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Even Order Half-Linear Differential Equations with Regularly Varying Coefficients

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Abstract: We establish nonoscillation criterion for the even order half-linear differential equation $(-1)^n\left(f_n(t)\Phi\left(x^{(n)}\right)\right)^{(n)}+\sum_{l=1}^{n}(-1)^{n-l}\beta_{n-l}\left(f_{n-l}(t)\Phi\left(x^{(n-l)}\right)\right)^{(n-l)}=0$, where $\beta_0,\beta_1,\dots,\beta_{n-1}$ are real numbers, $n\in\mathbb{N}$, $\Phi(s)=|s|^{p-1}\operatorname{sgn}s$ for $s\in\mathbb{R}$, $p\in(1,\infty)$ and f_{n-l} is a regularly varying (at infinity) function of the index $\alpha-lp$ for $l=0,1,\dots,n$ and $\alpha\in\mathbb{R}$. This equation can be understood as a generalization of the even order Euler type half-linear differential equation. We obtain this Euler type equation by rewriting the equation above as follows: the terms $f_n(t)$ and $f_{n-l}(t)$ are replaced by the t^{α} and $t^{\alpha-lp}$, respectively. Unlike in other texts dealing with the Euler type equation, in this article an approach based on the theory of regularly varying functions is used. We establish a nonoscillation criterion by utilizing the variational technique.

Keywords: higher order half-linear differential equation; nonoscillation criterion; variational principle; energy functional; regular variation

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1. Introduction

Consider the $2n$-th order half-linear differential equation

$$(-1)^n\left(f_n(t)\Phi\left(x^{(n)}\right)\right)^{(n)}+\sum_{l=1}^{n}(-1)^{n-l}\beta_{n-l}\left(f_{n-l}(t)\Phi\left(x^{(n-l)}\right)\right)^{(n-l)}=0, \tag{1}$$

where $\beta_0,\beta_1,\dots,\beta_{n-1}$ are real numbers, $n\in\mathbb{N}$, Φ is the odd power function defined by the relation $\Phi(s)=|s|^{p-1}\operatorname{sgn}s$ for $s\in\mathbb{R}$, $p\in(1,\infty)$ and for each $l\in\{0,1,\dots,n\}$ the function f_{n-l} is defined, positive and continuous on $[S,\infty)$, where $S\in\mathbb{R}$.

Moreover, we assume that f_{n-l} is a regularly varying (at infinity) function of the index $\alpha-lp$ (the definition is given later) for $l=0,1,\dots,n$ and $\alpha\in\mathbb{R}$. More briefly, we write $f_{n-l}\in\mathcal{RV}(\alpha-lp)$, where $\mathcal{RV}(\vartheta)$ for $\vartheta\in\mathbb{R}$ denotes the set of all regularly varying functions of the index ϑ. Denote $\mathcal{SV}:=\mathcal{RV}(0)$. Functions belonging to $\mathcal{SV}$ are called *slowly varying functions* and the function f_{n-l} can be equivalently described for $l=0,1,\dots,n$ as follows: there exists a function L_{n-l} defined and continuous on $[S,\infty)$ such that

$$L_{n-l}\in\mathcal{SV}\quad\text{and}\quad f_{n-l}(t)=t^{\alpha-lp}L_{n-l}(t),\ t\in[S,\infty).$$

The function L_{n-l} is called *component of* f_{n-l}.

In this article, we give sufficient conditions on the constants $\beta_0,\beta_1,\dots,\beta_{n-1}$ and on the slowly varying functions $L_0,L_1,\dots,L_{n-1}$ (the components of $f_0,f_1,\dots,f_{n-1}$), such that Equation (1) is nonoscillatory.

Equation (1) can be understood as a generalization of the $2n$-th order Euler type half-linear differential equation

$$(-1)^n \left(t^\alpha \Phi\left(x^{(n)}\right)\right)^{(n)} + \sum_{l=1}^{n} (-1)^{n-l} \beta_{n-l} \left(t^{\alpha-lp} \Phi\left(x^{(n-l)}\right)\right)^{(n-l)} = 0, \tag{2}$$

studied in [1,2]. The two-term even order (Euler type and more general) half-linear differential equations are studied in [1,3,4] and in the book [5] (Section 9.4).

The two-term $2n$-th order Euler type linear differential equation

$$(-1)^n \left(t^\alpha x^{(n)}\right)^{(n)} + \gamma t^{\alpha-2n} x = 0 \tag{3}$$

with $\gamma \in \mathbb{R}$ is a special case of Equation (2) since $\Phi(s) = s$ for $s \in \mathbb{R}$ and $p = 2$. Equation (3) with $\alpha \in \mathbb{R} \setminus \mathcal{M}_{n,2}$ is nonoscillatory if and only if $\gamma_{n,2,\alpha} + \gamma \geq 0$ (see [6] (p. 132) and for $\alpha = 0$ see also [7] (pp. 97–98)), where

$$\mathcal{M}_{n,p} := \{p-1, 2p-1, \ldots, np-1\}, \quad \gamma_{p,\alpha} := \left(\frac{|p-1-\alpha|}{p}\right)^p \quad \text{and} \quad \gamma_{n,p,\alpha} := \prod_{l=0}^{n-1} \gamma_{p,\alpha-lp}.$$

Equation (2) with $n = 1$ and $\beta_0 = \gamma$ is the second order Euler type half-linear differential equation

$$-\left(t^\alpha \Phi\left(x'\right)\right)' + \gamma t^{\alpha-p} \Phi(x) = 0, \tag{4}$$

which is nonoscillatory if and only if $\gamma_{p,\alpha} + \gamma \geq 0$, see [5] (Theorem 1.4.4) for $\alpha \in \mathbb{R} \setminus \mathcal{M}_{1,p}$ and for the proof see [8]. For the case $\alpha \in \mathcal{M}_{1,p}$ (that is $\alpha = p-1$) see Remark 2 in this article. Equation (4) with $\alpha = 0$ and its various perturbations are also studied in [9–14].

This article is organized as follows. In Section 2, we define the concept of nonoscillation for (1), we formulate the variational principle for (1) and we recall basic concepts of the theory of regularly varying functions. The main results are given in Section 3. We conclude the article with several examples and comments in the last two sections.

2. Preliminaries

First, we define the concept of nonoscillation for Equation (1). Similarly as in the linear case, real points t_1 and t_2 are said to be *conjugate relative to* Equation (1), if $t_1 \neq t_2$ and there exists a nontrivial solution x of Equation (1), such that t_1 and t_2 are its zero points of multiplicity n, i.e., t_1 and t_2 satisfying

$$x^{(i)}(t_1) = 0 \quad \text{and} \quad x^{(i)}(t_2) = 0$$

for $i = 0, 1, \ldots, n-1$.

Note that the concept of conjugate points does not need such strict assumptions on coefficients as they are given for Equation (1). Instead of $\beta_{n-l} f_{n-l}(t)$ in (1) we can take $r_{n-l}(t)$ defined and continuous on the interval $[S, \infty)$ for $l = 1, 2, \ldots, n$; and instead of $f_n(t)$ we can take $r_n(t)$ defined, continuous and positive on the interval $[S, \infty)$.

Definition 1. *Equation (1) is said to be nonoscillatory if there exists $T \in \mathbb{R}$ such that no pair t_1, t_2 of points from $[T, \infty)$ conjugate relative to Equation (1) exists. In the opposite case, Equation (1) is said to be oscillatory.*

Recall the definition of the Sobolev space. Denote

$$\begin{aligned} W_0^{n,p}[T,\infty) := \{y\colon [T,\infty) \to \mathbb{R} \mid\ & y^{(n-1)} \in \mathcal{AC}[T,\infty),\ y^{(n)} \in \mathcal{L}^p(T,\infty), \\ & y^{(i-1)}(T) = 0 \text{ for } i = 1,2,\dots,n, \\ & \exists m \in \mathbb{R} \text{ such that } m > T \\ & \text{and } y(t) = 0 \text{ for } t \in [m,\infty)\}, \end{aligned}$$

where $n \in \mathbb{N}$, $T \in \mathbb{R}$ and $p \in (1,\infty)$. The symbol $\mathcal{AC}[T,\infty)$ indicates the set of all absolutely continuous functions of the form $f\colon [T,\infty) \to \mathbb{R}$ and the symbol $\mathcal{L}^p(T,\infty)$ indicates the space of (Lebesgue) measurable functions (equivalence classes of functions) such that $f \in \mathcal{L}^p(T,\infty)$ if and only if $\int_T^\infty |f(t)|^p \,\mathrm{d}t < \infty$.

Suppose that $y \in W_0^{n,p}[T,\infty)$. If we say that y is nontrivial, we mean that the function y is not identically zero on the interval $[T,\infty)$.

The relation between Equation (1) and the energy functional $\mathcal{F}_n$, for which Equation (1) is its Euler–Lagrange equation, is formulated in the following lemma and is called the variational principle.

Lemma 1 ([5]). *Equation (1) is nonoscillatory if there exists $T \in \mathbb{R}$ such that for every nontrivial function $y \in W_0^{n,p}[T,\infty)$, $y = y(t)$ we have*

$$\mathcal{F}_n(y) := \int_T^\infty \left[f_n(t) \left| y^{(n)} \right|^p + \sum_{l=1}^{n} \beta_{n-l} f_{n-l}(t) \left| y^{(n-l)} \right|^p \right] \mathrm{d}t > 0. \tag{5}$$

If Equation (1) is of the second order, condition (5) is even equivalent to nonoscillation of (1). Consider the general second order half-linear differential equation

$$-\left(r(t)\Phi\left(x'\right)\right)' + c(t)\Phi(x) = 0, \tag{6}$$

where r and c are continuous functions defined on a neighborhood of infinity and r is positive.

Lemma 2 ([5]). *Equation (6) is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that*

$$\int_T^\infty \left[r(t) \left| y' \right|^p + c(t) \left| y \right|^p \right] \mathrm{d}t > 0$$

for every nontrivial function $y \in W_0^{1,p}[T,\infty)$, $y = y(t)$.

For any $f\colon M \to \mathbb{R}$, $M \subseteq \mathbb{R}$ we denote $f_-(t) := \min\{0, f(t)\}$, $t \in M$. If $p \in (1,\infty)$, the symbol q denotes the *conjugate number* of p, i.e., the number q is such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The auxiliary statement below is proved in [5] (Theorem 2.1.2).

Proposition 1. *Denote $\gamma_p := \gamma_{p,0}$. The following statements hold.*

(*a*) *Let $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$ and $\int^\infty c_-(t)\,\mathrm{d}t > -\infty$. Equation (6) is nonoscillatory provided*

$$\liminf_{t\to\infty} \left[\left(\int^t r^{1-q}(s)\,\mathrm{d}s \right)^{p-1} \int_t^\infty c_-(s)\,\mathrm{d}s \right] > -\frac{\gamma_p}{p-1}. \tag{7}$$

(*b*) *Let* $\int^\infty r^{1-q}(t)\,\mathrm{d}t < \infty$*. Equation* (6) *is nonoscillatory provided*

$$\liminf_{t\to\infty}\left[\left(\int_t^\infty r^{1-q}(s)\,\mathrm{d}s\right)^{p-1}\int^t c_-(s)\,\mathrm{d}s\right] > -\frac{\gamma_p}{p-1}.$$

Note that the assumptions of part (a) of Proposition 1 can be weakened, see [5] (Theorem 2.2.9). The function c_- is replaced by c and instead of $\int^\infty c_-(t)\,\mathrm{d}t > -\infty$ we assume that $\int^\infty c(t)\,\mathrm{d}t$ converges.

The last part of this section is devoted to the theory of regular varying functions. A comprehensive study of regular variation can be found in [15], where the proofs of the presented statements can be found.

Definition 2. *Let* ϑ *and* S *be real numbers. A (Lebesgue) measurable function* $f\colon [S,\infty) \to (0,\infty)$ *is said to be regularly varying (more precisely, regularly varying at* ∞*) of index* ϑ *if*

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^\vartheta$$

for every $\lambda \in (0,\infty)$.

Positive constant functions defined on $[S,\infty)$ are trivial examples of slowly varying functions (elements of $\mathcal{RV}(0)$). The logarithm defined on $[S,\infty)$ is also an element of $\mathcal{SV}$ (where $\mathcal{SV} = \mathcal{RV}(0)$) if $S > 1$. Let $\mu_1, \mu_2, \ldots, \mu_k$ be real numbers, $k \in \mathbb{N}$ and $\ln_{i+1} t := \ln\ln_i t$ for $i \in \mathbb{N}$, where $\ln_1 t := \ln t$. Then, the function defined by the relation

$$K(t) = \prod_{i=1}^{k}(\ln_i t)^{\mu_i}, \quad t \in [S,\infty)$$

is slowly varying for sufficiently large S. Examples of regularly varying functions of index ϑ have the form $t^\vartheta L(t)$, where L is a slowly varying function; see Lemma 3.

Let f and g be real-valued functions, which are positive in a neighborhood of infinity. The functions f and g are said to be *asymptotically equivalent* if $\lim_{t\to\infty} f(t)/g(t) = 1$; we write $f(t) \sim g(t)$ as $t \to \infty$.

Lemma 3. *Let* ϑ *and* S *be real numbers. Then following statements hold.*

(*a*) *A measurable function* $f\colon [S,\infty) \to (0,\infty)$ *belongs to* $\mathcal{RV}(\vartheta)$ *if and only if there exists a measurable function* $L\colon [S,\infty) \to (0,\infty)$ *such that* $L \in \mathcal{SV}$ *and* $f(t) = t^\vartheta L(t)$ *for* $t \in [S,\infty)$.
(*b*) *If* $L \in \mathcal{SV}$ *and* $K\colon [S,\infty) \to (0,\infty)$ *is a measurable function such that* $K(t) \sim L(t)$ *as* $t \to \infty$*, then* $K \in \mathcal{SV}$.
(*c*) *If* $f \in \mathcal{RV}(\vartheta)$*, then* $f^\beta \in \mathcal{RV}(\vartheta\beta)$ *for every* $\beta \in \mathbb{R}$.
(*d*) *Let* $f \in \mathcal{RV}(\vartheta_1)$ *and* $g \in \mathcal{RV}(\vartheta_2)$*. Then* $fg \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

The following statement allows us to include equations with regularly varying coefficients in our investigation.

Proposition 2 (Karamata's theorem [15])**.** *Let* S *be a real number and* L *be a slowly varying function defined on* $[S,\infty)$*. The following statements hold.*

(*a*) *If* $\vartheta < -1$*, then*

$$\int_t^\infty s^\vartheta L(s)\,\mathrm{d}s \sim -\frac{t^{\vartheta+1}}{\vartheta+1}L(t) \quad \text{as} \quad t \to \infty.$$

(*b*) *If* $\vartheta > -1$*, then*

$$\int_S^t s^\vartheta L(s)\,\mathrm{d}s \sim \frac{t^{\vartheta+1}}{\vartheta+1}L(t) \quad \text{as} \quad t \to \infty.$$

Note that the case $\vartheta = -1$ is not included in any part of Karamata's theorem since the integral $\int_S^\infty s^{-1} L(s)\, ds$ may or may not converge.

3. Equations with Regularly Varying Coefficients

It is worthy to note that the methods presented in this section have been previously used in [16], where we are dealing with the discrete case. As far as we know, our result in this section is new even in the linear case ($p = 2$).

We use the following notation, which greatly simplifies the formulation of the main result. Recall

$$\gamma_{p,\alpha-lp} = \left(\frac{|(l+1)p - 1 - \alpha|}{p} \right)^p \quad \text{for} \quad l = 0, 1, \dots, n-1$$

and denote

$$\gamma^{[n]}(0) := 1 \quad \text{and} \quad \gamma^{[n]}(k+1) := \gamma^{[n]}(k)\gamma_{p,\alpha-kp} + \beta_{n-1-k}$$

for $k = 0, 1, 2, \dots, n-1$. If $n > 1$, then

$$\begin{aligned}
\gamma^{[n]}(1) &= \gamma_{p,\alpha} + \beta_{n-1}, \\
\gamma^{[n]}(2) &= \gamma_{p,\alpha}\gamma_{p,\alpha-p} + \beta_{n-1}\gamma_{p,\alpha-p} + \beta_{n-2}, \\
&\vdots \\
\gamma^{[n]}(n) &= \gamma_{n,p,\alpha} + \sum_{k=1}^{n-1} \left[\prod_{l=k}^{n-1} \gamma_{p,\alpha-lp} \right] \beta_{n-k} + \beta_0.
\end{aligned}$$

Now we formulate the main theorem. It is an extension of the result in [1] (Theorem 3.3) obtained for Equation (2). The result in [1] is also extended in [2]. The extension from [2] generalizes the conditions on the coefficients of Equation (2). In this paper, moreover, a more general Equation (1) is considered.

Theorem 1. *Let $\alpha \in \mathbb{R} \setminus \mathcal{M}_{n,p}$. If*

$$L_0(t) \sim L_1(t) \sim \cdots \sim L_n(t) \quad as \quad t \to \infty$$

and $\gamma^{[n]}(k) > 0$ for every $k \in \{1, 2, \dots, n\}$, then Equation (1) is nonoscillatory.

The difference in the approach of this article and our previous articles [1,2] is that we do not utilize the so-called Wirtinger inequality, see [5] (Lemma 2.1.1). Consequently, we can consider more general coefficients, but we lose some potentially critical states of the constants $\beta_0, \beta_1, \dots, \beta_{n-1}$ (especially, the case $\gamma^{[n]}(k) = 0$ for k from an arbitrary subset of $\{1, 2, \dots, n-1\}$ and $\gamma^{[n]}(k) > 0$ for k from the complement of this subset with respect to $\{1, 2, \dots, n\}$). Oscillation properties in the case $\gamma^{[n]}(n) = 0$ are completely unknown to us.

Consider a special case of Equation (1), namely the second order half-linear differential equation

$$-\left(f(t)\Phi(x')\right)' + \gamma g(t)\Phi(x) = 0, \tag{8}$$

where $\gamma \in \mathbb{R}$ and functions f and g are such that

$$f(t) = t^\alpha K(t) \text{ for } t \in [S, \infty) \quad \text{and} \quad g(t) = t^{\alpha-p} L(t) \text{ for } t \in [S, \infty)$$

for some $\alpha \in \mathbb{R}$ and some slowly varying functions K and L.

We start with the auxiliary nonoscillation criterion for Equation (8). In its proof, both parts of Propositions 1 and 2 (Karamata's theorem) are used.

Lemma 4. *Let $\alpha \in \mathbb{R} \setminus \mathcal{M}_{1,p}$. If*

$$\gamma_{p,\alpha} + \gamma > 0 \quad \text{and} \quad K(t) \sim L(t) \text{ as } t \to \infty,$$

then Equation (8) is nonoscillatory.

Proof. From Lemma 1 it follows that Equation (8) is nonoscillatory for $\gamma \geq 0$.

Let $\gamma < 0$ and $\alpha < p-1$. We verify the assumptions of part (a) of Proposition 1 for Equation (8). By part (b) of Proposition 2 we have

$$\int_S^t s^{\alpha(1-q)} K^{1-q}(s)\,\mathrm{d}s \sim \frac{t^{\alpha(1-q)+1}}{\alpha(1-q)+1} K^{1-q}(t) \quad \text{as} \quad t \to \infty.$$

Indeed, $\alpha(1-q) > -1$ if and only if $\alpha < p-1$ and $K^{1-q} \in \mathcal{SV}$ by part (c) of Lemma 3. Hence, $\int_S^\infty t^{\alpha(1-q)} K^{1-q}(t)\,\mathrm{d}t = \infty$ by the limit comparison test. Now, denote $c(t) = \gamma t^{\alpha-p} L(t)$. Then $c_- \equiv c$ for $\gamma < 0$ and by part (a) of Proposition 2 we get

$$\int_t^\infty s^{\alpha-p} L(s)\,\mathrm{d}s \sim -\frac{t^{\alpha-p+1}}{\alpha-p+1} L(t) \quad \text{as} \quad t \to \infty.$$

Therefore, $\int_S^\infty \gamma t^{\alpha-p} L(t)\,\mathrm{d}t > -\infty$ holds.

Further, the left-hand side of inequality (7) admits the form

$$\liminf_{t\to\infty} \left[\left(\int_S^t s^{\alpha(1-q)} K^{1-q}(s)\,\mathrm{d}s \right)^{p-1} \int_t^\infty \gamma s^{\alpha-p} L(s)\,\mathrm{d}s \right] = \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} \lim_{t\to\infty} \frac{L(t)}{K(t)}$$

and

$$\frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} > -\frac{\gamma_p}{p-1} \quad \text{if and only if} \quad \gamma_{p,\alpha} + \gamma > 0$$

for $\alpha < p-1$.

The proof of the case $\alpha < p-1$ (with $\gamma < 0$) is analogous to the one of the case $\alpha(1-q) > -1$ and it uses part (b) of Proposition 1. □

Remark 1. *The oscillation complement of Lemma 4 holds too. Indeed, instead of the parts* (a) *and* (b) *of Proposition 1, we use their oscillation complements (see [5] (Theorem 2.3.2 (ii)) in case of $\alpha < p-1$ and [5] (Theorem 3.1.4) in case of $\alpha > p-1$). Nevertheless, in this paper we only need the nonoscillation criterion shown in Lemma 4, and therefore we do not prove the oscillation complement explicitly.*

Remark 2. *Due to the note below Proposition 2, we cannot decide on the convergence of integrals $\int^\infty t^{\alpha(1-q)} K^{1-q}(t)\,\mathrm{d}t$ and $\int^\infty t^{\alpha-p} L(t)\,\mathrm{d}t$ if $\alpha \in \mathcal{M}_{1,p}$. However, if we set $K \equiv L \equiv 1$ and $\alpha = p-1$ in Equation (8), then Equation (8) is the second order Euler type half-linear differential equation and it is nonoscillatory if and only if $\gamma_{p,p-1} + \gamma \geq 0$ ($\gamma_{p,p-1} = 0$). Indeed, the "if" part immediately follows from Lemma 2 and the "only if" part follows from the half-linear version of the Leighton–Wintner oscillation criterion (see [5] (Theorem 1.2.9)).*

The variational principle formulated in Lemma 2 allows obtaining a certain inequality from the knowledge of nonoscillation of an equation. This way, we obtain the inequalities, as shown in the following lemma.

Lemma 5. *Let $m \in \mathbb{N}$, $p \in (1,\infty)$, $\alpha \in \mathbb{R} \setminus \mathcal{M}_{m,p}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}$ be arbitrary positive real numbers. Further let*

$$L_i \in \mathcal{SV} \quad \text{and} \quad L_{m-j}(t) \sim L_{m-j-1}(t) \text{ as } t \to \infty$$

for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, m-1$. Then there exists $T \in \mathbb{R}$ such that

$$\int_T^\infty \left[t^{\alpha - jp} L_{m-j}(t) \left| y^{(m-j)} \right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp} \right) t^{\alpha-(j+1)p} L_{m-j-1}(t) \left| y^{(m-j-1)} \right|^p \right] \mathrm{d}t > 0$$

for every nontrivial function $y \in W_0^{m,p}[T, \infty)$ and for every $j \in \{0, 1, \dots, m-1\}$.

Proof. Let the assumptions of Lemma 5 hold. Then the equation

$$-\left(t^{\alpha-jp} L_{m-j}(t) \Phi\left(x'\right) \right)' + (\varepsilon_j - \gamma_{p,\alpha-jp}) t^{\alpha-(j+1)p} L_{m-j-1}(t) \Phi(x) = 0 \tag{9}$$

is nonoscillatory for every $j \in \{0, 1, \dots, m-1\}$. Indeed, let $j \in \{0, 1, \dots, m-1\}$ be arbitrary, then

$$\gamma_{p,\alpha-jp} + \varepsilon_j - \gamma_{p,\alpha-jp} = \varepsilon_j > 0, \quad \alpha - jp \in \mathbb{R} \setminus \mathcal{M}_{1,p}$$

and

$$L_{m-j}(t) \sim L_{m-j-1}(t) \quad \text{as} \quad t \to \infty.$$

Hence, by Lemma 4, Equation (9) is nonoscillatory for every $j \in \{0, 1, \dots, m-1\}$.

Due to Lemma 2, for every $j \in \{0, 1, \dots, m-1\}$ there exists $T_j \in \mathbb{N}$ such that

$$\int_{T_j}^\infty \left[t^{\alpha-jp} L_{m-j}(t) \left|z'\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp} \right) t^{\alpha-(j+1)p} L_{m-j-1}(t) \left|z\right|^p \right] \mathrm{d}t > 0 \tag{10}$$

for every nontrivial $z \in W_0^{1,p}[T_j, \infty)$.

Denote $T = \max\{T_0, T_1, \dots, T_{m-1}\}$, then for an arbitrary $z \in W_0^{1,p}[T, \infty)$ a function z_j defined by the relation

$$z_j(t) = \begin{cases} 0 & \text{for } t \in [T_j, T), \\ z(t) & \text{for } t \in [T, \infty) \end{cases}$$

belongs to $W_0^{1,p}[T_j, \infty)$ for $j = 0, 1, \dots, m-1$ (if any function of $z, z_1, z_2, \dots, z_{m-1}$ is nontrivial, then all the others are nontrivial). Hence,

$$\int_{T_j}^T \left[t^{\alpha-jp} L_{m-j}(t) \left|z_j'\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp} \right) t^{\alpha-(j+1)p} L_{m-j-1}(t) \left|z_j\right|^p \right] \mathrm{d}t = 0$$

for every $z \in W_0^{1,p}[T, \infty)$ and for every $j \in \{0, 1, \dots, m-1\}$, therefore, by (10),

$$\int_T^\infty \left[t^{\alpha-jp} L_{m-j}(t) \left|z'\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp} \right) t^{\alpha-(j+1)p} L_{m-j-1}(t) \left|z\right|^p \right] \mathrm{d}t > 0$$

for every nontrivial $z \in W_0^{1,p}[T, \infty)$ and for every $j \in \{0, 1, \dots, m-1\}$.

Choose any $j \in \{0, 1, \dots, m-1\}$ and any nontrivial function $y \in W_0^{m,p}[T, \infty)$. Then the function z defined by the relation $z(t) = y^{(m-j-1)}(t)$ for $t \in [T, \infty)$ is nontrivial and belongs to the set $W_0^{1,p}[T, \infty)$. Hence,

$$\int_T^\infty \left[t^{\alpha-jp} L_{m-j}(t) \left| y^{(m-j)} \right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp} \right) t^{\alpha-(j+1)p} L_{m-j-1}(t) \left| y^{(m-j-1)} \right|^p \right] \mathrm{d}t > 0$$

for every nontrivial $y \in W_0^{m,p}[T, \infty)$ and for every $j \in \{0, 1, \dots, m-1\}$. □

Proof of Theorem 1. By Lemma 1, it is sufficient to prove that for some $T \in \mathbb{N}$ the energy functional

$$\widetilde{\mathcal{F}}_n(y) := \int_T^\infty \left[f_n(t) \left| y^{(n)} \right|^p + \beta_{n-1} f_{n-1}(t) \left| y^{(n-1)} \right|^p + \ldots + \beta_0 f_0(t) \left|y\right|^p \right] \mathrm{d}t \tag{11}$$

is positive for every nontrivial $y \in W_0^{n,p}[T,\infty)$. Note that Theorem 1 for $n = 1$ had already been proved, see Lemma 4.

Assume that $n > 1$. We estimate functional (11) by using inequalities obtained via Lemma 5. Let $\varepsilon \in (0,\infty)$ be such that

$$\varepsilon < \gamma^{[n]}(n) \quad \text{and} \quad \varepsilon < 2^{n-l-1}\gamma^{[n]}(l) \prod_{j=l}^{n-1} \gamma_{p,\alpha-jp} \tag{12}$$

for every $l = 1,2,\dots,n-1$. Define real numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n-2}$ by the relations

$$\varepsilon_{2l-1} = \frac{\varepsilon}{2^{n-l}\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}} \quad \text{and} \quad \varepsilon_{2l} = \frac{\varepsilon\gamma_{p,\alpha-lp}}{2^{n-l}\gamma^{[n]}(l)\left[\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}\right]-\varepsilon}$$

for $l = 1,2,\dots,n-1$. From conditions (12) we have the inequalities

$$\gamma_{p,\alpha-lp} > \varepsilon_{2l} > 0 \quad \text{and} \quad \gamma^{[n]}(l) > \varepsilon_{2l-1} > 0$$

for $l = 1,2,\dots,n-1$.

By Lemma 5, $T \in \mathbb{R}$ exists such that the relations

$$\int_T^\infty \left[f_n(t)\left|y^{(n)}\right|^p + (\varepsilon_1 - \gamma_{p,\alpha})\, f_{n-1}(t)\left|y^{(n-1)}\right|^p\right] \mathrm{d}t > 0 \tag{13}$$

and

$$\int_T^\infty \left[f_{n-j}(t)\left|y^{(n-j)}\right|^p + (\varepsilon_{2j} - \gamma_{p,\alpha-jp})\, f_{n-j-1}(t)\left|y^{(n-j-1)}\right|^p\right] \mathrm{d}t > 0 \tag{14}$$

hold for every nontrivial $y \in W_0^{n,p}[T,\infty)$ and for every $j = 1,2,\dots,n-1$.

By direct evaluation we can verify that for $l = 1,2,\dots,n-2$ we have

$$\left(\gamma^{[n]}(l) - \varepsilon_{2l-1}\right)\left(\gamma_{p,\alpha-lp} - \varepsilon_{2l}\right) = \left[\gamma^{[n]}(l+1) - \beta_{n-1-l}\right] - \varepsilon_{2(l+1)-1} \tag{15}$$

and

$$\left(\gamma^{[n]}(n-1) - \varepsilon_{2n-3}\right)\left(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}\right) = \left[\gamma^{[n]}(n) - \beta_0\right] - \varepsilon. \tag{16}$$

Now we prove the positivity of functional (11). Among others, we use the relations

$$\gamma^{[n]}(l) > \varepsilon_{2l-1} \quad \text{and} \quad \gamma^{[n]}(n) - \varepsilon > 0$$

for $l = 1,2,\dots,n-1$. Using (13)–(15) we have

$$\begin{aligned}
&\int_T^\infty \left[f_n(t)\left|y^{(n)}\right|^p + \beta_{n-1}f_{n-1}(t)\left|y^{(n-1)}\right|^p\right] \mathrm{d}t \\
&\quad > \left[(\gamma_{p,\alpha} - \varepsilon_1) + \beta_{n-1}\right]\int_T^\infty f_{n-1}(t)\left|y^{(n-1)}\right|^p \mathrm{d}t \\
&\quad = \left(\gamma^{[n]}(1) - \varepsilon_1\right)\int_T^\infty f_{n-1}(t)\left|y^{(n-1)}\right|^p \mathrm{d}t \\
&\quad > \left(\gamma^{[n]}(1) - \varepsilon_1\right)(\gamma_{p,\alpha-p} - \varepsilon_2)\int_T^\infty f_{n-2}(t)\left|y^{(n-2)}\right|^p \mathrm{d}t \\
&\quad = \left(\gamma^{[n]}(2) - \beta_{n-2} - \varepsilon_3\right)\int_T^\infty f_{n-2}(t)\left|y^{(n-2)}\right|^p \mathrm{d}t
\end{aligned}$$

for every nontrivial $y \in W_0^{n,p}[T,\infty)$. Therefore, using (14) and (15),

$$\begin{aligned}
&\int_T^\infty \left[f_n(t) \left|y^{(n)}\right|^p + \beta_{n-1} f_{n-1}(t) \left|y^{(n-1)}\right|^p + \beta_{n-2} f_{n-2}(t) \left|y^{(n-2)}\right|^p \right] \mathrm{d}t \\
&\quad > \left[\left(\gamma^{[n]}(2) - \beta_{n-2} - \varepsilon_3 \right) + \beta_{n-2} \right] \int_T^\infty f_{n-2}(t) \left|y^{(n-2)}\right|^p \mathrm{d}t \\
&\quad > \left(\gamma^{[n]}(2) - \varepsilon_3 \right) \left(\gamma_{p,\alpha-2p} - \varepsilon_4 \right) \int_T^\infty f_{n-3}(t) \left|y^{(n-3)}\right|^p \mathrm{d}t \\
&\quad = \left(\gamma^{[n]}(3) - \beta_{n-3} - \varepsilon_5 \right) \int_T^\infty f_{n-3}(t) \left|y^{(n-3)}\right|^p \mathrm{d}t
\end{aligned}$$

for every nontrivial $y \in W_0^{n,p}[T, \infty)$. Stepwise by (14) and (16) we obtain

$$\begin{aligned}
&\int_T^\infty \left[f_n(t) \left|y^{(n)}\right|^p + \ldots + \beta_2 f_2(t) \left|y''\right|^p + \beta_1 f_1(t) \left|y'\right|^p \right] \mathrm{d}t \\
&\quad > \left[\left(\gamma^{[n]}(n-1) - \beta_1 - \varepsilon_{2n-3} \right) + \beta_1 \right] \int_T^\infty f_1(t) \left|y'\right|^p \mathrm{d}t \\
&\quad > \left(\gamma^{[n]}(n-1) - \varepsilon_{2n-3} \right) \left(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2} \right) \int_T^\infty f_0(t) \left|y\right|^p \mathrm{d}t \\
&\quad = \left(\gamma^{[n]}(n) - \beta_0 - \varepsilon \right) \int_T^\infty f_0(t) \left|y\right|^p \mathrm{d}t
\end{aligned}$$

for every nontrivial $y \in W_0^{n,p}[T, \infty)$. Hence, the functional $\widetilde{\mathcal{F}}_n(y)$ is greater than the expression

$$\left[\left(\gamma^{[n]}(n) - \beta_0 - \varepsilon \right) + \beta_0 \right] \int_T^\infty f_0(t) \left|y\right|^p \mathrm{d}t,$$

which is positive for every nontrivial $y \in W_0^{n,p}[T, \infty)$. This implies that the energy functional $\widetilde{\mathcal{F}}_n(y)$ is positive for every nontrivial $y \in W_0^{n,p}[T, \infty)$. □

Remark 3. *We believe that the oscillation behavior of (1) in the case $\gamma^{[n]}(n) = 0$ cannot be obtained under the general (remaining) assumptions of Theorem 1. More precisely, we conjecture that if $\alpha \in \mathbb{R} \setminus \mathcal{M}_{n,p}$, $L_0(t) \sim L_1(t) \sim \cdots \sim L_n(t)$ as $t \to \infty$, $\gamma^{[n]}(k) \geq 0$ for every $k \in \{1, 2, \ldots, n-1\}$ and $\gamma^{[n]}(n) = 0$, then Equation (1) may or may not be nonoscillatory (nonoscillation of (1) depends on the functions $L_0, L_1, \ldots, L_n$).*

4. Examples

Example 1. *Consider the full-term fourth order half-linear differential equation*

$$\begin{aligned}
&\left(\left[\ln^2(\ln t) + \ln(\ln^2 t) \right] \Phi\left(x''\right) \right)'' \\
&\quad - \beta_1 \left(t^{-p} \left[\ln^2(\ln t) + \mathrm{e}^{-t} \right] \Phi\left(x'\right) \right)' + \beta_0 t^{-2p} \ln^2(1 + \ln t) \Phi(x) = 0. \qquad (17)
\end{aligned}$$

It is easy to verify that function $L(t) = \ln^2(\ln t) + \ln(\ln^2 t)$ defined in some neighborhood of infinity is slowly varying, and that

$$\left[\ln^2(\ln t) + \ln(\ln^2 t) \right] \sim \left[\ln^2(\ln t) + \mathrm{e}^{-t} \right] \sim \ln^2(1 + \ln t) \quad \text{as} \quad t \to \infty.$$

According to Theorem 1, Equation (17) is nonoscillatory if

$$\begin{aligned}
\left(\frac{p-1}{p} \right)^p + \beta_1 > 0, \\
\left(\frac{p-1}{p} \right)^p \left(\frac{2p-1}{p} \right)^p + \left(\frac{2p-1}{p} \right)^p \beta_1 + \beta_0 > 0.
\end{aligned}$$

If $p = 2$ (Equation (17) is linear), then these inequalities take the form $\frac{1}{4} + \beta_1 > 0$ and $\frac{9}{16} + \frac{9}{4}\beta_1 + \beta_0 > 0$.

To our knowledge, oscillation properties of the full-term Euler type linear differential equation (Equation (2) with $p = 2$) were only studied for the case $n = 2$ (fourth order), see [17].

Example 2. *Consider the eighth order half-linear differential equation with a middle term*

$$\left(t^{4p}\Phi\left(x^{(4)}\right)\right)^{(4)} + \beta_2\left((t+1)^{2p}\Phi\left(x''\right)\right)'' + \beta_0\left(1 + \frac{1}{\ln^p t}\right)\Phi(x) = 0. \tag{18}$$

According to Theorem 1, Equation (18) is nonoscillatory provided

$$\gamma_{p,4p}\gamma_{p,3p} + \beta_2 > 0, \qquad i.e., \qquad \left(\frac{3p+1}{p}\right)^p\left(\frac{2p+1}{p}\right)^p + \beta_2 > 0,$$

$$\gamma_{4,p,4p} + \gamma_{p,2p}\gamma_{p,p}\beta_2 + \beta_0 > 0, \qquad i.e., \qquad \prod_{i=1}^{4}\left(\frac{(4-i)p+1}{p}\right)^p + \left(\frac{p+1}{p}\right)^p\frac{1}{p^p}\beta_2 + \beta_0 > 0.$$

The next example shows a special case of our result for a two-term equation.

Example 3. *Let $\alpha \in \mathbb{R} \setminus \mathcal{M}_{n,p}$. By Theorem 1, the two-term half-linear differential equation*

$$(-1)^n\left(t^\alpha K(t)\Phi\left(x^{(n)}\right)\right)^{(n)} + \gamma t^{\alpha-np}L(t)\Phi(x) = 0$$

(the two-term version of (1) with $L_n \equiv K$, $L_0 \equiv L$ and $\beta_0 = \gamma$) is nonoscillatory if

$$-\gamma_{n,p,\alpha} < \gamma \quad \text{and} \quad K(t) \sim L(t) \text{ as } t \to \infty.$$

5. Conclusions

We derived a general nonoscillation criterion formulated in Theorem 1. The coefficients are not completely generic functions, which are usually considered in even order half-linear differential equations, but they are still from a "large" class of regularly varying functions. Additionally, we believe the methods in our proofs may result in a new approach for obtaining nonoscillation criteria for more general even ordered half-linear differential equations considered as a perturbation of Equation (1). However, as we conjecture in Remark 3, oscillatory behavior of the coefficients' boundary states in (1) might not be unambiguously determined without additional assumptions.

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