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Article

q-Generalized Linear Operator on Bounded Functions of Complex Order

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Abstract: This article presents a q-generalized linear operator in Geometric Function Theory (GFT) and investigates its application to classes of analytic bounded functions of complex order $S_q(c;M)$ and $C_q(c;M)$ where $0 < q < 1, 0 \neq c \in \mathbb{C}$, and $M > \frac{1}{2}$. Integral inclusion of the classes related to the q-Bernardi operator is also proven.

Keywords: q-difference operator; subordinating factor sequence; bounded analytic functions of complex order; q-generalized linear operator

MSC: Primary 30C45; Secondary 30C50; 30H05

1. Introduction

Quantum calculus or q-calculus is attributed to the great mathematicians L.Euler and C. Jacobi, but it became popular when Albert Einstein used it in quantum mechanics in his paper [1] published in 1905. F.H. Jackson [2,3] introduced and studied the q-derivative and q-integral in a proper way. Later, quantum groups gave the geometrical aspects to q-calculus. It is pertinent to mention that q-calculus can be considered an extension of classical calculus discovered by I. Newton and G.W. Leibniz. In fact, the operators defined as:

$$d_h f(z) = \frac{f(z+h) - f(z)}{h}$$

and:

$$d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \; 0 < q < 1,$$

where $z \in \mathbb{C}$ and $h > 0$ are the h-derivative and q-derivative, respectively, where h is Planck's constant, are related as: $q = e^{ih} = e^{2\pi i \bar{h}}$ where $\bar{h} = h/2\pi$. Srivastava [4] applied the concepts of q-calculus by using the basic (or q-) hypergeometric functions in Geometric Function Theory (GFT). Ismail [5] and Agarwal [6] introduced the class of q-starlike functions by using the q-derivative. The q-close-to-convex functions were defined in [7], and Sahoo and Sharma [8] obtained several interesting results for q-close-to-convex functions. Several convolution and fractional calculus q-operators were defined by the researchers, which were reposited by Srivastava in [9]. Darus [10] defined a new differential operator called the q-generalized operator by using q-hypergeometric functions. Let A be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

analytic in the open unit disc $E = \{z : |z| < 1\}$.

Let $f(z)$ be given by (1) and $g(z)$ defined as:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let f, h be analytic functions. Then, f is subordinate to h, written as $f \prec h$ or $f(z) \prec h(z)$, $z \in E$, if there exists a Schwartz function $\omega(z)$ analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$, such that $f(z) = h(\omega(z))$. If h is univalent in E, then $f \prec h$, if and only if $f(0) = h(0)$ and $f(E) \subset h(E)$.

A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is a subordinating factor if, whenever $f(z) = \sum_{k=1}^{\infty} a_k z^k, a_1 = 1$ is regular, univalent, and convex in E, we have $\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), z \in E$ [11].

We recall some basic concepts from q-calculus that are used in our discussion and refer to [2,3,12] for more details.

A subset $B \subset \mathbb{C}$ is called q-geometric if $zq \in B$ whenever $z \in B$, and it contains all the geometric sequences $\left\{zq^k\right\}_0^{\infty}$. In GFT, the q-derivative of $f(z)$ is defined as:

$$d_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \; q \in (0,1), \quad (z \in B \setminus \{0\}),$$

and $d_q f(0) = f'(0)$. For a function $g(z) = z^k$, the q-derivative is:

$$d_q g(z) = [k] z^{k-1},$$

where $[k] = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}$.

We note that as $q \to 1^-$, $d_q f(z) \to f'(z)$, which is the ordinary derivative. From (1), we deduce that:

$$d_q f(z) = 1 + \sum_{k=2}^{\infty} [k] \, a_k z^k.$$

Let $f(z)$ and $g(z)$ be defined on a q-geometric set B. Then, for complex numbers a, b, we have:

$$d_q(af(z) \pm bg(z)) = a d_q f(z) \pm b d_q g(z).$$

$$d_q(f(z)g(z)) = f(qz) d_q g(z) + g(z) d_q f(z).$$

$$d_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\, d_q f(z) - f(z)\, d_q g(z)}{g(z)g(qz)}, \; g(z)g(qz) \neq 0.$$

$$d_q(\log f(z)) = \frac{\ln q^{-1}}{1-q} \frac{d_q f(z)}{f(z)}.$$

Jackson [2] introduced the q-integral of a function f, given by:

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(q^k z),$$

provided that the series converges.

For any non-negative integer n, the q-number shift factorial is defined as:

$$[n]! = \begin{cases} [1]\,[2]\,...\,[n] & \text{if } n \neq 0, \\ 1 & \text{if } n = 0 \end{cases}$$

Let $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$; the q-generalized Pochhammer symbol is defined as:

$$[\lambda]_n = [\lambda]\,[\lambda+1]\,[\lambda+2] \ldots . [\lambda+n-1]\,.$$

The q-Gamma function is defined for $\lambda > 0$ as:

$$\Gamma_q(\lambda+1) = [\lambda]\Gamma_q(\lambda) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

For complex parameters $a_i\ (1 \le i \le l), b_j \ne 0, -1, -2, ...(1 \le j \le m)$ with $l \le m+1$, the basic q-hypergeometric function is defined as,

$$ {}_lF_m(a_1, ...a_l; b_1, ..., b_m, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_l)_k}{(q)_k (b_1)_k ... (b_m)_k} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+m-l} z^k. \tag{2}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$ and $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, the q-shifted factorial is defined for $a \in \mathbb{C}$ as:

$$(a)_k = \begin{cases} (1-a)(1-aq) \ldots \left(1-aq^{k-1}\right) & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0. \end{cases}$$

Let $l = m+1$, $a_1 = q^{\lambda+1} (\lambda > -1)$, $a_i = q\ (\forall\, 2 \le i \le l)$, and $b_j = q\ (\forall\, 1 \le j \le m)$, and by using the property $(q^a)_k = \Gamma_q(a+k)\,(1-q)^k / \Gamma_q(a)$, from (2), we get the function,

$$F_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(\lambda+k)}{[k-1]!\Gamma_q(\lambda+1)} z^k = z + \sum_{k=2}^{\infty} \frac{[\lambda+1]_{k-1}}{[k-1]!} z^k,\ z \in E.$$

In [13], the q-Srivastava–Attiya convolution operator is defined as:

$$G^s_{q,a}(z) = z + \sum_{k=2}^{\infty} \left(\frac{[1+a]}{[k+a]} \right)^s z^k, z \in E,$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \mathrm{Re}(s) > 1 \text{ when } |z| = 1).$$

Using convolution, the operator $D^s_{q,a,\lambda}$ for $\lambda > -1$ is defined as:

$$\begin{aligned} D^s_{q,a,\lambda} f(z) &= J^s_{q,a,\lambda}(z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \left(\frac{[k+a]}{[1+a]} \right)^s \frac{[\lambda+1]_{k-1}}{[k-1]!} a_k z^k, z \in E, \end{aligned}$$

where:

$$J^s_{q,a,\lambda}(z) = \left(G^s_{q,a}(z) \right)^{-1} * F_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty} \left(\frac{[k+a]}{[1+a]} \right)^s \frac{[\lambda+1]_{k-1}}{[k-1]!} z^k.$$

It is a convergent series with a radius of convergence of one. We observe that $D^0_{q,a,0} f(z) = f(z)$ and $D^1_{q,0,0} f(z) = z d_q f(z)$. The operator $D^s_{q,a,\lambda}$ reduces to known linear operators for different values of parameters a, s, and λ as:

(i) If $q \to 1^-$, it reduces to the operator $D^s_{a,\lambda}$ discussed by Noor et al. in [14].
(ii) For $s = 0$, it is a q-Ruscheweyh differential operator [15].
(iii) If $s = -1$, $\lambda = 0$, and $q \to 1^-$, it is an Owa–Srivastava integral operator [16].
(iv) If $s \in \mathbb{N}_0$, $a = 1$, $\lambda = 0$, and $q \to 1^-$, it reduces to the generalized Srivastava–Attiya integral operator [17].
(v) If $s \in \mathbb{N}_0$, $a = 0$, $\lambda = 0$, it is a q-Salagean differential operator [18].
(vi) For $s, \lambda \in \mathbb{N}_0$, and $a = 0$, it is the operator defined in [19].

The following identities hold for the operator $D^{s}_{q,a,\lambda}f(z)$,

$$zd_q\left(D^{s}_{q,a,\lambda}f(z)\right)=\left(\frac{[1+a]}{q^a}\right)D^{s+1}_{q,a,\lambda}f(z)-\frac{[a]}{q^a}D^{s}_{q,a,\lambda}f(z) \tag{3}$$

$$zd_q(D^{s}_{q,a,\lambda}f(z))=\left(\frac{[1+\lambda]}{q^\lambda}\right)D^{s}_{q,a,\lambda+1}f(z)-\frac{[\lambda]}{q^\lambda}D^{s}_{q,a,\lambda}f(z). \tag{4}$$

Let $P(q)$ be the class of functions of the form $p(z) = 1+c_1z+c_2z^2+....$, analytic in E, and satisfying:

$$\left|p(z)-\frac{1}{1-q}\right|\leq\frac{1}{1-q}, \qquad (z\in E, q\in(0,1)).$$

It is known from [20] that $p\in P(q)$ implies $p(z)\prec\frac{1+z}{1-qz}$. It follows immediately that Re $p(z)>0$, $z\in E$.

The classes of bounded q-starlike functions $S_q(c,M)$ and bounded q-convex functions $C_q(c,M)$ of complex order c were defined in [21], respectively, as:

$$S_q(c,M)=\left\{f\in A:\left|\frac{c-1+\frac{zd_qf(z)}{f(z)}}{c}-M\right|<M\right\},$$
$$\left(c\in\mathbb{C}^*;\ M>\frac{1}{2}, z\in E\right),$$

or equivalently,

$$S_q(c,M)=\left\{f\in A:\frac{zd_qf(z)}{f(z)}\prec\frac{1+\{c(1+m)-m\}z}{1-mz}\right\},$$
$$\left(c\in\mathbb{C}^*;\ m=1-\frac{1}{M};\ M>\frac{1}{2}\right).$$

The class of bounded q-convex functions $C_q(c,M)$ of complex order c is defined as:

$$C_q(c,M)=\left\{f\in A:\left|\frac{c-1+\frac{d_q(zd_qf(z))}{d_qf(z)}}{c}-M\right|<M\right\},$$
$$\left(c\in\mathbb{C}^*;\ M>\frac{1}{2},\ z\in E\right),$$

or equivalently,

$$C_q(c,M)=\left\{f\in A:\frac{d_q(zd_qf(z))}{d_qf(z)}\prec\frac{1+\{c(1+m)-m\}z}{1-mz}\right\}$$
$$\left(c\in^*;\ m=1-\frac{1}{M};\ M>\frac{1}{2}\right).$$

Using the operator $D^{s}_{q,a,\lambda}f(z)$, we now define the following new classes $S_{q,a,s,\lambda}(c,M)$ and $C_{q,a,s,\lambda}(c,M)$ as:

$$S_{q,a,s,\lambda}(c,M) = \left\{ f \in A : \frac{z(d_q D^s_{q,a,\lambda}(f(z)))}{D^s_{q,a,\lambda}(f(z))} \prec \frac{1+\{c(1+m)-m\}z}{1-mz},\ z \in E \right\},$$
$$\left(0 < q < 1, c \in^*; m = 1 - \frac{1}{M}; \ M > \frac{1}{2} \right).$$

Special cases:

(i) If $c = 1$, $m = 1$, and $q \to 1^-$, then $S_{q,a,s,\lambda}(c,M)$ reduces to class $S^s(a,\lambda)$ discussed in [22].
(ii) If $c = 1$, $s = 0$, $\lambda = 0$, $m = -q$, then $S_{q,a,s,\lambda}(c,M)$ reduces to class S^*_q introduced by Noor et al. [23].
(iii) If $s = 0$, $c = \frac{m}{1+m}$ $(-1 < m < 0)$, $m = -q$, then $S_{q,a,s,\lambda}(c,M)$ reduces to class ST_q studied by Noor [24].
(iv) If $s = 0, \lambda = 0$, $c = ae^{-i\beta}\cos\beta$ $(a \in \mathbb{C}^*, |\beta| < \frac{\pi}{2})$, and $q \to 1^-$, then $S_{q,a,s,\lambda}(c,M)$ becomes special cases of Janowski β-spiral like functions of complex order $S^\beta(A,B,a)$ discussed in [25].
(v) If $s \in \mathbb{N}_0$, $\lambda = 0, a = 0$, and $q \to 1^-$, then $S_{q,a,s,\lambda}(c,M)$ reduces to class $H_n(c,M)$ discussed by Aouf et al. in [26].
(vi) If $0 < c \le 1$, $-1 < m < 0$, and $q \to 1^-$, then $S_{q,a,s,\lambda}(c,M)$ becomes a special case of the class $S^s_{a,\lambda}(\eta, A,B)$ with $\eta = 0$ discussed in [19].

A function $f \in A$ is in the class $S_{q,a,s,\lambda}(c,M)$ if and only if:

$$\left| \frac{\frac{zd_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)} - 1}{A - B\left\{\frac{zd_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)}\right\}} \right| < 1, \tag{5}$$

where $A = c(1+m) - m$ and $B = -m$.

The class $C_{q,a,s,\lambda}(c,M)$ is defined as:

$$C_{q,a,s,\lambda}(c,M) = \left\{ f \in A : \frac{d_q(zd_q(D^s_{q,a,\lambda}f(z))}{d_q(D^s_{q,a,\lambda}f(z))} \prec \frac{1+\{c(1+m)-m\}z}{1-mz}, z \in E \right\},$$
$$\left(0 < q < 1, c \in \mathbb{C}^*; m = 1 - \frac{1}{M}; \ M > \frac{1}{2} \right).$$

It is easy to see that $f \in C_{q,a,s,\lambda}(c,M) \Leftrightarrow zd_q f \in S_{q,a,s,\lambda}(c,M)$. In order to develop results for the classes $S_{q,a,s,\lambda}(c,M)$ and $C_{q,a,s,\lambda}(c,M)$, we need the following:

Lemma 1 ([27])**.** *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be regular in E with $h(0) = 1$ and* $\mathrm{Re}[\beta h(z) + \gamma] > 0$. *If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in E, then $p(z) + \frac{zd_q p(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z)$.*

Lemma 2 ([11])**.** *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if:*

$$\mathrm{Re}\left\{ 1 + 2\sum_{k=1}^{\infty} b_k z^k \right\} > 0,\ z \in E.$$

2. Properties of Classes $S_{q,a,s,\lambda}(c,M)$ and $C_{q,a,s,\lambda}(c,M)$

We start the section with the necessary and sufficient condition for a function to be in the class $S_{q,a,s,\lambda}(c,M)$.

Theorem 1. *Let $f \in A$. Then, $f \in S_{q,a,s,\lambda}(c,M)$ if and only if:*

$$\sum_{k=2}^{\infty} \{[k]-1+|c(1+m)+m([k]-1)|\} \frac{[\lambda+1]_{k-1}}{[k-1]!} \left| \left(\frac{[k+a]}{[1+a]} \right)^{s} \right| |a_k| < |c(1+m)|, \tag{6}$$

where $m = 1 - \frac{1}{M}$, $(M > \frac{1}{2})$.

Proof. Let us assume first that Inequality (6) holds. To show $f \in S_{q,a,s,\lambda}(c,M)$, we need to prove Inequality (5).

$$\begin{aligned}
\left| \frac{\frac{z(d_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)} - 1}{A - B\left\{ \frac{zd_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)} \right\}} \right| &= \left| \frac{\sum_{k=2}^{\infty} \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!}([k]-1)a_k z^k}{(A-B)z + \sum_{k=2}^{\infty} (A - B\,[k]) \left(\frac{[1+a]}{[k+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!} a_k z^k} \right| \\
&\leq \frac{\sum_{k=2}^{\infty} \left| \left(\frac{[k+a]}{[1+a]} \right)^{s} \right| \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!}([k]-1)\,|a_k|}{|A-B| - \left| \sum_{k=2}^{\infty} (A - B\,[k]) \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!} a_k \right|} \\
&\leq \frac{\sum_{k=2}^{\infty} \left| \left(\frac{[k+a]}{[1+a]} \right)^{s} \right| \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!}([k]-1)|a_k|}{|c(1+m)| - \sum_{k=2}^{\infty} |c(1+m)+m([k]-1)| \frac{[\lambda+1]_{k-1}}{[k-1]!} \left| \left(\frac{[k+a]}{[1+a]} \right)^{s} \right| |a_k|} \\
&< 1.
\end{aligned}$$

Hence, $f \in S_{q,a,s,\lambda}(c,M)$ by using Inequality (6). Conversely, let $f \in S_{q,a,s,\lambda}(c,M)$ be of the form (1), then:

$$\left| \frac{\frac{z(d_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}(f(z))} - 1}{A - B\left\{ \frac{zd_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)} \right\}} \right| = \left| \frac{\sum_{k=2}^{\infty} \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!}([k]-1)a_k z^k}{(A-B)z + \sum_{k=2}^{\infty} (A - B\,[k]) \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!} a_k z^k} \right|.$$

Since $|\mathrm{Re}\, z| \leq |z|$,, we have:

$$\mathrm{Re} \left\{ \left| \frac{\sum_{k=2}^{\infty} \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!}([k]-1)a_k z^k}{(A-B)z + \sum_{k=2}^{\infty} (A - B\,[k]) \left(\frac{[k+a]}{[1+a]} \right)^{s} \cdot \frac{[\lambda+1]_{k-1}}{[k-1]!} a_k z^k} \right| \right\} < 1.$$

Now, we choose values of z on the real axis such that $zd_q(D^s_{q,a,\lambda}f(z))/D^s_{q,a,\lambda}f(z)$ is real. Letting $z \to 1^-$ through real values, after some calculations, we obtain Inequality (6). □

Remark 1. *(i) If $q \to 1^-$, $s \in \mathbb{N}_0$, $a = 0$, and $\lambda = 0$, the above result reduces to the sufficient condition for $f(z)$ to be in class $H_n(c,M)$ $(c \in \mathbb{C}^*, M > \frac{1}{2})$ discussed in [26]. (ii) If $c = 1 - \alpha$ $(\alpha \in [0,1))$, $m = 0$, $\lambda = 0$, and $q \to 1^-$, the above result reduces to the sufficient condition for $f(z)$ to be in class $S^*_{s,a}(\alpha)$ discussed in [28].*

Theorem 2. *Let $f_i \in S_{q,a,s,\lambda}(c,M)$ having the form:*

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i} z^k, \qquad \text{for } i = 1,2,3,..,l.$$

Then, $F \in S_{q,a,s,\lambda}(c,M)$, where $F(z) = \sum_{i=1}^{l} c_i f_i(z)$ with $\sum_{i=1}^{t} c_i = 1$.

Proof. From Theorem 1, we can write:

$$\sum_{k=2}^{\infty}\left\{\frac{\{[k]-1+|b(1+m)+m([k]-1)|\}\frac{[\lambda+1]_{k-1}}{[k-1]!}\left|\left(\frac{[k+a]}{[1+a]}\right)^{s}\right|}{|b(1+m)|}\right\}a_{k,i}<1. \tag{7}$$

Therefore:

$$\begin{aligned}F(z)&=\sum_{i=1}^{l}c_i\left(z+\sum_{k=2}^{\infty}a_{k,i}z^k\right)\\&=z+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{l}c_ia_{k,i}\right)z^k;\end{aligned}$$

where however due to (7), we have:

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\{[k]-1+|b(1+m)+m([k]-1)|\}\frac{[\lambda+1]_{k-1}}{[k-1]!}\left|\left(\frac{[k+a]}{[1+a]}\right)^{s}\right|}{|b(1+m)|}\left(\sum_{i=2}^{l}c_ia_{k,i}\right)\\&=\sum_{i=2}^{l}\left[\frac{\{[k]-1+|b(1+m)+m([k]-1)|\}\frac{[\lambda+1]_{k-1}}{[k-1]!}\left|\left(\frac{[k+a]}{[1+a]}\right)^{s}\right|}{|b(1+m)|}\right]c_i\leq 1;\end{aligned}$$

Therefore, $F\in S_{q,a,s,\lambda}(c,M)$. □

Theorem 3. *Let f_i with $i=1,2,...,v$ belong to the class $S_{q,a,s,\lambda}(c,M)$. The arithmetic mean h of f_i is given by:*

$$h(z)=\frac{1}{v}\sum_{i=1}^{v}f_i(z) \tag{8}$$

belonging to class $S_{q,a,s,\lambda}(c,M)$.

Proof. From (8), we can write:

$$h(z)=\frac{1}{v}\sum_{i=1}^{v}\left(z+\sum_{k=2}^{\infty}a_{k,i}z^k\right)=z+\sum_{k=2}^{\infty}\left(\frac{1}{v}\sum_{i=1}^{v}a_{k,i}\right)z^k. \tag{9}$$

Since $f_i\in S_{q,a,s,\lambda}(c,M)$ for every $i=1,2,...,v$, using (6) and (9), we have:

$$\begin{aligned}&\sum_{k=2}^{\infty}\{[k]-1+|b(1+m)+m([k]-1)|\}\frac{[\lambda+1]_{k-1}}{[k-1]!}\left|\left(\frac{[k+a]}{[1+a]}\right)^{s}\right|\left(\frac{1}{v}\sum_{i=1}^{v}a_{k,i}\right)\\&=\frac{1}{v}\sum_{i=1}^{v}\left(\sum_{k=2}^{\infty}\{[k]-1+|b(1+m)+m([k]-1)|\}\frac{[\lambda+1]_{k-1}}{[k-1]!}\left|\left(\frac{[k+a]}{[1+a]}\right)^{s}\right|a_{k,i}\right)\\&\leq\frac{1}{v}\sum_{i=1}^{v}(|b(1+m)|)=|b(1+m)|,\end{aligned}$$

and this completes the proof. □

Now, we give the subordination relation for the functions in class $S_{q,a,s,\lambda}(c,M)$ by using the subordination theorem.

Theorem 4. *Let $m=1-\frac{1}{M}$ $(M>\frac{1}{2})$. Furthermore, $c\neq 0$ with $\mathrm{Re}(c)>\frac{-m}{2(1+m)}$ when $m>0$ and $\mathrm{Re}(c)<\frac{-m}{2(1+m)}$ when $m<0$ and $\lambda\geq 0$. If $f\in S_{q,a,s,\lambda}(c,M)$, then:*

$$\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}(f*g)(z)\prec g(z) \tag{10}$$

where $g(z)$ *is a convex function in* E, $C_{\lambda,k}=\frac{[\lambda+1]_{k-1}}{[k-1]!}$, $B_{s,a}(k)=\left|\left(\frac{[k+a]}{[1+a]}\right)^s\right|$, *and:*

$$\operatorname{Re} f(z) > -1-\frac{(1+m)\,|c|}{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}. \tag{11}$$

The constant $\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}$ *is the best estimate.*

Proof. Let $f(z)\in S_{q,a,s,\lambda}(c,M)$ and $g(z)=z+\sum_{k=2}^{\infty}c_kz^k$. Then:

$$\begin{aligned}&\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}(f*g)(z)\\&=\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}\left(z+\sum_{k=2}^{\infty}a_kc_kz^k\right).\end{aligned} \tag{12}$$

Thus, (10) holds true if:

$$\left\{\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}a_k\right\}_{k=1}^{\infty} \tag{13}$$

is a subordinating factor sequence with $a_1=1$. From Lemma 2, it suffices to show:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}a_kz^k\right\}>0. \tag{14}$$

Now, as $\{[k]-1+|c(1+m)+m([k]-1)|\}\,C_{\lambda,k}B_{s,a}(k)$ is an increasing function of k $(k\geq 2)$, we have:

$$\begin{aligned}&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}a_kz^k\right\}\\
&=\operatorname{Re}\left\{1+\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}z+\right.\\
&\left.+\frac{\sum_{k=2}^{\infty}\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)a_kz^k}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}\right\}\\
&\geq 1-\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}r-\\
&\frac{\sum_{k=2}^{\infty}\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)\,|a_k|\,r^k}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}\\
&>1-\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}r-\\
&\frac{(1+m)\,|c|}{[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}r\\
&>0.\qquad (|z|=r<1)\end{aligned}$$

Hence, (14) holds true in E, and the subordination result (10) is affirmed by Theorem 4. The inequality (11) follows by taking $g(z)=\frac{z}{1-z}=\sum_{k=1}^{\infty}z^k$ in (10).

Let us consider the function:

$$\phi(z) = z - \frac{|c(1+m)|}{[\{q + |c(1+m) + mq|\}C_{\lambda,2}B_{s,a}(2) + |c(1+m)|]} z^2 \; (z \in E)$$

which is a member of $S_{q,a,s,\lambda}(c, M)$. Then. by using (10), we have:

$$\frac{\{q + |c(1+m) + mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q + |c(1+m) + mq|\}C_{\lambda,2}B_{s,a}(2) + |c(1+m)|]}\phi(z) \prec \frac{z}{1-z} \quad .$$

It is easily verified that:

$$\min \operatorname{Re}\left\{\frac{\{q + |c(1+m) + mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q + |c(1+m) + mq|\}C_{\lambda,2}B_{s,a}(2) + |c(1+m)|]}\phi(z)\right\} = -\frac{1}{2} \quad (z \in E),$$

then the constant $\frac{\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)}{2[\{q+|c(1+m)+mq|\}C_{\lambda,2}B_{s,a}(2)+|c(1+m)|]}$ cannot be replaced by a larger one. □

Remark 2. *If $s \in \mathbb{N}_0$, $a = 0$, $\lambda = 0$, and $q \to 1^-$, Theorem 4 reduces to the subordination result proven in [29].*

Now, we discuss the inclusion results pertaining to classes $S_{q,a,s,\lambda}(c, M)$ and $C_{q,a,s,\lambda}(c, M)$ in reference to parameters s and λ.

Theorem 5. *For any complex number s, $S_{q,a,s+1,\lambda}(c, M) \subset S_{q,a,s,\lambda}(c, M)$ if $\operatorname{Re}(\frac{1+\{c(1+m)-m\}z}{1-mz}) > \frac{1}{q^{a_1}(1-q)}\{1 - \cos(a_2 \ln q)\}$ where $a = a_1 + ia_2$.*

Proof. Let $f \in S_{q,a,s+1,\lambda}(c, M)$, then:

$$\frac{zd_q(D^{s+1}_{q,a,\lambda}f(z))}{D^{s+1}_{q,a,\lambda}f(z)} \prec \frac{1 + \{c(1+m) - m\}z}{1 - mz}, \tag{15}$$

Let:

$$h(z) = \frac{1 + \{c(1+m) - m\}z}{1 - mz}$$

and:

$$r(z) = \frac{zd_q(D^{s}_{q,a,\lambda}f(z))}{D^{s}_{q,a,\lambda}f(z)}.$$

We will show:

$$r(z) \prec h(z),$$

which would prove $S_{q,a,s,\lambda}(c, M) \subset S_{q,a,s+1,\lambda}(c, M)$. From the identity relation (3), after a few calculations, we have:

$$\frac{zd_q(D^{s}_{q,a,\lambda}f(z))}{D^{s}_{q,a,\lambda}f(z)} = \frac{[1+a]}{q^a}.\frac{D^{s+1}_{q,a,\lambda}f(z)}{D^{s}_{q,a,\lambda}f(z)} - \frac{[a]}{q^a}.$$

After some calculations, we have:

$$\frac{D^{s+1}_{q,a,\lambda}f(z)}{D^{s}_{q,a,\lambda}f(z)} = \frac{1}{[1+a]}\left\{\frac{q^{a}zd_q(D^{s}_{q,a,\lambda}f(z))}{D^{s}_{q,a,\lambda}f(z)} + [a]\right\}$$
$$= \frac{1}{[1+a]}\left\{q^{a}r(z) + [a]\right\}.$$

Applying logarithmic q-differentiation, we have:

$$\frac{zd_q(D^{s+1}_{q,a,\lambda}f(z))}{D^{s+1}_{q,a,\lambda}f(z)} = r(z) + \frac{zd_q r(z)}{r(z) + q^{-a}[a]}. \tag{16}$$

From (15) and (16), we have:

$$r(z) + \frac{z[d_q r(z)]}{r(z) + q^{-a}[a]} \prec \frac{1+\{c(1+m)-m\}z}{1-mz}.$$

If $\mathrm{Re}(h(z)) > \frac{1}{q^{a_1}(1-q)}\{1-\cos(a_2 \ln q)\}$, then from Lemma 1, it implies:

$$r(z) \prec h(z),$$

which implies $f(z) \in S_{q,a,s,\lambda}(c,M)$. Therefore, $S_{q,a,s,\lambda}(c,M) \subset S_{q,a,s+1,\lambda}(c,M)$. □

Theorem 6. *For any complex number s, $C_{q,a,s+1,\lambda}(c,M) \subset C_{q,a,s,\lambda}(c,M)$ if $\mathrm{Re}(\frac{1+\{c(1+m)-m\}z}{1-mz}) > \frac{1}{q^{a_1}(1-q)}\{1-\cos(a_2 \ln q)\}$ where $a = a_1 + ia_2$.*

Proof. It is obvious from the fact $f \in C_{q,a,s,\lambda}(c,M) \Leftrightarrow zd_q f \in S_{q,a,s,\lambda}(c,M)$. □

Theorem 7. *For any complex number s, $S_{q,a,s,\lambda+1}(c,M) \subset S_{q,a,s,\lambda}(c,M)$ if $\mathrm{Re}(\frac{1+\{c(1+m)-m\}z}{1-mz}) > \frac{1-q^{-\lambda}}{1-q}$, $\lambda > -1$.*

Proof. Let $f \in S_{q,a,s,\lambda+1}(c,M)$, then:

$$\frac{zd_q(D^{s}_{q,a,\lambda+1}f(z))}{D^{s}_{q,a,\lambda+1}f(z)} \prec \frac{1+\{c(1+m)-m\}z}{1-mz}. \tag{17}$$

Consider:

$$h(z) = \frac{1+\{c(1+m)-m\}z}{1-mz}$$

and:

$$q(z) = \frac{zd_q(D^{s}_{q,a,\lambda}f(z))}{D^{s}_{q,a,\lambda}f(z)}.$$

We will show:

$$q(z) \prec h(z),$$

which would conveniently prove $S_{q,a,s,\lambda+1}(c,M) \subset S_{q,a,s,\lambda}(c,M)$. From the identity relation (4), after a few calculations, we have:

$$\frac{zd_q(D^{s}_{q,a,\lambda}f(z))}{D^{s}_{q,a,\lambda}f(z)} = \frac{[1+\lambda]}{q^{\lambda}}\frac{D^{s}_{q,a,\lambda+1}f(z)}{D^{s}_{q,a,\lambda}f(z)} - \frac{[\lambda]}{q^{\lambda}}.$$

After some calculations, we have:

$$\frac{D^s_{q,a,\lambda+1}f(z)}{D^s_{q,a,\lambda}f(z)} = \frac{1}{[1+\lambda]}\left\{\frac{q^a.zd_q(D^s_{q,a,\lambda}f(z))}{D^s_{q,a,\lambda}f(z)} + [\lambda]\right\}$$
$$= \frac{1}{[1+\lambda]}\left\{q^\lambda q(z) + [\lambda]\right\}.$$

Applying logarithmic q-differentiation, we have:

$$\frac{zd_q(D^s_{q,a,\lambda+1}f(z))}{D^s_{q,a,\lambda+1}f(z)} = q(z) + \frac{zd_q q(z)}{q(z)+q^{-\lambda}[\lambda]} \tag{18}$$

From (17) and (18), we have:

$$q(z) + \frac{z[d_q q(z)]}{q(z)+q^{-\lambda}[\lambda]} \prec \frac{1+\{c(1+m)-m\}z}{1-mz}.$$

If $\mathrm{Re}(h(z)) > \frac{1-q^{-\lambda}}{1-q}$ for any value of $\lambda > -1$, so by Lemma 1, we have $q(z) \prec h(z)$, which implies $f(z) \in S_{q,a,s,\lambda}(c, M)$. Therefore, $S_{q,a,s,\lambda+1}(c, M) \subset S_{q,a,s,\lambda}(c, M)$. □

Remark 3. *If we consider $q \to 1^-$ with $\mathrm{Re}\, a \geq 0$, $c = 1, m = 1$ in Theorem 5 and $\lambda \geq 0, c = 1, m = 1$ in Theorem 7, we obtain the special cases of the inclusion results, Theorems 2.4 and 2.5 in [19].*

In [30], the q-Bernardi integral operator $L_b f(z)$ is defined as:

$$L_b f(z) = \frac{[1+b]}{z^b}\int_0^z t^{b-1} f(t) d_q t$$
$$= z + \sum_{k=2}^{\infty}\left(\frac{[1+b]}{[k+b]}\right) a_k z^k,\ b = 1, 2, 3, ...$$

Now, we apply the generalized operator $D^s_{q,a,\lambda}$ on $L_b f(z)$ as:

$$D^s_{q,a,\lambda}(L_b f(z)) = z + \sum_{k=2}^{\infty}\left(\frac{[k+a]}{[1+a]}\right)^s . \frac{[\lambda+1]_{k-1}}{[k-1]!}\left(\frac{[1+b]}{[k+b]}\right) a_k z^k.$$

The identity relation of $D^s_{q,a,\lambda}(L_b f(z))$ is given as:

$$zd_q\left[D^s_{q,a,\lambda}\{L_b f(z)\}\right] = \left(\frac{[1+b]}{q^b}\right) D^s_{q,a,\lambda} f(z) - \frac{[b]}{q^b} D^s_{q,a,\lambda}\{L_b f(z)\}. \tag{19}$$

The following theorems are the integral inclusions of the classes $S_{q,a,s,\lambda}(c, M)$ and $C_{q,a,s,\lambda}(c, M)$ with respect to the q-Bernardi integral operator.

Theorem 8. *If $f(z) \in S_{q,a,s,\lambda}(c, M)$ then $L_b f(z) \in S_{q,a,s,\lambda}(c, M)$ if $\mathrm{Re}(\frac{1+\{c(1+m)-m\}z}{1-mz}) > \frac{1-q^{-b}}{1-q}$ for any complex number s.*

Proof. Let $g(z) \in S_{q,a,s,\lambda}(c, M)$, then:

$$\frac{zd_q(D^s_{q,a,\lambda}g(z))}{D^s_{q,a,\lambda}g(z)} \prec \frac{1+\{c(1+m)-m\}z}{1-mz}. \tag{20}$$

Consider:

$$h(z) = \frac{1 + \{c(1+m) - m\}z}{1 - mz}$$

and:

$$u(z) = \frac{zd_q(D^s_{q,a,\lambda} L_b g(z))}{D^s_{q,a,\lambda} L_b g(z)}.$$

We will show:

$$u(z) \prec h(z),$$

which would prove $L_b g(z) \in S_{q,a,s,\lambda}(c, M)$. From the identity relation (19), after some calculations, we have:

$$\frac{zd_q(D^s_{q,a,\lambda} L_b g(z))}{D^s_{q,a,\lambda} L_b g(z)} = \left(\frac{[1+b]}{q^b}\right) \frac{D^s_{q,a,\lambda} g(z)}{(D^s_{q,a,\lambda} L_b g(z))} - \frac{[b]}{q^b}.$$

After some calculations, we have:

$$\frac{D^s_{q,a,\lambda} g(z)}{D^s_{q,a,\lambda} L_b g(z)} = \frac{1}{[1+b]} \left[\frac{q^b.zd_q(D^s_{q,a,\lambda} L_b g(z))}{D^s_{q,a,\lambda} L_b g(z)} + [b] \right]$$

Applying logarithmic q-differentiation, we have:

$$\frac{zd_q(D^s_{q,a,\lambda} g(z))}{D^s_{q,a,\lambda} g(z)} = u(z) + \frac{z[d_q u(z)]}{u(z) + q^{-b}[b]} \tag{21}$$

From (20) and (21), we have:

$$u(z) + \frac{z[d_q u(z)]}{u(z) + q^{-b}[b]} \prec \frac{1 + \{c(1+m) - m\}z}{1 - mz}$$

If $\mathrm{Re}(h(z)) > \frac{1-q^{-b}}{1-q}$, so by Lemma 1, we have $u(z) \prec h(z)$, which implies $L_b g(z) \in S_{q,a,s,\lambda}(c, M)$. □

Theorem 9. *If $f(z) \in C_{q,a,s,\lambda}(c, M)$, then $L_b f(z) \in C_{q,a,s,\lambda}(c, M)$ for any complex number s.*

Proof. It is an immediate consequence of the fact $C_{q,a,s,\lambda}(c, M) \Leftrightarrow zd_q f \in S_{q,a,s,\lambda}(c, M)$. □

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