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A Weak Convergence Self-Adaptive Method for Solving Pseudomonotone Equilibrium Problems in a Real Hilbert Space

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Abstract: In this paper, we presented a modification of the extragradient method to solve pseudomonotone equilibrium problems involving the Lipschitz-type condition in a real Hilbert space. The method uses an inertial effect and a formula for stepsize evaluation, that is updated for each iteration based on some previous iterations. The key advantage of the algorithm is that it is achieved without previous knowledge of the Lipschitz-type constants and also without any line search procedure. A weak convergence theorem for the proposed method is well established by assuming mild cost bifunction conditions. Many numerical experiments are presented to explain the computational performance of the method and to equate them with others.

Keywords: equilibrium problem; Lipschitz-type conditions; pseudomonotone bifunction; weak convergence; variational inequality problems; Nash-Cournot oligopolistic equilibrium model

1. Introduction

An equilibrium problem (EP) consider various mathematical problems as a special case i.e., the variational inequality problems (VIP), the optimization problems, the fixed point problems, the complementarity problems, Nash equilibrium of non-cooperative games, the saddle point and the vector minimization problem (see [1–3] for more details). To the best of our knowledge the term “equilibrium problem” was initially established in 1992 by Muu and Oettli [2] and has been further developed by Blum and Oettli [1]. The problem of equilibrium is also considered as the Ky Fan inequality [4]. It is interesting and useful to develop new iterative schemes and study their convergence analysis. Many methods have been well-established to figure out the solution of an equilibrium problem (1) in finite and infinite-dimensional spaces. Some of these algorithms involve projection methods [5–8], the proximal point methods [9,10], the extragradient methods with or without line searches [11–18], the methods using the inertial effect [19–22] and other methods in [23–28].

The proximal point method (PPM) is one of the well-established methods to deal with equilibrium problems. Martinet [29] initially developed this strategy for the case of monotone variational inequality

problems, and afterwards, they extended it to monotone operators by Rockafellar [30]. Moudafi [9] presented the proximal point method to solve monotone equilibrium. Konnov [31] also proposed another modification of the proximal point method, with weaker assumptions to deal with equilibrium problems involving monotone bifunction. Normally, the proximal point method applies to monotone equilibrium problems. Then, each regularized sub-problem converts into strongly monotone and accordingly its unique solution exists. Another well-known technique is the auxiliary problem principle, i.e., formulate a new equivalent problem that is usually simpler and easier to figure out compared to the initial problem. This idea was originally initiated by Cohen [32] for solving optimization problems and thereafter applied to variational inequality problems [33]. In addition, Mastroeni [34] extended the auxiliary problem principle to solve equilibrium problems involving a strongly monotone bifunction.

Next, we consider a well-known two-step extragradient method [35]. Tran et al. [36] and Flam et al. [10] used the auxiliary problem principle and extend the extragradient method in [35] for monotone equilibriums problems. The first drawback of the extragradient method is to compute two projections on the feasible set C to achieve the next iteration. So, in the case, when the computation of a projection on a constraint set C is difficult to compute, then it is hard to solve two minimal distance problems. A fact that could affect the performance and applicability of the method. To overcome this situation, Censor et al. introduced a subgradient extragradient method [37] wherein the second projection in the extragradient method is replaced by a specific half-plane projection that can be computed easily and efficiently. An iterative sequence developed due to extragradient-like methods essentially needs a stepsize that depends upon the Lipschitz-type constants of a cost bifunction. The prior information of such constants imposes some restrictions on implementing these methods because these Lipschitz-type constants are normally not known or hard to compute.

In this paper, we are introducing a new algorithm to solve the pseudomonotone equilibrium problem (1) incorporating the Lipschitz-type condition (2) on a bifunction in a real Hilbert space. The algorithm can be viewed as a combination of Algorithm 1 in [16] with inertial effects. The stepsize used in the proposed scheme is not fixed but updated on the basis of the previous iterations using an explicit formula. Under certain mild conditions, a weak convergence theorem is established which ensures that the sequence of iterations converges towards a solution to the problem of equilibrium. We are also examine the computational performance of the new method on various test problems in comparison to Algorithm 1 in [16].

The rest of this paper is arranged in the following way: in Section 2, we provide some definitions and necessary results that will be needed throughout this article. Section 3 contains our algorithm and the weak convergence theorem. Section 4 sets out the numerical experiments to show the computational performance of our proposed method on different test problems.

2. Preliminaries

We use C to be a closed, convex subset of the real Hilbert space $\mathbb{H}$. The notion $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and induced norm on the Hilbert space, respectively. The notion $EP(f, C)$ is the solution set of an equilibrium problem over C and q represent any arbitrary element of $EP(f, C)$.

Definition 1 ([1]). Let C be a nonempty, close and convex subset of $\mathbb{H}$ and $f : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$. The equilibrium problem for the bifunction f on C is formulated as follows:

$$\text{Determine } q \in C \text{ such that } f(q, y) \geq 0, \forall y \in C. \quad (1)$$

Definition 2 ([38]). The metric projection $P_C(x)$ of x onto a closed, convex subset C of $\mathbb{H}$ is defined as follows:

$$P_C(x) = \arg \min_{y \in C} \{\|y - x\|\}.$$

Now, we recall classical concepts of monotonicity of nonlinear operators (see [39,40]).

Definition 3. An operator $F : \mathbb{H} \rightarrow \mathbb{H}$ is said to be

(1) *strongly monotone on C if*

$$\langle F(x) - F(y), x - y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in C;$$

(2) *monotone on C if*

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(3) *strongly pseudomonotone on C if*

$$\langle F(x), y - x \rangle \geq 0 \implies \langle F(y), y - x \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in C;$$

(4) *pseudomonotone on C if*

$$\langle F(x), y - x \rangle \geq 0 \implies \langle F(y), y - x \rangle \geq 0, \quad \forall x, y \in C;$$

(5) *L -Lipschitz continuous on C if there exists a constant $L > 0$ such that*

$$\|F(x) - F(y)\| \leq L \|x - y\|, \quad \forall x, y \in C.$$

Now, we consider different notions of the bifunction monotonicity (see [1,41]).

Definition 4. A bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ on C for $\gamma > 0$ is said to be

(1) *strongly monotone if*

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C;$$

(2) *monotone if*

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

(3) *strongly pseudomonotone if*

$$f(x, y) \geq 0 \implies f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C;$$

(4) *pseudomonotone if*

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad \forall x, y \in C.$$

Remark 1. It is clear to see that the following consequences are as follows:

$$(1) \implies (2) \implies (4) \text{ and } (1) \implies (3) \implies (4).$$

Let a bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ satisfy the Lipschitz-type condition [34] on C if there are $c_1, c_2 > 0$ such that

$$f(x, z) \leq f(x, y) + f(y, z) + c_1 \|x - y\|^2 + c_2 \|y - z\|^2, \quad \forall x, y, z \in C. \quad (2)$$

The normal cone of C at $x \in C$ is defined by

$$N_C(x) = \{w \in \mathbb{H} : \langle w, y - x \rangle \leq 0, \forall y \in C\}.$$

Let $g : C \rightarrow \mathbb{R}$ is a convex function and *subdifferential* of g at $x \in C$ is defined as follows:

$$\partial g(x) = \{w \in \mathbb{H} : g(y) - g(x) \geq \langle w, y - x \rangle, \forall y \in C\}.$$

Lemma 1 ([42]). Let $g : C \rightarrow \mathbb{R}$ be a convex, sub-differentiable and lower semi-continuous function on C , where C to be a nonempty, convex and closed subset of a Hilbert space $\mathbb{H}$. An element $x \in C$ is said to be a minimizer of a function g if and only if $0 \in \partial g(x) + N_C(x)$, while $\partial g(x)$ and $N_C(x)$ are serve as the sub-differential of g on x and the normal cone of C at x , respectively.

Lemma 2 ([43]). For every $x, y \in \mathbb{H}$ and $\eta \in \mathbb{R}$, then the following relation is true:

$$\|\eta x + (1 - \eta)y\|^2 = \eta\|x\|^2 + (1 - \eta)\|y\|^2 - \eta(1 - \eta)\|x - y\|^2.$$

Lemma 3 ([44]). Let a_n, b_n and c_n be sequences in $[0, +\infty)$ such that

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + c_n, \quad \forall n \geq 1, \quad \text{with} \quad \sum_{n=1}^{+\infty} c_n < +\infty,$$

and also with $b > 0$ such that $0 \leq b_n \leq b < 1$ for all $n \in \mathbb{N}$. Then the following relations are hold.

- (i) $\sum_{n=1}^{+\infty} [a_n - a_{n-1}]_+ < \infty$, with $[s]_+ := \max\{s, 0\}$;
- (ii) $\lim_{n \rightarrow +\infty} a_n = a^* \in [0, \infty)$.

Lemma 4 ([45]). Let $\{x_n\}$ be a sequence in $\mathbb{H}$ and $C \subset \mathbb{H}$ such that the following conditions hold.

- (i) For each $x \in C$, $\lim_{n \rightarrow \infty} \|x_n - x\|$ exists;
- (ii) All sequentially weak cluster point of $\{x_n\}$ lies in C .

Then, $\{x_n\}$ weakly converges to a point in C .

3. Main Results

In this section, we present an inertial method to solve pseudomonotone equilibrium problem involving the Lipschitz-type bifunction condition. The detailed method is described below.

Assumption 1. Let a bifunction $f : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ satisfy the following conditions:

- (A1) $f(y, y) = 0, \forall y \in C$ and also f is pseudomonotone on C ;
- (A2) f satisfies the Lipschitz-type condition on $\mathbb{H}$ through positive constants c_1 and c_2 ;
- (A3) $\limsup_{n \rightarrow \infty} f(x_n, y) \leq f(x^*, y)$ for every $y \in C$ and $\{x_n\} \subset C$ satisfying $x_n \rightharpoonup x^*$;
- (A4) $f(x, \cdot)$ needs to be convex and subdifferentiable on C for each fixed $x \in \mathbb{H}$.

Remark 2. It is noted that T_n represent a half-space and $C \subset T_n$. From value of y_n , we have

$$y_n = \arg \min_{y \in C} \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}.$$

By using Lemma 1, we obtain

$$0 \in \partial_2 \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\} (y_n) + N_C(y_n).$$

Thus, there exists $v_n \in \partial f(w_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\xi_n v_n + y_n - w_n + \bar{w} = 0.$$

Thus, we have

$$\langle w_n - y_n, y - y_n \rangle = \xi_n \langle v_n, y - y_n \rangle + \langle \bar{w}, y - y_n \rangle, \quad \forall y \in C.$$

Since $\bar{w} \in N_C(y_n)$, then $\langle \bar{w}, y - y_n \rangle \leq 0$ for all $y \in C$. Therefore, we have

$$\langle w_n - y_n, y - y_n \rangle \leq \xi_n \langle v_n, y - y_n \rangle, \quad \forall y \in C.$$

The above implies that

$$\langle w_n - \xi_n v_n - y_n, y - y_n \rangle \leq 0, \quad \forall y \in C.$$

This shows that $C \subset T_n$ for every $n \in \mathbb{N}$.

Lemma 5. If $y_n = w_n$ in Algorithm 1 then $w_n \in EP(f, C)$.

Algorithm 1 Inertial Explicit Extragradient Algorithm for Pseudomonotone EP

Initialization: Choose $x_{-1}, x_0 \in \mathbb{H}$, $\xi_0 > 0$ and nondecreasing sequence $0 \leq \theta_n \leq \theta < \sqrt{5} - 2$. Set

$$w_0 = x_0 + \theta_0(x_0 - x_{-1}).$$

Iterative steps: Let $\xi_n > 0$ and $x_{n-1}, x_n \in \mathbb{H}$ are known for $n \geq 0$.

Step 1: Compute

$$y_n = \arg \min_{y \in C} \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\},$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$. If $y_n = w_n$ then stop and w_n is the solution of equilibrium problem. Otherwise, go to **Step 2**.

Step 2: Create a half space first

$$T_n = \{z \in \mathbb{H} : \langle w_n - \xi_n v_n - y_n, z - y_n \rangle \leq 0\},$$

where $v_n \in \partial_2 f(w_n, y_n)$. Compute

$$z_n = \arg \min_{y \in T_n} \left\{ \xi_n f(y_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}.$$

Step 3: Next, take $\beta_n \in (0, 1]$ and compute

$$x_{n+1} = (1 - \beta_n)w_n + \beta_n z_n.$$

Step 4: Assume $\mu(\theta) > 0$ and set $d = f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)$ with

$$\xi_{n+1} = \begin{cases} \min \left\{ \xi_n, \frac{\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2d} \right\} & \text{if } d > 0, \\ \xi_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back **Step 1**.

Proof. By definition of y_n and Lemma 1, we have

$$0 \in \partial_2 \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\}(y_n) + N_C(y_n).$$

Thus, there exists $v_n \in \partial f(w_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\xi_n v_n + y_n - w_n + \bar{w} = 0.$$

Thus, we have

$$\langle w_n - y_n, y - y_n \rangle = \xi_n \langle v_n, y - y_n \rangle + \langle \bar{w}, y - y_n \rangle, \quad \forall y \in C.$$

Given that $w_n = y_n$ and $\bar{\omega} \in N_C(y_n)$ implies that

$$\xi_n \langle v_n, y - y_n \rangle \geq 0. \quad (3)$$

Due to $v_n \in f(w_n, y_n)$ and using subdifferential definition, we obtain

$$f(w_n, y) - f(w_n, y_n) \geq \langle v_n, y - y_n \rangle, \quad \forall y \in \mathbb{H}. \quad (4)$$

Combining expressions (3), (4) and due to $\xi_n > 0$, we get

$$f(w_n, y) - f(w_n, y_n) \geq 0. \quad (5)$$

By $y_n = w_n$ and condition (A1) implies that $f(w_n, y) \geq 0$ for all $y \in C$. $\square$

Lemma 6. Let $f : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ is a bifunction satisfying the conditions (A1)-(A4). Let $q \in EP(f, C) \neq \emptyset$, we have

$$\|z_n - q\|^2 \leq \|w_n - q\|^2 - \left(1 - \frac{\mu\xi_n}{\xi_{n+1}}\right) \|w_n - y_n\|^2 - \left(1 - \frac{\mu\xi_n}{\xi_{n+1}}\right) \|z_n - y_n\|^2.$$

Proof. By value of z_n and using Lemma 1, we have

$$0 \in \partial_2 \left\{ \xi_n f(y_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\} (z_n) + N_{T_n}(z_n).$$

Thus, for $\omega \in \partial f(y_n, z_n)$ there exists $\bar{\omega} \in N_{T_n}(z_n)$ such that

$$\xi_n \omega + z_n - w_n + \bar{\omega} = 0.$$

Therefore, we have

$$\langle w_n - z_n, y - z_n \rangle = \xi_n \langle \omega, y - z_n \rangle + \langle \bar{\omega}, y - z_n \rangle, \quad \forall y \in T_n.$$

Since $\bar{\omega} \in N_{T_n}(z_n)$, then $\langle \bar{\omega}, y - z_n \rangle \leq 0$ for all $y \in T_n$. This implies that

$$\xi_n \langle \omega, y - z_n \rangle \geq \langle w_n - z_n, y - z_n \rangle, \quad \forall y \in T_n. \quad (6)$$

Due to $\omega \in \partial f(y_n, z_n)$, we have

$$f(y_n, y) - f(y_n, z_n) \geq \langle \omega, y - z_n \rangle. \quad (7)$$

Combining expressions (6) and (7), we obtain

$$\xi_n f(y_n, y) - \xi_n f(y_n, z_n) \geq \langle w_n - z_n, y - z_n \rangle, \quad \forall y \in T_n. \quad (8)$$

By substituting $y = q$ in (8), we get

$$\xi_n f(y_n, q) - \xi_n f(y_n, z_n) \geq \langle w_n - z_n, q - z_n \rangle, \quad \forall y \in T_n. \quad (9)$$

Since $q \in EP(f, C)$, then $f(q, y_n) \geq 0$ and it implies that $f(y_n, q) \leq 0$ due to pseudomonotonicity of a bifunction f . Thus, from (9) we get

$$\langle w_n - z_n, z_n - q \rangle \geq \xi_n f(y_n, z_n). \quad (10)$$

From the definition of ξ_{n+1} , we get

$$f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n) \leq \frac{\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\xi_{n+1}},$$

which after multiplying both sides by $\xi_n > 0$, gives that

$$\begin{aligned} \xi_n f(y_n, z_n) &\geq \xi_n \{f(w_n, z_n) - f(w_n, y_n)\} \\ &\quad - \frac{\mu\xi_n}{2\xi_{n+1}} \|w_n - y_n\|^2 - \frac{\mu\xi_n}{2\xi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \quad (11)$$

From expressions (10) and (11), we obtain

$$\begin{aligned} \langle w_n - z_n, z_n - q \rangle &\geq \xi_n \{f(w_n, z_n) - f(w_n, y_n)\} \\ &\quad - \frac{\mu\xi_n}{2\xi_{n+1}} \|w_n - y_n\|^2 - \frac{\mu\xi_n}{2\xi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \quad (12)$$

Since $z_n \in T_n$ and by the definition of T_n , we have

$$\langle w_n - \xi_n v_n - y_n, z_n - y_n \rangle \leq 0.$$

Thus, above implies that

$$\langle w_n - y_n, z_n - y_n \rangle \leq \xi_n \langle v_n, z_n - y_n \rangle. \quad (13)$$

Due to $v_n \in \partial_2 f(w_n, y_n)$ and with definition of subdifferential, we have

$$f(w_n, y) - f(w_n, y_n) \geq \langle v_n, y - y_n \rangle, \quad \forall y \in \mathbb{H}.$$

Substituting $y = z_n$ in expression (13), we have

$$f(w_n, z_n) - f(w_n, y_n) \geq \langle v_n, z_n - y_n \rangle. \quad (14)$$

From (13) and (14), we obtain

$$\xi_n \{f(w_n, z_n) - f(w_n, y_n)\} \geq \langle w_n - y_n, z_n - y_n \rangle. \quad (15)$$

Thus, expressions (12) and (15) implies that

$$\begin{aligned} \langle w_n - z_n, z_n - q \rangle &\geq \langle w_n - y_n, z_n - y_n \rangle \\ &\quad - \frac{\mu\xi_n}{2\xi_{n+1}} \|w_n - y_n\|^2 - \frac{\mu\xi_n}{2\xi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \quad (16)$$

We have the following facts:

$$2\langle w_n - z_n, z_n - q \rangle = \|w_n - q\|^2 - \|z_n - w_n\|^2 - \|z_n - q\|^2.$$

$$2\langle y_n - w_n, y_n - z_n \rangle = \|w_n - y_n\|^2 + \|z_n - y_n\|^2 - \|w_n - z_n\|^2.$$

From the above two facts and expression (16), we obtain

$$\|z_n - q\|^2 \leq \|w_n - q\|^2 - \left(1 - \frac{\mu\xi_n}{\xi_{n+1}}\right) \|w_n - y_n\|^2 - \left(1 - \frac{\mu\xi_n}{\xi_{n+1}}\right) \|z_n - y_n\|^2.$$

□

Theorem 1. Let sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ generated by Algorithm 1 are converges weakly to the solution q of the problem (EP) where

$$0 < \mu < \frac{\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2}{\frac{1}{2} - \theta + \frac{1}{2}\theta^2} \text{ and } 0 \leq \theta < \sqrt{5} - 2.$$

Proof. From Lemma 6, we have

$$\begin{aligned} \|z_n - q\|^2 &\leq \|w_n - q\|^2 - \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|w_n - y_n\|^2 - \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|z_n - y_n\|^2 \\ &\leq \|w_n - q\|^2 - \frac{1}{2} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|z_n - w_n\|^2. \end{aligned} \quad (17)$$

From the definition of x_{n+1} and using Lemma 2, we obtain

$$\|x_{n+1} - q\|^2 \leq (1 - \beta_n) \|w_n - q\|^2 + \beta_n \|z_n - q\|^2. \quad (18)$$

From expressions (17) and (18), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \beta_n) \|w_n - q\|^2 + \beta_n \|w_n - q\|^2 - \frac{\beta_n}{2} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|z_n - w_n\|^2 \\ &\leq \|w_n - q\|^2 - \frac{\beta_n}{2} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|z_n - w_n\|^2 \\ &\leq \|w_n - q\|^2 - \frac{1}{2\beta_n} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|x_{n+1} - w_n\|^2 \\ &\leq \|w_n - q\|^2 - \frac{1}{2} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right) \|x_{n+1} - w_n\|^2. \end{aligned} \quad (19)$$

Due to the definition of x_{n+1} we have $\|x_{n+1} - w_n\| = \beta_n \|z_n - w_n\|$ which is used to obtain the last inequality. From the value of w_n , we obtain

$$\begin{aligned} \|w_n - q\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - q\|^2 \\ &= \|(1 + \theta_n)(x_n - q) - \theta_n(x_{n-1} - q)\|^2 \\ &= (1 + \theta_n) \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 + \theta_n(1 + \theta_n) \|x_n - x_{n-1}\|^2. \end{aligned} \quad (20)$$

By the definition of w_n and using Cauchy inequality, we get

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_{n+1} - x_n, x_n - x_{n-1} \rangle \end{aligned} \quad (21)$$

$$\begin{aligned} &\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \|x_{n+1} - x_n\| \|x_n - x_{n-1}\| \\ &\geq \|x_{n+1} - x_n\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 - \theta_n \|x_{n+1} - x_n\|^2 - \theta_n \|x_n - x_{n-1}\|^2 \\ &= (1 - \theta_n) \|x_{n+1} - x_n\|^2 + (\theta_n^2 - \theta_n) \|x_n - x_{n-1}\|^2. \end{aligned} \quad (22)$$

Combining expressions (19), (20) and (22), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 + \theta_n) \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 + \theta_n(1 + \theta_n) \|x_n - x_{n-1}\|^2 \\ &\quad - \varrho_n(1 - \theta_n) \|x_{n+1} - x_n\|^2 - \varrho_n(\theta_n^2 - \theta_n) \|x_n - x_{n-1}\|^2 \end{aligned} \quad (23)$$

$$\begin{aligned} &= (1 + \theta_n) \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 - \varrho_n(1 - \theta_n) \|x_{n+1} - x_n\|^2 \\ &\quad + \left[\theta_n(1 + \theta_n) - \varrho_n(\theta_n^2 - \theta_n) \right] \|x_n - x_{n-1}\|^2 \\ &= (1 + \theta_n) \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 - Q_n \|x_{n+1} - x_n\|^2 \\ &\quad + R_n \|x_n - x_{n-1}\|^2, \end{aligned} \quad (24)$$

where $\varrho_n := \frac{1}{2} \left(1 - \frac{\mu\tilde{\xi}_n}{\xi_{n+1}}\right)$, $Q_n := \varrho_n(1 - \theta_n) \geq 0$ and $R_n := \theta_n(1 + \theta_n) - \varrho_n(\theta_n^2 - \theta_n)$.

Next, we substitute

$$\Psi_n = \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 + R_n \|x_n - x_{n-1}\|^2.$$

We have compute

$$\begin{aligned} \Psi_{n+1} - \Psi_n &= \|x_{n+1} - q\|^2 - \theta_{n+1} \|x_n - q\|^2 + R_{n+1} \|x_{n+1} - x_n\|^2 \\ &\quad - \|x_n - q\|^2 + \theta_n \|x_{n-1} - q\|^2 - R_n \|x_n - x_{n-1}\|^2 \\ &\leq \|x_{n+1} - q\|^2 - (1 + \theta_n) \|x_n - q\|^2 + \theta_n \|x_{n-1} - q\|^2 \\ &\quad + R_{n+1} \|x_{n+1} - x_n\|^2 - R_n \|x_n - x_{n-1}\|^2. \end{aligned} \quad (25)$$

Combining expressions (24) and (25), we obtain

$$\begin{aligned} \Psi_{n+1} - \Psi_n &\leq -Q_n \|x_{n+1} - x_n\|^2 + R_{n+1} \|x_{n+1} - x_n\|^2 \\ &= -(Q_n - R_{n+1}) \|x_{n+1} - x_n\|^2. \end{aligned} \quad (26)$$

Next, we need to compute

$$\begin{aligned} Q_n - R_{n+1} &= \varrho_n(1 - \theta_n) - \theta_{n+1}(1 + \theta_{n+1}) + \varrho_{n+1}(\theta_{n+1}^2 - \theta_{n+1}) \\ &\geq \varrho_n(1 - \theta_{n+1}) - \theta_{n+1}(1 + \theta_{n+1}) + \varrho_n(\theta_{n+1}^2 - \theta_{n+1}) \\ &= \varrho_n(1 - \theta_{n+1})^2 - \theta_{n+1}(1 + \theta_{n+1}) \\ &\geq \varrho_n(1 - \theta)^2 - \theta(1 + \theta) \\ &= (\varrho_n - \theta - \theta^2) + \varrho_n\theta^2 - 2\varrho_n\theta \\ &= \left(\frac{1}{2} - \theta - \theta^2 + \frac{\theta^2}{2} - \theta\right) - \mu\left(\frac{\xi_n}{2\xi_{n+1}} + \frac{\xi_n}{2\xi_{n+1}}\theta^2 - \frac{\xi_n}{\xi_{n+1}}\theta\right) \\ &= \left(\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2\right) - \mu\left(\frac{\xi_n}{2\xi_{n+1}} - \frac{\xi_n}{\xi_{n+1}}\theta + \frac{\xi_n}{2\xi_{n+1}}\theta^2\right). \end{aligned} \quad (27)$$

Since from our hypothesis, we have

$$0 < \mu < \frac{\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2}{\frac{1}{2} - \theta + \frac{1}{2}\theta^2} \text{ and } 0 \leq \theta < \sqrt{5} - 2.$$

Since $\frac{\xi_n}{\xi_{n+1}} \rightarrow 1$, then there exist $n_0 \geq 1$ such that

$$\epsilon \in \left(0, \frac{1}{2} - 2\theta - \frac{1}{2}\theta^2 - \mu\left(\frac{1}{2} - \theta + \frac{1}{2}\theta^2\right)\right).$$

Thus, expression (27) implies that

$$Q_n - R_{n+1} \geq \epsilon, \quad \forall n \geq n_0. \quad (28)$$

Combining expressions (26) and (28) implies that for all $n \geq n_0$, we have

$$\Psi_{n+1} - \Psi_n \leq -(Q_n - R_{n+1}) \|x_{n+1} - x_n\|^2 \leq 0. \quad (29)$$

The sequence $\{\Psi_n\}$ is nonincreasing for $n \geq n_0$. From definition of Ψ_{n+1} , we have

$$\begin{aligned} \Psi_{n+1} &= \|x_{n+1} - q\|^2 - \theta_{n+1} \|x_n - q\|^2 + R_{n+1} \|x_{n+1} - x_n\|^2 \\ &\geq -\theta_{n+1} \|x_n - q\|^2. \end{aligned} \quad (30)$$

From Ψ_n , we have

$$\begin{aligned}\Psi_n &= \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2 + R_n \|x_n - x_{n-1}\|^2 \\ &\geq \|x_n - q\|^2 - \theta_n \|x_{n-1} - q\|^2.\end{aligned}\quad (31)$$

The expression (31) implies that for $n \geq n_0$, we have

$$\begin{aligned}\|x_n - q\|^2 &\leq \Psi_n + \theta_n \|x_{n-1} - q\|^2 \\ &\leq \Psi_{n_0} + \theta \|x_{n-1} - q\|^2 \\ &\leq \dots \leq \Psi_{n_0} (\theta^{n-n_0} + \dots + 1) + \theta^{n-n_0} \|x_{n_0} - q\|^2 \\ &\leq \frac{\Psi_{n_0}}{1-\theta} + \theta^{n-n_0} \|x_{n_0} - q\|^2.\end{aligned}\quad (32)$$

Combining expressions (30) and (32), we obtain

$$\begin{aligned}-\Psi_{n+1} &\leq \theta_{n+1} \|x_n - q\|^2 \\ &\leq \theta \|x_n - q\|^2 \\ &\leq \theta \frac{\Psi_{n_0}}{1-\theta} + \theta^{n-n_0+1} \|x_{n_0} - q\|^2.\end{aligned}\quad (33)$$

It follows from (29) and (33) such that

$$\begin{aligned}\epsilon \sum_{n=n_0}^k \|x_{n+1} - x_n\|^2 &\leq \Psi_{n_0} - \Psi_{k+1} \\ &\leq \Psi_{n_0} + \theta \frac{\Psi_{n_0}}{1-\theta} + \theta^{k-n_0+1} \|x_{n_0} - q\|^2 \\ &\leq \frac{\Psi_{n_0}}{1-\theta} + \|x_{n_0} - q\|^2.\end{aligned}\quad (34)$$

By letting $k \rightarrow \infty$ in (34) implies that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$. This implies that

$$\|x_{n+1} - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (35)$$

From the expressions (21) and (35), we obtain

$$\|x_{n+1} - w_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (36)$$

Moreover, by Lemma 3 with expression (23) and $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$, implies that

$$\lim_{n \rightarrow \infty} \|x_n - q\|^2 = b. \quad (37)$$

Thus, expressions (20), (35) and (37) implies that

$$\lim_{n \rightarrow \infty} \|w_n - q\|^2 = b. \quad (38)$$

We also have

$$0 \leq \|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty. \quad (39)$$

Next, need to show that $\lim_{n \rightarrow \infty} \|y_n - q\|^2 = b$. Using Lemma 6 for $n \geq n_0$, we have

$$\begin{aligned} \left(1 - \frac{\mu \tilde{\xi}_n}{\tilde{\xi}_{n+1}}\right) \|w_n - y_n\|^2 &\leq \|w_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\leq (\|w_n - q\| + \|x_{n+1} - q\|)(\|w_n - q\| - \|x_{n+1} - q\|) \\ &\leq (\|w_n - q\| + \|x_{n+1} - q\|)\|x_{n+1} - w_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (40)$$

Further, it implies that

$$0 \leq \|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (41)$$

Combining expressions (35), (37) and (41), we obtain

$$\|x_{n+1} - y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \lim_{n \rightarrow \infty} \|y_n - q\|^2 = b. \quad (42)$$

The above discussion implies that the sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded and for every $q \in EP(f, C)$, and the limit of $\|w_n - q\|$, $\|x_n - q\|$ and $\|z_n - q\|$ exists. Next, we determine that the set of all sequentially weak limit point of the sequence $\{x_n\}$ belongs to $EP(f, C)$. To confirm this, consider element z to be sequentially weak limit point of the sequence $\{x_n\}$ which yields a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weakly converge to z . Thus, the sequence $\{y_{n_k}\}$ also converges weakly to $z \in C$, due to $\|x_n - y_n\| \rightarrow 0$. Next, the remaining part is to prove that element z belongs to the solution set $EP(f, C)$. By (8), the definition of $\tilde{\xi}_{n+1}$ and (15), we have

$$\begin{aligned} \tilde{\xi}_{n_k} f(y_{n_k}, y) &\geq \tilde{\xi}_{n_k} f(y_{n_k}, z_{n_k}) + \langle w_{n_k} - z_{n_k}, y - z_{n_k} \rangle \\ &\geq \tilde{\xi}_{n_k} f(w_{n_k}, z_{n_k}) - \tilde{\xi}_{n_k} f(w_{n_k}, y_{n_k}) - \frac{\mu \tilde{\xi}_{n_k}}{2 \tilde{\xi}_{n_k+1}} \|w_{n_k} - y_{n_k}\|^2 \\ &\quad - \frac{\mu \tilde{\xi}_{n_k}}{2 \tilde{\xi}_{n_k+1}} \|y_{n_k} - z_{n_k}\|^2 + \langle w_{n_k} - z_{n_k}, y - z_{n_k} \rangle \\ &\geq \langle w_{n_k} - y_{n_k}, z_{n_k} - y_{n_k} \rangle - \frac{\mu \tilde{\xi}_{n_k}}{2 \tilde{\xi}_{n_k+1}} \|w_{n_k} - y_{n_k}\|^2 \\ &\quad - \frac{\mu \tilde{\xi}_{n_k}}{2 \tilde{\xi}_{n_k+1}} \|y_{n_k} - z_{n_k}\|^2 + \langle w_{n_k} - z_{n_k}, y - z_{n_k} \rangle, \end{aligned} \quad (43)$$

where y is an any element in T_n . It follows from expressions (36), (40), (42) and the boundedness of $\{x_n\}$ that the right-hand side of the last inequality tends to zero. Using $\tilde{\xi}_{n_k} > 0$, condition (A3) and $y_{n_k} \rightharpoonup z$, we have

$$0 \leq \limsup_{k \rightarrow \infty} f(y_{n_k}, y) \leq f(z, y), \quad \forall y \in T_n.$$

Since $C \subset T_n$ and $z \in C$ implies $f(z, y) \geq 0, \forall y \in C$. This shows that $z \in EP(f, C)$. Thus Lemma 4, provides that $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ weakly converge to q as $n \rightarrow \infty$. $\square$

Note: Let consider the cost bifunction in the following way

$$f(x, y) := \langle F(x), y - x \rangle, \quad \forall x, y \in C, \quad (44)$$

where $F : C \rightarrow \mathbb{H}$ is an operator. Then, the problem (1) is translated into the variational inequality problem and the value of Lipschitz constant is $L = 2c_1 = 2c_2$. From the value of y_n in the Algorithm 1 and expression (44), we have

$$\begin{aligned}
y_n &= \arg \min_{y \in C} \left\{ \xi_n f(w_n, y) + \frac{1}{2} \|w_n - y\|^2 \right\} \\
&= \arg \min_{y \in C} \left\{ \xi_n \langle F(w_n), y - w_n \rangle + \frac{1}{2} \|w_n - y\|^2 + \frac{\xi_n^2}{2} \|F(w_n)\|^2 - \frac{\xi_n^2}{2} \|F(w_n)\|^2 \right\} \\
&= \arg \min_{y \in C} \left\{ \frac{1}{2} \|y - (w_n - \xi_n F(w_n))\|^2 \right\} \\
&= P_C(w_n - \xi_n F(w_n)).
\end{aligned} \tag{45}$$

The value of z_n translate into

$$z_n = P_{T_n}(w_n - \xi_n F(y_n)).$$

Let consider that F meets the following conditions:

(F1) F satisfy the following condition on C for some positive constant $L > 0$

$$\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in C;$$

(F2) The solution set $VI(F, C) \neq \emptyset$ and F is pseudomonotone operator on C ;

(F3) $\limsup_{n \rightarrow \infty} \langle F(u_n), y - u_n \rangle \leq \langle F(u^*), y - u^* \rangle$ for every $y \in C$ and $\{u_n\} \subset C$ satisfying $u_n \rightharpoonup u^*$.

Corollary 1. Suppose that $F : C \rightarrow \mathbb{H}$ follows the requirement (F1)-(F3). Assume $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ are the sequences formed in the following way:

(i) Choose $x_{-1}, x_0 \in \mathbb{H}$ and nondecreasing sequence $0 < \theta_n \leq \theta < \sqrt{5} - 2$. Set

$$w_0 = x_0 + \theta_0(x_0 - x_{-1}).$$

(ii) Let $\xi_n > 0$ and $x_{n-1}, x_n \in \mathbb{H}$ are known for $n \geq 0$. Compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \xi_n F(w_n)), \\ z_n = P_{T_n}(w_n - \xi_n F(y_n)), \\ x_{n+1} = (1 - \beta_n)w_n + \beta_n z_n, \end{cases}$$

where $\beta_n \subset (0, 1]$ and $T_n = \{z \in \mathbb{H} : \langle w_n - \xi_n F(w_n) - y_n, z - y_n \rangle \leq 0\}$.

(iii) Moreover, assume that

$$0 < \mu < \frac{\frac{1}{2} - 2\theta - \frac{1}{2}\theta^2}{\frac{1}{2} - \theta + \frac{1}{2}\theta^2} \text{ and } 0 \leq \theta < \sqrt{5} - 2,$$

and revise the stepsize ξ_{n+1} as follows:

$$\xi_{n+1} = \begin{cases} \min \left\{ \xi_n, \frac{\mu(\|w_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\langle F(w_n) - F(y_n), z_n - y_n \rangle} \right\} & \text{if } \langle F(w_n) - F(y_n), z_n - y_n \rangle > 0, \\ \xi_n & \text{otherwise.} \end{cases}$$

Then, the sequence $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ converge weakly to the solution q of $VI(F, C)$.

4. Numerical Experiments

For numerical experiments, we analyzed four examples by allowing different dimensions to see the effectiveness of our proposed method. We used the Matlab Quadprog program

(<https://www.mathworks.com/help/optim/ug/quadprog.html>) to solve optimization problems and computed on a (Matlab R2018b) PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB.

Example 1. Assume that there have been n companies that manufacture the same commodities and this model taken as extension of the Nash-Cournot Oligopolistic equilibrium model [36]. Let x correspond to a vector in which every element x_i indicates the amount of the commodity generated by the company i . So, in this case the price function P has been taken the form as $P_i(S) = \phi_i - \psi_i S$, where $\phi_i > 0$, $\psi_i > 0$ and $S = \sum_{i=1}^n x_i$ is the sum of total commodity generated by all companies. The profit function for each firm i is defined by $F_i(x) = P_i(S)x_i - t_i(x_i)$ where $t_i(x_i)$ is the value of tax and fee for producing x_i . Suppose that $C_i = [x_i^{\min}, x_i^{\max}]$ is the set of strategies belongs to each firm i and the strategy scheme for the whole model take the form as $C := C_1 \times C_2 \times \cdots \times C_n$. In particular, each organization desires to attain its optimum return by considering that the resulting amount of production is an input parameter and is dependent on the engagement of all the other parties. The methodology frequently managed to work on this type of model primarily concentrates on the well-known Nash equilibrium principle. A point $x^* \in C = C_1 \times C_2 \times \cdots \times C_n$ is an equilibrium point of the above-discussed model if

$$F_i(x^*) \geq F_i(x^*[x_i]) \quad \forall x_i \in C_i, \quad \forall i = 1, 2, \dots, n,$$

while the vector $x^*[x_i]$ represent the vector get from x^* by taking x_i^* with x_i . Finally, we take $f(x, y) := \varphi(x, y) - \varphi(x, x)$ with $\varphi(x, y) := -\sum_{i=1}^n F_i(x[y_i])$ and the problem of finding the Nash equilibrium point of the model can be taken as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C.$$

It follows from [36,46], the bifunction $f : C \times C \rightarrow \mathbb{R}$ could be taken in the following form

$$f(x, y) = \langle Px + Qy + q, y - x \rangle,$$

while $C \subset \mathbb{R}^n$ is defined as

$$C := \{x \in \mathbb{R}^n : -5 \leq x_i \leq 5\}.$$

The matrices P, Q are being taken arbitrary (Choosing two diagonal matrices randomly A_1 and A_2 with entries from $[0, 2]$ and $[-2, 0]$ respectively. Two random orthogonal matrices B_1 and B_2 are able to generate a positive semidefinite matrix $M_1 = B_1 A_1 B_1^T$ and negative semidefinite matrix $M_2 = B_2 A_2 B_2^T$. Finally, set $Q = M_1 + M_1^T$, $S = M_2 + M_2^T$ and $P = Q - S$) and vector q entries taken from interval $[-n, n]$ (for more details see [36,47]). The starting points are $x_{-1} = x_0 = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ and $\mu = 0.22$, $\theta_n = 0.20$ with $\beta_n = 0.85$. Figures 1–4 and Table 1 demonstrates the numerical performance of Algorithm 1 (InExEgA) comparison with Algorithm 1 (ExEgA) in [16] relative to number of iterations.

Table 1. Example 1: The numerical results for Figures 1–4.

n	ξ_0	TOL	ExEgA		InExEgA	
			Iter.	CPU(s)	Iter.	CPU(s)
5	0.1	10^{-12}	89	0.8725	71	0.5018
	0.2	10^{-12}	54	0.4895	40	0.2080
10	0.1	10^{-12}	106	1.1755	79	0.6562
	0.2	10^{-12}	73	0.8509	58	0.4067
20	0.1	10^{-12}	122	1.5985	98	0.9870
	0.2	10^{-12}	91	1.0032	76	0.7689
50	0.1	10^{-12}	162	2.4567	134	1.4567
	0.2	10^{-12}	107	1.4356	98	1.0067

Figure 1. Algorithm 1 comparison with Algorithm 1 in [16] in term of iterations when $n = 5$.

Figure 2. Algorithm 1 comparison with Algorithm 1 in [16] in term of iterations when $n = 10$.

Figure 3. Algorithm 1 comparison with Algorithm 1 in [16] in term of iterations when $n = 20$.

Figure 4. Algorithm 1 comparison with Algorithm 1 in [16] in term of iterations when $n = 50$.

Example 2. Consider the following problem of fractional programming as in [48]:

$$\min g(x) = \frac{x^T Qx + a^T x + a_0}{b^T x + b_0}$$

subject to $x \in C := \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\}$ where $a_0 = -2, b_0 = 4$ and

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

It is easy to verify that Q is symmetric and positive definite in $\mathbb{R}^4$ and consequently g is pseudoconvex on C . We minimize g over $C := \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10, i = 1, 2, 3, 4\}$ by using both algorithms with bifunction $f(x, y) = \langle G(x), y - x \rangle$ on $C \times C$ to $\mathbb{R}$ with unknown Lipschitz constants where

$$G(x) := \nabla g(x) = \frac{((b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0))}{(b^T x + b_0)^2}.$$

This problem has a unique solution $x^* = (1, 1, 1, 1)^T \in C$. Choose initial points are $x_{-1} = x_0 = (1, 1, 1, 1)^T \in \mathbb{R}^4$ and $\mu = 0.22, \theta_n = 0.20, \beta_n = 0.85, \zeta_0 = 0.5$. Figures 5 and 6 demonstrates the numerical performance of Algorithm 1 (InExEgA) comparison with Algorithm 1 (ExEgA) in [16] in term of number of iterations and elapsed time respectively.

Figure 5. Algorithm 1 comparison with Algorithm 1 in [16] and number of iterations are 204 and 175 respectively.

Figure 6. Algorithm 1 comparison with Algorithm 1 in [16] and elapsed time in seconds are 2.9404 and 0.9194 respectively.

Example 3. Let the bifunction f define on the convex set C as

$$f(x, y) = \langle (BB^T + S + D)x, y - x \rangle,$$

where B is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative. The feasible set $C \subset \mathbb{R}^m$ is closed and convex and defined as

$$C = \{x \in \mathbb{R}^m : Ax \leq b\},$$

while A is an $l \times m$ matrix and b is a nonnegative vector. It is clear that bifunction f is monotone and its Lipschitz-type constants are $c_1 = c_2 = \frac{\|BB^T + S + D\|}{2}$. The initial points are $x_{-1} = x_0 = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ and $\mu = 0.22$, $\theta_n = 0.20$ with $\beta_n = 0.85$. Figures 7–12 and Table 2 demonstrates the numerical performance of Algorithm 1 comparison with Algorithm 1 in [16] in term of number of iterations and elapsed time in seconds.

Table 2. Example 3: The numerical results for Figures 7–12.

n	ξ_0	TOL	ExEgA		InExEgA	
			Iter.	CPU(s)	Iter.	CPU(s)
5	0.5	10^{-4}	205	7.2979	131	1.8722
10	0.5	10^{-4}	544	23.0891	262	4.6898
20	0.5	10^{-4}	1562	65.7050	1084	23.1766

Figure 7. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 5$.

Figure 8. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 5$.

Figure 9. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 10$.

Figure 10. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 10$.

Figure 11. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 20$.

Figure 12. Algorithm 1 comparison with Algorithm 1 in [16] when $m = 20$.

Example 4. Consider a bifunction $f : C \times C \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \langle F(x), y - x \rangle,$$

with unknown Lipschitz constants c_1, c_2 and $C = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 5, i = 1, 2, 3, 4\}$ and

$$F(x) = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 - 4x_2x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_3 \end{pmatrix}.$$

The initial values are $x_{-1} = x_0 = (1, 1, 1, 1)^T \in \mathbb{R}^n$ and $\mu = 0.22, \theta_n = 0.20$ with $\beta_n = 0.85$. Figures 13 and 14 demonstrates the numerical performance of Algorithm 1 comparison with Algorithm 1 in [16].

Figure 13. Algorithm 1 comparison with Algorithm 1 in [16] and number of iterations are 108 and 49 respectively.

Figure 14. Algorithm 1 comparison with Algorithm 1 in [16] and elapsed time in seconds are 1.3294 and 0.1701 respectively.

5. Conclusions

The paper proposed an inertial extragradient-like approach to addressing the problem of equilibrium of a pseudomonotone bifunction with a Lipschitz-like condition. A stepsize rule has been presented that does not depend on the Lipschitz-type constant information. Several experimental studies have been reported to demonstrate the numerical behavior of our method and compare it with other methods. Such numerical experiments indicate that inertial influences normally extend the performance of iterative sequences in this sense.

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