}-\Theta y\|_{\gamma,\ln} \\ &\leq L_{f}(\ln T)\,Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\|f(\cdot,y_{n}(\cdot))-f(\cdot,y(\cdot))\|_{\gamma,\ln}\to 0\end{aligned}$$

as $n \to \infty$.

Step 2. Θ maps bounded sets into bounded sets in $C_{\gamma,\ln}[1,T]$.

Step 3. Θ is equicontinuous of $C_{\gamma,\ln}[1,T]$.

Let $1 < t_1 < t_2 \leq T$, $y \in \mathcal{B}_r$. Using the assumption (A_3), we have

$$\begin{aligned}
&\|(\Theta y)(t_2) - (\Theta y)(t_1)\| \\
&\leq \left\| Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_2,1)\,a - Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_1,1)\,a \right\| \\
&+ \int_1^h \left\| \left(Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_2,s) - Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_1,s) \right) \right\| \left\| \mathbb{B}\varphi\left(\frac{s}{h}\right) \right\| \frac{ds}{s} \\
&+ \int_1^{t_1} \left\| \left(Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_1,s) - Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_2,s) \right) \right\| \|f(s,y(s))\| \frac{ds}{s} \\
&+ \int_{t_1}^{t_2} \left\| Y_{h,\alpha,\alpha}^{\mathbb{A},\mathbb{B}}(t_2,s) \right\| \|f(s,y(s))\| \frac{ds}{s}.
\end{aligned}$$

The case $1 = t_1 < t_2 \leq T$ is similar. By Lemmas 5 and 7, as $t_2 \to t_1$, the right hand side tends to zero, so Θ is equicontinuous.

Steps 1–3 imply that Θ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$W = \left\{ y \in C_{\gamma,\ln}[1,T] : y = \lambda\Theta y, \text{ for some } 0 < \lambda < 1 \right\}$$

is bounded. Assume that $y \in W$, then $y = \lambda\Theta y$ for some $0 < \lambda < 1$. Thus, for any $t \in [1,T]$, we have

$$\begin{aligned}
&\left\| (\ln t)^\gamma y(t) \right\| = \lambda \left\| (\ln t)^\gamma (\Theta y)(t) \right\| \\
&\leq (\ln t)^\gamma Y_{1,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(t,1)\,\|a\| + (\ln h)^{-\gamma+1} (\ln t)^\gamma Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1) \left\| \mathbb{B}\varphi\left(\frac{s}{h}\right) \right\|_{\gamma,\ln} \\
&+ M_f (\ln t)^\gamma Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1)\,\|y\|_{\gamma,\ln}
\end{aligned}$$

Since $M_f (\ln T)^\gamma Y_{h,\alpha,\alpha}^{\|\mathbb{A}\|,\|\mathbb{B}\|}(T,1) < 1$, this shows that the set W is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Θ has a fixed point which is a solution of the Cauchy problem (3). $\square$

4. Example

In this section, we give an examples to illustrate the obtained theoretical result.

Example 1. *Let* $\alpha = 0.3$, $h = 1.2$, $k = 4$. *Consider*

$$\begin{aligned}
\left({}^H D_{1^+}^{0.3} y\right)(t) &= \mathbb{B}y\left(\frac{t}{1.2}\right), \quad t \in (1, 2.0736], \\
y(t) &= \varphi(t), \quad \frac{1}{1.2} \leq t \leq 1, \\
\left({}^H I_{\frac{1}{1.2}^+}^{0.7} y\right)\left(\frac{1}{1.2}^+\right) &= a \in \mathbb{R}^2,
\end{aligned} \tag{25}$$

where

$$\mathbb{A} = \Theta, \quad \mathbb{B} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The solution of (25) can be represented by $Y_{h,\alpha,\alpha}^{\Theta,\mathbb{B}}(t,s)$

$$y(t) = Y_{h,\alpha,\alpha}^{\Theta,\mathbb{B}}(t,1)\,a + \int_1^h Y_{h,\alpha,\alpha}^{\Theta,\mathbb{B}}(t,s)\,\mathbb{B}\varphi\left(\frac{s}{h}\right)\frac{ds}{s},$$

where

$$Y_{h,\alpha,\alpha}^{\Theta,\mathbb{B}}(t,s)=\begin{cases}0, & -\infty<t<s,\\ I, & t=s,\\ I\frac{(\ln t-\ln s)^{-0.7}}{\Gamma(0.3)}, & s<t\leq 1.2s,\\ I\frac{(\ln t-\ln s)^{-0.7}}{\Gamma(0.3)}+\mathbb{B}\frac{((\ln t-\ln 1.2s))^{-0.4}}{\Gamma(0.6)}, & 1.2s<t\leq(1.2)^2 s,\\ I\frac{(\ln t-\ln s)^{-0.7}}{\Gamma(0.3)}+\mathbb{B}\frac{((\ln t-\ln 1.2s))^{-0.4}}{\Gamma(0.6)}+\mathbb{B}^2\frac{\left(\ln t-\ln(1.2)^2 s\right)^{-0.1}}{\Gamma(0.9)} & (1.2)^2 s<t\leq(1.2)^3 s,\\ I\frac{(\ln t-\ln s)^{-0.7}}{\Gamma(0.3)}+\mathbb{B}\frac{((\ln t-\ln 1.2s))^{-0.4}}{\Gamma(0.6)}+\mathbb{B}^2\frac{\left(\ln t-\ln(1.2)^2 s\right)^{-0.1}}{\Gamma(0.9)} & \\ +\mathbb{B}^3\frac{\left(\ln t-\ln(1.2)^3 s\right)^{0.2}}{\Gamma(1.2)} & (1.2)^3 s<t\leq(1.2)^4 s.\end{cases}$$

Example 2. *Consider*

$$\left({}^H D_{1^+}^{0.3}y\right)(t)=\mathbb{B}y\left(\frac{t}{1.2}\right)+L_f\begin{pmatrix}\sin y_1(t)\\ t\sin y_2(t)-y_1(t)\cos t\end{pmatrix},\ t\in(1,2.0736],$$

$$y(t)=\varphi(t),\ \frac{1}{1.2}\leq t\leq 1,$$

$$\left({}^H I_{\frac{1}{1.2}^+}^{0.7}y\right)\left(\frac{1}{1.2}^+\right)=a\in\mathbb{R}^2.$$

Clearly, the function

$$f(t,y)=L_f\begin{pmatrix}\sin y_1\\ t\sin y_2-y_1\cos t\end{pmatrix}$$

is jointly continuous and Lipschitz continuous with respect to y*. We can choose* $L_f>0$ *so that the conditions of Theorems 3 and 4 are satisfied. Thus, the above problem has a unique solution which is Ulam–Hyers stable.*

5. Conclusions

In this paper, we have introduced delayed perturbation of the M-L matrix exponential with logarithms, to get a representation formula for time-delay Hadamard type fractional differential equations with non-commutative linear part. Using this representation formula we have obtained several existence results for an initial value problem of time-delay Hadamard-type fractional differential equations. Furthermore, we have presented a sufficient condition for stability in the Ulam–Hyers sense. In our future work, we are planing to investigate the existence, stability and controllability of solutions to an initial value problem for time-delay fractional differential equations involving a combination of Caputo and Hadamard fractional derivatives.

Author Contributions: All authors contributed equally in this research paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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