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A Multi-Attribute Decision-Making Algorithm Using Q-Rung Orthopair Power Bonferroni Mean Operator and Its Application

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Abstract: The process of decision-making is subject to various influence factors and environmental uncertainties, which makes decision become a very complex task. As a new type of decision processing tool, the q-rung orthopair fuzzy sets can effectively deal with complex uncertain information arising in the decision process. To this end, this study proposes a new multi-attribute decision-making algorithm based on the power Bonferroni mean operator in the context of q-rung orthopair fuzzy information. In this method, in view of multi-attribute decision-making problem of internal relationship between multiple variables and extreme evaluation value, the Bonferroni mean operator is combined with power average operator. Then, the integrated operator is introduced into the q-rung orthopair fuzzy set to develop a new q-rung orthopair power Bonferroni mean operator, and some relevant properties of this new operator are discussed. Secondly, a multi-attribute decision-making method is established based on this proposed operator. Finally, the feasibility and superiority of our method are testified via a numerical example of investment partner selection in the tourism market.

Keywords: q-rung orthopair fuzzy set; power Bonferroni mean operator; multi-attribute decision-making algorithm; complex uncertain information

1. Introduction

With the rapid economic development, increasingly more practical decision-making problems with uncertainty information, like supplier selection [1,2], venture capital project evaluation [3,4], consumption decision [5,6], emergency decision-making [7,8], depend on expert's subjective decision. It is well known that optimal decision-making often needs multi-aspect consideration on decision schemes. However, in complex environment, decision-making information, decision-making conditions and decision-making processes involve abundant uncertain factors and decision-makers' preferences. Hence, conducting an optimal decision process is an extremely difficult task for decision-makers. In response to these problems, many scholars proposed the fuzzy multi-attribute decision-making (MADM) methods, which are widely used in engineering construction, design, finance and management. Nevertheless, with the increasing complexity of the decision-making environment, how to improve accuracy in information expression of decision attributes and reliability of decision-making results are the key concerns in MADM theory. Where, the tools used by many scholars to handle the above problems include MADM methods based on intuitionistic fuzzy set (IFS) [9–11], Pythagorean fuzzy set (PFS) [12–14], picture fuzzy set (PtFS) [15–17] and q-rung orthopair fuzzy set (q-ROFS) [18–23]. Of which, q-ROFS is a powerful tool to solve issues of information representation and information aggregation.

The concept of q-ROFS proposed by Yager [24], characterized by greater information representation space than traditional IFS and PFS. MADM methods based on q-ROFS have higher flexibility. Since the advantages of q-ROFS, many scholars have extended it. For example, Liu and Liu [25] firstly developed the family of q-rung orthopair fuzzy (q-ROF) information aggregation operators. Ju et al. [26] studied the q-ROF weighted average operator with interval value and constructed a corresponding multi-attribute group decision-making method. In addition, to investigate correlation between multiple attributes, Liu and Liu [27] applied the classic Bonferroni mean (BM) operator to q-ROFS and proposed the q-ROF Bonferroni mean operator. Based on Minkowski distance measure, Du [28] studied Minkowski distance of q-ROF sets. Wei et al. [29] proposed a series of Heornian average operators under q-ROFS to reflect the correlation between input variables. After that, Peng et al. [30] further proposed exponential operation laws of q-ROFS and its exponential integration operator. In the latest research report, Liu and Wang [31] established two q-ROFS-based MADM methods using the improved generalized weighted Maclaurin symmetric average operator and generalized weighted geometric Maclaurin symmetric average operator. Garg and Chen [32] studied the characteristics of membership coefficients and interaction between the membership functions of q-ROFS. Combining previous studies, Garg [33] defined a new sine triangle q-ROF weighted average geometric operator. Moreover, to solve the problems that the existing operators cannot effectively aggregate some information, Shao and Zhuo [34] proposed two new curve integrals under the q-ROF environment, thus providing a new method for solving the MADM problem under discrete or continuous q-ROF information. Pinar and Boran [35] proposed a new distance measure applicable to q-ROFSs, etc.

In overall, compared with IFS and PFS, the current researches on q-ROFS are still in its infancy. Its relevant researches focused on the information aggregation operators and operational rules of q-ROFS, and produced a series of rich results. However, the existing research does not fully consider the extreme value of the evaluation information and the heterogeneous relationship between the attributes and their influence on the decision-making results. As mentioned above, complex MADM under uncertain environment relies on the subjective evaluation of experts or decision makers. Therefore, the evaluation information based on the expert's preference will inevitably have some extreme values that are too high or too low, which will affect the rationality of the decision result. In addition, since many attributes (factors) are involved in actual decision-making, attributes interacting with each other will also affect the decision maker's evaluation, and then influence the ranking results of alternatives. In order to avoid the above-mentioned unreasonable phenomena, this study uses the power average (PA) operator to eliminate the adverse effect of extreme values given by experts on the model's output. Meanwhile, the BM operator is embedded in the q-ROF environment to investigate the interaction relationship between attributes. Subsequently, a new multi-attribute decision-making algorithm using q-rung orthopair power Bonferroni mean operator (q-ROFPBM) is developed.

The contributions of the study are to propose the q-ROF power Bonferroni mean aggregation operator and established a new proposed operator-based multi-attribute decision making algorithm.

The remainder of this paper are as follows: Section 2 explains the necessary basic knowledge of q-ROFSs, including algorithm rules, size comparison method and distance measure. Section 3 proposes the q-ROF power Bonferroni mean (q-ROFPBM) operator and proves the boundedness, idempotence and monotonicity of the operator. Section 4 establishes a MADM algorithm based on the q-ROFPBM operator. Section 5 gives a numeric example to testify the effectiveness of the proposed method. Section 6 summarizes some conclusions and proposes future development directions of this study.

2. Preliminaries Knowledge

Definition 1 [24]. *Suppose X is a non-empty general set, then the expression of the q-ROFS A is defined as:*

$$A = \{< x, u_A(x), v_A(x) > | x \in X\}$$

where, $u_A(x): X \to [0,1]$ *and* $v_A(x): X \to [0,1]$ *respectively represent the membership function and non-membership function of the element* $x \in X$ *to* A *, and there is* $\forall x \in X$ *,* $0 \leq u_A(x)^q + v_A(x)^q \leq 1 (q \geq 1)$*. Then, the hesitancy degree is defined as* $\pi_A(x) = \sqrt[q]{1 - u_A(x)^q - v_A(x)^q}$*. Liu and Wang [36] define* $A = (u_A, v_A)$ *as a q-ROF number.*

Definition 2 [25]. *Suppose* $\alpha_1 = (u_1, v_1)$*,* $\alpha_2 = (u_2, v_2)$ *and* $\alpha = (u, v)$ *are three q-ROF numbers,* λ*,* λ_1*,* λ_2 *are any real numbers greater than or equal to zero and the following algorithm is specified:*

(1) $\alpha_1 \oplus \alpha_2 = (\sqrt[q]{u_1{}^q + u_2{}^q - u_1{}^q u_2{}^q}, v_1 v_2)$

(2) $\alpha_1 \otimes \alpha_2 = (u_1 u_2, \sqrt[q]{v_1{}^q + v_2{}^q - v_1{}^q v_2{}^q})$

(3) $\lambda\alpha = (\sqrt[q]{1 - (1 - u^q)^\lambda}, v^\lambda)$

(4) $\alpha^\lambda = \left(u^\lambda, \sqrt[q]{1 - (1 - v^q)^\lambda}\right)$

The above-mentioned operators have some properties as follows:

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$

(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$

(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$

(4) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$

(5) $\lambda_1\alpha + \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$

(6) $\alpha^{\lambda 1} \otimes \alpha^{\lambda 2} = \alpha^{\lambda 1 + \lambda 2}$

Definition 3 [25]. *Suppose* $\alpha = (u, v)$ *is a q-ROF number, then the score function of* α *is defined as* $S_{(\alpha)} = u^q - v^q$*, the exact function of* α *is defined as* $H_{(\alpha)} = u^q + v^q$*. For two arbitrary q-ROF numbers* $\alpha_1 = (u_1, v_1)$*,* $\alpha_2 = (u_2, v_2)$*, we can get*

(1) *If* $S_{(\alpha 1)} > S_{(\alpha 2)}$*, then* $\alpha_1 > \alpha_2$

(2) *If* $S_{(\alpha 1)} = S_{(\alpha 2)}$*, and if* $H_{(\alpha 1)} = H_{(\alpha 2)}$*, then* $\alpha_1 = \alpha_2$*; if* $H_{(\alpha 1)} < H_{(\alpha 2)}$*, then* $\alpha_1 < \alpha_2$

Definition 4 [25]. *Suppose* $\alpha_1 = (u_1, v_1)$*,* $\alpha_2 = (u_2, v_2)$ *are any two q-ROF numbers, then the standard Hamming distance* $d(\alpha_1, \alpha_2)$ *of* α_1 *and* α_2 *can be defined as :*

$$d(\alpha_1, \alpha_2) = \frac{|u_1{}^q - u_2{}^q| + |v_1{}^q - v_2{}^q| + |\pi_1{}^q - \pi_2{}^q|}{2} \quad (1)$$

where, $\pi_1 = \sqrt[q]{1 - u_1{}^q - v_1{}^q}$*,* $\pi_2 = \sqrt[q]{1 - u_2{}^q - v_2{}^q}$*.*

3. q-ROFPBM Operator

3.1. The Concept and Demonstration of q-ROFPBM Operator

Definition 5 [37]. *Suppose* $\alpha_k (k = 1,2,\cdots,n)$ *is a series of non-negative real numbers, s and t are non-negative real numbers with they are not 0 at the same time, we call*

$$PBM^{s,t}(a_1, a_2, a_3, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1, i \neq j \\ i,j \in k}}^{n} \left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))} a_i\right)^s \cdot \left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))} a_j\right)^t\right)\right)^{\frac{1}{s+t}} \quad (2)$$

the power Bonferroni mean (PBM) operator.

The expression form of $T(a_i)$ *in Formula (2) is as follows:*

$$T(a_i) = \sum_{\substack{i,j=1, i\neq j \\ i,j\in k}}^{n} Sup(a_i, a_j)\ (i, j \in k = 1,2,\cdots,n),$$

where $Sup(a_i, a_j)$ *means the support degree of* a_i *and* a_j*, which meet the following conditions:*

(1) $Sup(a_i, a_j) \in [0,1]$
(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$
(3) $Sup(a, b) \geq Sup(c, d)$ *if and only if* $|a - b| \geq |c - d|$.
(4) *If* $d(a_1, a_2) \leq d(a_3, a_4)$*, then* $Sup(a_1, a_2) \geq Sup(a_3, a_4)$*, where* d *is the distance between* a_i *and* a_j.

Definition 6. *Suppose* $\alpha_k (k = 1,2,\cdots,n)$ *is a set of q-ROF numbers,* s *and* t *are non-negative real numbers that are not 0 at the same time. Where,* $a_k = (u_k, v_k)(k = 1,2,\ldots,n)$ *and* $q \geq 1$*. Then, q-ROFPBM* $(q-ROFPBM)$ *operator of* $a_k (k = 1,2,\cdots,n)$ *can be defined as:*

$$q-ROFPBM^{s,t}(a_1, a_2, a_3,\cdots, a_n) = \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1, i\neq j \\ i,j\in k}}^{n} \left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))} a_i\right)^s \otimes \left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))} a_j\right)^t\right)\right)^{\frac{1}{s+t}} \quad (3)$$

where,

$$T(a_i) = \sum_{\substack{i,j=1, i\neq j \\ i,j\in k}}^{n} Sup(a_i, a_j)\ (i, j \in k = 1,2,\cdots,n),$$

$Sup(a_i, a_j)$ *means support degree of* a_i *and* a_j*, which meet the following conditions:*

(1) $Sup(a_i, a_j) \in [0,1]$
(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$
(3) $Sup(a, b) \geq Sup(c, d)$ *if and only if* $|a - b| \geq |c - d|$,
(4) *If* $d(a_1, a_2) \leq d(a_3, a_4)$*, then* $Sup(a_1, a_2) \geq Sup(a_3, a_4)$*, where* d *is the distance between* a_i *and* a_j.

Definition 7. *Suppose* $\alpha_k = (u_k, v_k)(k = 1,2,\cdots,n;\ i, j \in k = 1,2,\cdots,n)$ *is a q-ROF number,* s *and* t *are non-negative real numbers that are not 0 at the same time, then the integration value obtained by using* $q-ROFPBM$ *operator is still a q-ROF number, and*

$$q-ROFPBM^{s,t}(a_1, a_2,\cdots, a_n)$$
$$= \left(\sqrt[q]{\{1 - [\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1 - (1 - (1 - {u_i}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s (1 - (1 - (1 - {u_j}^q)^{\frac{n(1+T(a_j))}{\sum_{i=1}^{n}(1+T(a_j))}})^t)^q]^{\frac{1}{n(n-1)}}\}^{\frac{1}{s+t}}},\right.$$

$$\left.\sqrt[q]{1 - \{1 - [\prod_{\substack{i,j=1 \\ i\neq j}}^{n}(1 - (1 - {v_i}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s (1 - {v_j}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]^{\frac{1}{n(n-1)}}\}^{\frac{1}{s+t}}}\right) \quad (4)$$

Proof:

Firstly, according to the operation rule of (3) in Definition 2, we can get

$$\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))} a_i = (\sqrt[q]{1 - (1 - {u_i}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}}, {v_i}^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})$$

and

$$\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j=(\sqrt[q]{1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}},\ v_j{}^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})$$

Then, based on the Formula (4) in Definition 2, we can obtain

$$(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s=((\sqrt[q]{1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}})^s,\ \sqrt[q]{1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s}\ ),$$

and

$$(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t=((\sqrt[q]{1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}})^t,\ \sqrt[q]{1-(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t}\ ).$$

Meanwhile, we use the Equation (2) in Definition 2 to get

$$(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t$$

$$=(\ (\sqrt[q]{1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}})^s(\sqrt[q]{1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}})^t,$$

$$\sqrt[q]{\begin{array}{l}2-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s-(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t-\\ {[1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s][1-(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\end{array}}\ )$$

$$\overset{n}{\underset{i,j=1,i\neq j}{\oplus}}((\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t)$$

$$=(\sqrt[q]{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}[1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s\,(1-(1-u_j{}^q)^{\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\ ,$$

$$\sqrt[q]{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\ [1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\ ).$$

Next, we prove the above equation is always true for any i, j.

(1) If $n=2$, we can get

$$\overset{2}{\underset{i,j=1,i\neq j}{\oplus}}\ ((\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t\ )\ =$$

$$\left[(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_1)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_2)^t\right]\oplus\left[(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_2)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_1)^t\right]=$$

$$(\sqrt[q]{\begin{array}{l}1-(1-(1-(1-u_1{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_2{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)\\ (1-(1-(1-u_2{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_1{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)\end{array}}\ ,$$

$$\sqrt[q]{\begin{array}{l}(1-(1-v_1{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_2{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t\ )\\ (1-(1-v_2{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_1{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t\ )\end{array}}\ )$$

Therefore, we know that if $n=2$, the Definition 7, that is

$$\overset{n}{\underset{i,j=1,i\neq j}{\oplus}}\ ((\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t\ )\ =$$

$$\left(\sqrt[q]{1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left[1-\left(1-\left(1-u_i{}^q\right)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-\left(1-u_j{}^q\right)^{\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]},\right.$$

$$\left.\sqrt[q]{\prod_{\substack{i,j=1\\i\neq j}}^{n}\left[1-\left(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]}\right).$$

is true.

(2) If n = k (k > 2), that is

$$\bigoplus_{i,j=1,i\neq j}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)=$$

$$\left(\sqrt[q]{1-\prod_{\substack{i,j=1\\i\neq j}}^{k}\left[1-\left(1-\left(1-u_i{}^q\right)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-\left(1-u_j{}^q\right)^{\frac{n(1+T(a_{ij}))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]},\right.$$

$$\left.\sqrt[q]{\prod_{\substack{i,j=1\\i\neq j}}^{k}\left[1-\left(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]}\right).$$

If n = k + 1, then, we can get

$$\bigoplus_{i,j=1,i\neq j}^{k+1}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)=$$

$$\left(\bigoplus_{i,j=1,i\neq j}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)\right.$$

$$\oplus\left(\bigoplus_{i=1}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_{k+1}\right)^t\right)\right)$$

$$\left.\oplus\left(\bigoplus_{j=1}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_{k+1}\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)\right)\right)$$

Further, we can infer that

$$\bigoplus_{i=1}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_{k+1}\right)^t\right)$$

$$=\left(\sqrt[q]{1-\prod_{i=1}^{k}\left[1-\left(1-\left(1-u_i{}^q\right)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-\left(1-u_{k+1}{}^q\right)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]},\right.$$

$$\left.\sqrt[q]{\prod_{i=1}^{k}\left[1-\left(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-v_{k+1}{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]}\right).$$

and

$$\bigoplus_{j=1}^{k}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_{k+1}\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)=$$

$$\left(\sqrt[q]{1-\prod_{j=1}^{k}\left[1-\left(1-\left(1-u_{k+1}{}^q\right)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-\left(1-u_j{}^q\right)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]},\right.$$

$$\left.\sqrt[q]{\prod_{j=1}^{k}\left[1-\left(1-v_{k+1}{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}}\right)^s\left(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}}\right)^t\right]}\right).$$

Then, we can get

$$\underset{i,j=1,i\neq j}{\overset{k+1}{\oplus}}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)$$

$$=\left(\sqrt[q]{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}[1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]},\right.$$

$$\left.\sqrt[q]{\prod_{\substack{i,j=1\\ i\neq j}}^{k}[1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\right)\oplus$$

$$\left(\sqrt[q]{1-\prod_{i=1}^{k}[1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_{k+1}{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]},\right.$$

$$\left.\sqrt[q]{\prod_{i=1}^{k}[1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_{k+1}{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\right)\oplus$$

$$\left(\sqrt[q]{1-\prod_{j=1}^{k}[1-(1-(1-u_{k+1}{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]},\right.$$

$$\left.\sqrt[q]{\prod_{j=1}^{k}[1-(1-v_{k+1}{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\right)=$$

$$\left(\sqrt[q]{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}[1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]},\right.$$

$$\left.\sqrt[q]{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}[1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t]}\right).$$

That is, the equation is also true for $n = k + 1$ or any n.

Again, according to the Formula (3) in Definition 2, we can get

$$\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)\right)=$$

$$\left(\sqrt[q]{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)\right)^{\frac{1}{n(n-1)}}}\right.$$

$$\left.\left(\sqrt[q]{\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)}\right)^{\frac{1}{n(n-1)}}\right)$$

Again, based on the Formula (4) in Definition 2, we can obtain

$$\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\left(\left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i\right)^s\otimes\left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j\right)^t\right)\right)\right)^{\frac{1}{s+t}}=$$

$$\left(\frac{1}{n(n-1)}\left(\sqrt[q]{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(1-u_i{}^q)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-(1-u_j{}^q)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)},\right.\right.$$

$$\left.\left.\sqrt[q]{\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}})^s(1-v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}})^t)}\right)\right)^{\frac{1}{s+t}}=$$

$$= \left(\left(\sqrt[q]{1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left[1 - \left(1 - \left(1 - u_i{}^q \right)^{\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \right)^s \left(1 - \left(1 - u_j{}^q \right)^{\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}} \right)^t \right] \right)^{\frac{1}{n(n-1)}}} \right)^{\frac{1}{s+t}}, \right.$$

$$\left. \sqrt[q]{1 - \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left[1 - \left(1 - v_i{}^{\frac{qn(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}} \right)^s \left(1 - v_j{}^{\frac{qn(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}} \right)^t \right] \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{s+t}}} \right) =$$

$$q - ROFPBM^{s,t}(a_1, a_2, \cdots, a_n)$$

is true, that is the Definition 7 is always true. □

3.2. Some Properties of the q-ROFPBM Operator

The basic properties of the q-ROFPBM operator are discussed below:

(1) (Idempotence). Suppose $\alpha_i = (u_i, v_i)(i = 1,2,\cdots,n)$ is a q-ROF number, and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then

$$q - ROFPBM^{s,t}(a_1, a_2, a_3, \cdots, a_n) = \alpha\,. \tag{5}$$

Proof:

$$q - ROFPBM^{s,t}(a_1, a_2, a_3, \cdots, a_n)$$

$$= \left(\frac{1}{n(n-1)} \left(\mathop{\oplus}_{i,j=1, i \neq j}^{n} \left\{ \left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))} a_i \right)^s \otimes \left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))} a_j \right)^t \right\} \right) \right)^{\frac{1}{s+t}}$$

$$= \left(\frac{1}{n(n-1)} \left(\mathop{\oplus}_{i,j=1, i \neq j}^{n} \left\{ \left(\frac{n(1+T(a))}{\sum_{i=1}^{n}(1+T(a))} a \right)^s \otimes \left(\frac{n(1+T(a))}{\sum_{j=1}^{n}(1+T(a))} a \right)^t \right\} \right) \right)^{\frac{1}{s+t}}$$

$$= \left(\frac{1}{n(n-1)} \mathop{\oplus}_{i,j=1, i \neq j}^{n} a^{s+t} \right)^{\frac{1}{s+t}}$$

$$= \mathrm{a}$$

□

(2) (Permutation invariability). Suppose $\alpha_i = (u_i, v_i)(i = 1,2,\cdots,n)$ is a q-ROF number, and $\beta_1, \beta_2, \beta_3 \ldots \beta_n$ is any permutation and combination of α_1, α_2, $\alpha_3 \ldots \alpha_n$, then

$$q - ROFPBM^{s,t}(a_1, a_2, a_3, \cdots, a_n) = q - ROFPBM^{s,t}(\beta_1, \beta_2, \beta_3 \ldots \beta_n) \tag{6}$$

Proof:

$$q - ROFPBM^{s,t}(a_1, a_2, a_3, \cdots, a_n)$$

$$= \left(\frac{1}{n(n-1)} \left(\mathop{\oplus}_{i,j=1, i \neq j}^{n} \left\{ \left(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))} a_i \right)^s \otimes \left(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))} a_j \right)^t \right\} \right) \right)^{\frac{1}{s+t}}$$

$$= \left(\frac{1}{n(n-1)} \left(\mathop{\oplus}_{i,j=1, i \neq j}^{n} \left\{ \left(\frac{n(1+T(\beta_i))}{\sum_{i=1}^{n}(1+T(\beta_i))} \beta_i \right)^s \otimes \left(\frac{n(1+T(\beta_j))}{\sum_{j=1}^{n}(1+T(\beta_j))} \beta_j \right)^t \right\} \right) \right)^{\frac{1}{s+t}}$$

$$= q - ROFPBM^{s,t}(\beta_1, \beta_2, \beta_3 \ldots \beta_n)$$

□

(3) (Boundedness). Suppose $\alpha_i = (u_i, v_i)(i = 1,2,\cdots,n)$ is a q-ROF number. Let $\beta_i = \frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))} a_i$, $\beta^+ = \max(\beta_i)$, $\beta^- = \min(\beta_i)$, then

$$\beta^- \leq q-ROFPBM^{s,t}(a_1,a_2,a_3,\cdots,a_n) \leq \beta^+ \quad (7)$$

Proof:

$$q-ROFPBM^{s,t}(a_1,a_2,a_3,\cdots,a_n)=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\{(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t\}\right)\right)^{\frac{1}{s+t}}$$

$$\leq\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\{(\beta^+)^s\otimes(\beta^+)^t\}\right)\right)^{\frac{1}{s+t}}=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}(\beta^+)^{s+t}\right)\right)^{\frac{1}{s+t}}$$

$$=((\beta^+)^{s+t})^{\frac{1}{s+t}}=\beta^+$$

$$q-ROFPBM^{s,t}(a_1,a_2,a_3,\cdots,a_n)=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\{(\frac{n(1+T(a_i))}{\sum_{i=1}^{n}(1+T(a_i))}a_i)^s\otimes(\frac{n(1+T(a_j))}{\sum_{j=1}^{n}(1+T(a_j))}a_j)^t\}\right)\right)^{\frac{1}{s+t}}$$

$$\geq\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}\{(\beta^-)^s\otimes(\beta^-)^t\}\right)\right)^{\frac{1}{s+t}}=\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,i\neq j}{\overset{n}{\oplus}}(\beta^-)^{s+t}\right)\right)^{\frac{1}{s+t}}$$

$$=((\beta^-)^{s+t})^{\frac{1}{s+t}}=\beta^-$$

We can get

$$\beta^- \leq q-ROFPBM^{s,t}(a_1,a_2,a_3,\cdots,a_n) \leq \beta^+$$

□

4. The MADM Algorithm Based on the q-ROFPBM Operator

In this section, we will establish a new MADM algorithm based on the q-ROFPBM operator. In the MADM problem, suppose there are n candidate solutions $X=(x_1,x_2,x_3\ldots x_n)$, m decision attributes $C=(C_1,C_2,C_3\ldots C_m)$. Let the attribute value of scheme x_i under the attribute c_j be a_{ij} $(i=1,2,3,\ldots,n; j=1,2,3,\ldots,m)$. Where, a_{ij} is the form of q-ROF number, so the q-ROF decision matrix $M=\left(a_{ij}\right)_{mn}$, $a_{ij}=\left(u_{ij},v_{ij}\right)$ can be obtained. u_{ij} and v_{ij} represent the values of the membership degree and non-membership degree of candidate scheme i with respect to attribute j, which satisfy $u_{ij}^q+v_{ij}^q\leq 1$, and $0\leq u_{ij},v_{ij}\leq 1$. u_{ij} indicates the degree to agree with the meaning of an attribute representation, v_{ij} denotes the degree of corresponding negation. The values of u_{ij} and v_{ij} are obtained by the experts or decision makers' subjective evaluation based on their own experience and knowledge. The closer the values of u_{ij},v_{ij} get to 1, the bigger the degree of approval or disapproval is, and vice versa. Next, the q-ROFPBM operator-based MADM algorithm steps are as follows:

Step 1: Normalize the decision information

There are different attribute properties in the q-ROF decision matrix $M=\left(a_{ij}\right)_{mn}$, including cost-related attribute and benefit-related attribute, so we must transform the cost-related attribute into benefit-related attribute by using the following equation [36]:

$$\hat{a}_{ij}=\left(\hat{u}_{ij},\hat{v}_{ij}\right)=\begin{cases}\left(u_{ij},v_{ij}\right) & \text{if the attribute } C_j \text{ is benefit type}\\ \left(v_{ij},u_{ij}\right) & \text{if the attribute } C_j \text{ is cost type}\end{cases} \quad (8)$$

Step 2: Use the q-ROFPBM operator to integrate attribute values $\hat{a}_{ij}$ into comprehensive attribute value $\hat{a}_i$ as follows:

$$\hat{a}_i=q-ROFPBM^{s,t}(\hat{a}_{i1},\hat{a}_{i2},\hat{a}_{i3},\cdots,\hat{a}_i) \quad (9)$$

Step 3: Calculate the score values $S_{(\hat{a}_i)}$ and $H_{(\hat{a}_i)}$ of each alternative according to Definition 3.

Step 4: Sort and select the optimal solution according to the comparing rules of score values in Definition 3.

5. Numerical Example

5.1. Decision Process

A certain place vigorously carried out investment promotion and introduced a tourism investment company to prepare for investment in some small and medium-sized tourism-related enterprises in the region. However, due to insufficient funds and policy restrictions, the tourism investment company intends to find partners in the tourism market to reduce costs, increase revenue opportunities and diversify investment risks. After a rigorous market survey, 5 partners are initially identified: x_1 car company, x_2 chain restaurant, x_3 software company, x_4 military supplies, x_5 home appliance manufacturer. Based on its own situation, an opinion group composed of senior management and the board of directors decided to examine the candidates from four factors. They are C_1 risk assessment (including the company's capital reserve, investment–oriented assessment, etc.), C_2 growth assessment (including company size, performance, development prospects, etc.), C_3 social impact analysis (including company social status, user groups, social reputation, etc.), C_4 environmental impact analysis (including corporate sustainable development strategies, environmental friendly rating, etc.), respectively. Based on the simulated evaluation data giving by virtual decision makers, a q-ROF information decision matrix $M = (a_{ij})_{5\times4}$ is established. Where, the evaluation value $a_{ij} = (u_{a_{ij}}, v_{a_{ij}})$ is the form of q-ROF number as shown in Table 1.

Table 1. q-rung orthopair fuzzy set (q-ROF) information decision matrix.

	C_1		C_2		C_3		C_4	
	u	v	u	v	u	v	u	v
x_1	0.4	0.5	0.5	0.4	0.2	0.7	0.2	0.5
x_2	0.6	0.4	0.6	0.3	0.6	0.3	0.3	0.6
x_3	0.5	0.5	0.4	0.5	0.4	0.4	0.5	0.4
x_4	0.7	0.2	0.5	0.4	0.2	0.5	0.6	0.7
x_5	0.5	0.3	0.3	0.4	0.6	0.2	0.4	0.4

Step 1: Normalize the decision information.

Since all the attributes in Table 1 are benefit type, the converted attributes' information by Formula (8) is the same with the original information in Table 1.

Step 2: Use the $q - ROFPBM^{s,t}$ operator to calculate the comprehensive attribute value as shown in Table 2. To distinguish q-ROFS from IFS and PFS, we set the parameter q = 3. Meanwhile, to facilitate comparison, we also calculate the comprehensive attribute values under the different parameters $s = t = 1$，2，3, respectively, as shown in Table 2.

Table 2. The comprehensive attribute value of each scheme.

s,t	x_1		x_2		x_3		x_4		x_5	
	u	v	u	v	u	v	u	v	u	v
1	0.352	0.536	0.549	0.414	0.453	0.453	0.540	0.483	0.464	0.335
2	0.389	0.532	0.571	0.411	0.458	0.453	0.574	0.477	0.485	0.335
3	0.408	0.528	0.581	0.408	0.464	0.452	0.590	0.470	0.499	0.334

Step 3: Calculate the score values $S_{(\hat{a}_i)}$ $(i = 1,2,3,4,5)$ with the different parameters $s = t = 1$，2，3, respectively, as follows:

When $s, t = 1$,

$s_{(x_1)} = -0.1103, s_{(x_2)} = 0.0941, s_{(x_3)} = -0.0002, s_{(x_4)} = 0.0449, s_{(x_5)} = 0.0621$;

when $s, t = 2$,

$s_{(x_1)} = -0.0916$，$s_{(x_2)} = 0.1164$，$s_{(x_3)} = 0.0035$，$s_{(x_4)} = 0.0805$，$s_{(x_5)} = 0.0765$;

when $s, t = 3$,

$s_{(x_1)} = -0.0790$，$s_{(x_2)} = 0.1283$，$s_{(x_3)} = 0.0072$，$s_{(x_4)} = 0.1013$，$s_{(x_5)} = 0.0871$;

Step 4: Rank the score values of alternatives under different s, t, as follows:

When $s, t = 1$, $s_{(x_2)} > s_{(x_5)} > s_{(x_4)} > s_{(x_3)} > s_{(x_1)}$, and x_2 is the optimal solution.

When $s, t = 2$, $s_{(x_2)} > s_{(x_4)} > s_{(x_5)} > s_{(x_3)} > s_{(x_1)}$, and x_2 is the optimal solution.
When $s, t = 3$, $s_{(x_2)} > s_{(x_4)} > s_{(x_5)} > s_{(x_3)} > s_{(x_1)}$, and x_2 is the optimal solution.

5.2. Comparative Analysis

In order to show the superiority of the q-ROFABM operator in this study, we compare the ranking results based on the q-ROF power average (q-ROFPA) operator and q-ROF Bonferroni mean (q-ROFBM) operator with that using the proposed operator, their calculation steps are as described in Section 4.

(1) When using the q-ROFPA operator.

Step 1: Normalize the data in Table 1.
Step 2: Use the q-ROFPA operator with q=3 in Formula (10)

$$q - ROFPA(a_1, a_2, a_3, \cdots, a_n) = \sum_{i=1}^{n} \frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))} a_i \tag{10}$$

to get the comprehensive information value of each scheme as follows:

$x_1 = (0.6084, 0.0432)$, $x_2 = (0.8401, 0.0106)$

$x_3 = (0.7222, 0.0221)$, $x_4 = (0.8477, 0.0151)$, $x_5 = (0.7524, 0.0041)$

Step 3: Calculate the score value of each scheme based on the Definition 3 as follows:

$$s_{(x_1)} = 0.2251,\ s_{(x_2)} = 0.5930,\ s_{(x_3)} = 0.3767,\ s_{(x_4)} = 0.6091,\ s_{(x_5)} = 0.4259$$

Step 4: Rank the score values of alternatives: $s_{(x_4)} > s_{(x_2)} > s_{(x_5)} > s_{(x_3)} > s_{(x_1)}$, and we can infer that the alternative x_4 is the optimal scheme based on the Definition 3.

It can be seen that the best choice based on the q-ROFPA operator is x_4, which is significantly different compared with the best choice x_2 using our q-ROFPBM operator. The main reason for the different ranking results is that q-ROFPA operator does not consider the heterogeneous relationship between different attributes, while the proposed method in this paper fully considers the relationship between different attributes and their impact on decision-making results. In the proposed q-ROFPBM operator, we embedded the classic Bonferroni mean operator into q-ROFS to investigate the relationship between attributes. In contrast, the q-ROFPA operator is based on the arithmetic mean operators, which consider the attributes are independent of each other. Above all, the proposed q-ROFPBM operator can effectively mine the characteristics of the relationships between attributes, and make the decision result more reasonable than the q-ROFPA operator does.

(2) When using the q-ROFBM operator.

Step 1: Normalize the data in Table 1.
Step 2: Use the q-ROFBM operator in Formula (11)

$$q - ROFBM^{s,t}(a_1, a_2, a_3, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^s\, a_j^t\right)^{\frac{1}{s+t}} \tag{11}$$

to calculate the comprehensive attribute value of each scheme with $q = 3$, $s = t = 1, 2, 3$, respectively, as shown in Table 3:

Table 3. The comprehensive attribute value of each scheme based on q-ROF Bonferroni mean (q-ROFBM) operator.

s, t	x_1		x_2		x_3		x_4		x_5	
	u	v	u	v	u	v	u	v	u	v
1	0.015	0.788	0.089	0.633	0.042	0.686	0.086	0.726	0.046	0.524
2	0.000	0.931	0.003	0.800	0.000	0.854	0.004	0.883	0.001	0.685
3	0.000	0.986	0.000	0.911	0.000	0.950	0.000	0.964	0.000	0.814

Step 3: Calculate the score values of each scheme:

When $s, t = 1$,

$s_{(x_1)} = -0.4886, s_{(x_2)} = -0.2530, s_{(x_3)} = -0.3233, s_{(x_4)} = -0.3823, \quad s_{(x_5)} = -0.1440;$

when $s, t = 2$,

$s_{(x_1)} = -0.8067, s_{(x_2)} = -0.5127, s_{(x_3)} = -0.6225, s_{(x_4)} = -0.6883, s_{(x_5)} = -0.3214;$

when $s, t = 3$,

$s_{(x_1)} = -0.9600, s_{(x_2)} = -0.7563, s_{(x_3)} = -0.8569, s_{(x_4)} = -0.8946, s_{(x_5)} = -0.5384;$

Step 4: Rank the score values under different s, t, then:

When $s, t = 1$, then $s_{(x_5)} > s_{(x_2)} > s_{(x_3)} > s_{(x_4)} > s_{(x_1)}$, then $s_{(x_5)}$ is the best alternative.

When $s, t = 2$, then $s_{(x_5)} > s_{(x_2)} > s_{(x_3)} > s_{(x_4)} > s_{(x_1)}$, then $s_{(x_5)}$ is the best alternative.

When $s, t = 3$, then $s_{(x_5)} > s_{(x_2)} > s_{(x_3)} > s_{(x_4)} > s_{(x_1)}$, then $s_{(x_5)}$ is the best alternative.

It can be seen that the optimization scheme based on the q-ROFBM operator is s_{x_5}, while s_{x_2} is the optimal scheme based on the proposed method in this study. There is significant difference between the two results. The main reason is that q-ROFBM operator-based method does not consider the impact of outliers in evaluation information on operation of the model. In expert-based MADM problems, the attribute values are subjectively evaluated by the experts or decision makers, so it is inevitable that the evaluation values of some attributes are too high or too low, which will affect the rationality of the decision result. Therefore, the q-ROFBM operator ignores the adverse influence and produces unreasonable alternative ranking. Whereas, in this paper, the PA operator is used to eliminate the adverse effects caused by outliers, and the proposed q-ROFPBM operator effectively avoids the negative influence of extreme evaluation value, and makes the decision result more objective and accurate.

Subsequently, compared with the methods based on q-ROFPA operator and q-ROFBM operator, the proposed q-ROFPBM operator fully examines the interaction between different attributes, and at the same time removes the adverse effects of extreme value information on the decision results in the evaluation process. It can be inferred that the proposed method has more advantages than the q-ROFPA and q-ROFBM operators do.

6. Conclusions

This paper proposes a new MADM method based on the q-ROFPBM operator. Firstly, starting from fuzzy set theory, it successively reviews operational rules and properties of the q-ROFS, as well as its size comparison method and distance measure. Secondly, the q-ROFPBM operator is defined, and the general properties of the operator are analyzed. Then, a MADM method based on q-ROFPBM operator is developed. Finally, a case of investment partner selection in the tourism market is given to analyze the effectiveness of the method. The method in this paper can not only effectively remove the influence of extreme values present in decision information, but also correlate each attribute value, making it more suitable for solving some decision-making problems under complex decision environment and information conditions.

The work done in this paper is mainly to propose an effective operator for solving MADM problems, promote the application of fuzzy set theory in MADM and provide a new idea for the research on addressing MADM issues, which carries practical significance. However, there are still some deficiencies in this paper: this paper only defines and proves the q-ROFPBM operator, but fails to study its extended forms, such as weighted q-ROFPBM operator, interval value type of q-ROFPBM operator, which needs further study by future generations. At the same time, the proposed method can be used in other practical decision-making fields, such as risk assessment, weather prediction and pattern recognition.

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