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A Five-Component Generalized mKdV Equation and Its Exact Solutions

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Abstract: In this paper, a 3×3 spectral problem is proposed and a five-component equation that consists of two different mKdV equations is derived. A Darboux transformation of the five-component equation is presented relating to the gauge transformations between the Lax pairs. As applications of the Darboux transformations, interesting exact solutions, including soliton-like solutions and a solution that consists of rational functions of e^x and t, for the five-component equation are obtained.

Keywords: gauge transformation; Darboux transformation; explicit solution

1. Introduction

In the study of nonlinear partial differential equations, the theory on solitons is an indispensable part [1]. A lot of systematic methods, such as the inverse scattering method [2,3], are developed to find exact explicit solutions for soliton equations. In the past few decades, as the soliton theory grew vigorously, solitons has been observed in solid physics, fluid physics, plasma physics, laser physics, condensed matter physics, etc. The research on soliton is an important topic in many physics laboratories. Many published articles are about soliton equations and integrability [4], Hamiltonian structures [5–7], various forms of solutions and their properties [8–10] and so on. Among the many ways to study soliton equations [11–17], the Darboux transformation method [18–25] is one of the most effective and fruitful tools. Darboux transformations can be used to obtain the rogue-wave solution [26,27], solutions on periodic backgrounds [28–30], and so on.

Inspired by the coupled mKdV equation [31,32], the complex modified Korteweg-de Vries equation [33], the coupled nonlinear Schrödinger equation [34,35], and so on, we propose a new five-component nonlinear integrable equation in this paper,

$$\begin{aligned} u_t =& - u_{xxx} - 6u^2u_x + (wv)_{xx} + (uv^2 - rv - \tfrac{1}{2}uw^2)_x + qv, \\ v_t =& v_{xxx} + 6v^2v_x - (wu)_{xx} - (u^2v + ru - \tfrac{1}{2}vw^2)_x - qu, \\ w_x =& - 2uv, \\ r_x =& (u^2 - v^2)w + (vu_x - uv_x), \\ q_x =& (u^2 - v^2)r + \tfrac{3}{2}w_x(u^2 + v^2) + \tfrac{1}{2}(u^2 + v^2)_x w + \tfrac{1}{6}(w^3)_x - (vu_{xx} + uv_{xx}), \end{aligned} \tag{1}$$

where $x, t \in \mathbb{R}$. Different from the traditional two-component mKdV equation, the five-component Equation (1) contains two different mKdV equations. That is, when $v = w = r = q = 0$ and when $u = w = r = q = 0$, the five-component Equation (1) is reduced to, respectively,

$$- u_t = u_{xxx} + 6u^2u_x, \tag{2}$$

and

$$v_t = v_{xxx} + 6v^2v_x. \tag{3}$$

Of course these are related by the reflection $x' = -x$ that replaces right-traveling waves by left-traveling waves. Resorting to the gauge transformations between the Lax pairs, we derive Darboux transformations and exact solutions for (1). As a result, we present several different type of solitons (including soliton solutions and a solution that is a rational function of e^x and t) for (1).

The structure of the paper is as follows. In Section 2, a Lax pair associated with the five-component Equation (1) is proposed, and a Darboux transformation is constructed. In Section 3, as applications of the Darboux transformations, some example solutions for (1) are constructed and illustrated.

2. Darboux Transformation

The five-component nonlinear integrable Equation (1) is associated with the spectral problem

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \quad \Phi = \begin{pmatrix} \phi \\ \psi \\ \varphi \end{pmatrix}, \quad U = \begin{pmatrix} \lambda & u & v \\ -u & 0 & 0 \\ -v & 0 & 0 \end{pmatrix}, \tag{4}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter independent of x and t, and $u = u(x,t), v = v(x,t)$ are potentials, the expression of V is

$$V = \begin{pmatrix} \lambda(-u^2+v^2) & -\lambda^2 u + \lambda(vw - u_x) - P & \lambda^2 v + \lambda(-uw + v_x) - Q \\ \lambda^2 u + \lambda(vw - u_x) + P & \lambda^3 + \lambda(u^2 + \frac{1}{2}w^2) & \lambda^2 w + \lambda r + q \\ -\lambda^2 v + \lambda(-uw + v_x) + Q & -\lambda^2 w + \lambda r - q & -\lambda^3 - \lambda(v^2 + \frac{1}{2}w^2) \end{pmatrix} \tag{5}$$

with

$$\begin{aligned} P &= 2u^3 + rv - uv^2 + \tfrac{1}{2}uw^2 - wv_x - vw_x + u_{xx}, \\ Q &= ru + u^2 v - 2v^3 - \tfrac{1}{2}vw^2 + wu_x + uw_x - v_{xx}. \end{aligned} \tag{6}$$

The soliton Equation (1) is yielded by the zero-curvature equation $U_t - V_x + [U, V] = 0$.

Letting $\hat{\Phi} = T\Phi$ we have

$$\hat{\Phi}_x = \hat{U}\hat{\Phi}, \quad \hat{\Phi}_t = \hat{V}\hat{\Phi}, \tag{7}$$

where $\hat{U} = (T_x + TU)T^{-1}$, $\hat{V} = (T_t + TV)T^{-1}$. In order to ensure that $\hat{U}, \hat{V}$ and U, V are the same form except that the old potentials u, v, w, q, r are replaced by the new potentials $\hat{u}, \hat{v}, \hat{w}, \hat{q}, \hat{r}$ respectively, we suppose that T has the following form,

$$T = \begin{pmatrix} \lambda + d & a & b \\ a & \lambda + g & c \\ b & c & \lambda + f \end{pmatrix}, \tag{8}$$

where

$$d = \frac{ab}{2c} - \frac{ac}{2b} - \frac{bc}{2a}, \quad g = \frac{ac}{2b} - \frac{ab}{2c} - \frac{bc}{2a}, \quad f = \frac{bc}{2a} - \frac{ac}{2b} - \frac{ab}{2c}. \tag{9}$$

It is easy to see that

$$\det T = (\lambda + \lambda_1)^2(\lambda - \lambda_1), \tag{10}$$

where

$$\lambda_1 = -\frac{bc}{2a} - \frac{ab}{2c} - \frac{ac}{2b}. \tag{11}$$

In view of

$$\begin{aligned}(\det T)_x &= \mathrm{tr}(T_x T^*) = \mathrm{tr}[(\hat{U}T - TU)T^*] = \mathrm{tr}(\hat{U}TT^*) - \mathrm{tr}(TUT^*)\\ &= (\det T)(\mathrm{tr}\hat{U}) - (\det T)(\mathrm{tr}U) = (\det T)\mathrm{tr}(\hat{U} - U) = 0,\\ (\det T)_t &= 0,\end{aligned}$$

where $T^* = T^{-1}\det T$, we find $\det T$ is independent of (x,t). As a result, λ_1 is a constant.

Suppose $\Phi_1 = (\phi_1, \psi_1, \varphi_1)^T$ is a solution of the Lax pair (4) and (5) when $\lambda = \lambda_1$, and denote $\beta_1 = \psi_1/\phi_1$ and $\gamma_1 = \varphi_1/\phi_1$. Assume the quantities a, b, c, d, f, g satisfy

$$\begin{pmatrix} \lambda_1 + d & a & b \\ a & \lambda_1 + g & c \\ b & c & \lambda_1 + f \end{pmatrix} \begin{pmatrix} 1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0, \tag{12}$$

or equivalently

$$\begin{aligned} -\tfrac{bc}{a} - \tfrac{ac}{b} + a\beta_1 + b\gamma_1 &= 0,\\ a + (2\lambda_1 + \tfrac{ac}{b})\beta_1 + c\gamma_1 &= 0,\\ b + c\beta_1 + (2\lambda_1 + \tfrac{bc}{a})\gamma_1 &= 0. \end{aligned} \tag{13}$$

Computing Equation (13), the expressions of a, b, c are obtained,

$$a = \frac{-2\lambda_1\beta_1}{1+\beta_1^2+\gamma_1^2}, \quad b = \frac{-2\lambda_1\gamma_1}{1+\beta_1^2+\gamma_1^2}, \quad c = \frac{-2\lambda_1\beta_1\gamma_1}{1+\beta_1^2+\gamma_1^2}. \tag{14}$$

Substituting (14) into (9), the expressions of d, e, f can be written as

$$d = \lambda_1 + \frac{-2\lambda_1}{1+\beta_1^2+\gamma_1^2}, \quad g = \lambda_1 + \frac{-2\lambda_1\beta_1^2}{1+\beta_1^2+\gamma_1^2}, \quad f = \lambda_1 + \frac{-2\lambda_1\gamma_1^2}{1+\beta_1^2+\gamma_1^2}. \tag{15}$$

Theorem 1. *Let the following conditions be satisfied:*

1. *u, v, w, q, r is a known solution of the five-component Equation (1);*
2. *$\lambda_1 \in \mathbb{R}$ is a fixed constant; and*
3. *$\Phi_1 = (\phi_1, \psi_1, \varphi_1)^T$ is a nonzero solution of the Lax pair (4) and (5) when $\lambda = \lambda_1$.*

Denote $\beta_1 = \psi_1/\phi_1$ and $\gamma_1 = \varphi_1/\phi_1$. Then the new potentials $\hat{u}, \hat{v}, \hat{w}, \hat{q}, \hat{r}$ given by the Darboux transformation

$$\begin{aligned} \hat{u} =& u - a, \quad \hat{v} = v - b, \quad \hat{w} = w - 2c,\\ \hat{r} =& r - ub + va + (g-f)w + (f-g)c,\\ \hat{q} =& q - auw - bvw + bu_x + av_x - (u^2+v^2+w^2)c + (g-f)r + (bc-af)v\\ &+ (ac-bg)u + (2c^2+a^2+g^2-gf)w - 2c(a^2+b^2+d^2) \end{aligned} \tag{16}$$

is a new solution of the five-component Equation (1), and the corresponding Darboux matrix is T. In the above equations, the quantities a, b, c, d, f, g are given by (14) and (15).

Proof. First of all, we prove $\hat{U}T = T_x + TU$, where $\hat{U} = U|_{u=\hat{u}, v=\hat{v}}$. Denote

$$A = \begin{pmatrix} d & a & b \\ a & g & c \\ b & c & f \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & u & v \\ -u & 0 & 0 \\ -v & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $T = \lambda E + A$, $U = \lambda B + C$. Denote $\hat{C} = C|_{u=\hat{u},v=\hat{v}}$, then $\hat{U} = \lambda B + \hat{C}$. Comparing the coefficients of λ^j in $\hat{U}T = T_x + TU$ we have

$$\hat{C} + BA = C + AB, \tag{17}$$
$$\hat{C}A = A_x + AC. \tag{18}$$

Now we prove that expressions (17) and (18) are true.

A direct computation shows that

$$\hat{C} + BA = \begin{pmatrix} d & \hat{u}+a & \hat{v}+b \\ -\hat{u} & 0 & 0 \\ -\hat{v} & 0 & 0 \end{pmatrix}, \qquad C + AB = \begin{pmatrix} d & u & v \\ -u+a & 0 & 0 \\ -v+b & 0 & 0 \end{pmatrix}.$$

From the first two expressions of the Darboux transformation (16), it is easy to verify that expression (17) holds.

Next, we prove this expression (18) is valid. From Equation (9), it is immediate that

$$ga + cb + ad = 0, \quad ca + fb + bd = 0, \quad ab + cg + cf = 0. \tag{19}$$

From $\Phi_{1,x} = (U|_{\lambda=\lambda_1})\Phi_1$ we have the Riccati equations

$$\beta_{1,x} = -u - \beta_1\lambda_1 - u\beta_1^2 - v\beta_1\gamma_1, \tag{20}$$
$$\gamma_{1,x} = -v - \gamma_1\lambda_1 - u\beta_1\gamma_1 - v\gamma_1^2. \tag{21}$$

According to Equations (14) and (15), we derive the first derivatives of a, b, c, d, g, f with respect to x.

$$\begin{aligned} &a_x = (g-d)u + vc + ad, \quad b_x = cu + (f-d)v + bd, \quad c_x = -bu - av + ab, \\ &d_x = 2(au+bv) - a^2 - b^2, \quad g_x = -2au + a^2, \quad f_x = -2bv + b^2. \end{aligned} \tag{22}$$

Then, we arrive at

$$\begin{aligned} A_x + AC &= \begin{pmatrix} d_x - au - bv & a_x + du & b_x + dv \\ a_x - gu - cv & g_x + au & c_x + av \\ b_x - cu - fv & c_x + bu & f_x + bv \end{pmatrix} \\ &= \begin{pmatrix} 2(au+bv) - a^2 - b^2 - au - bv & (g-d)u + vc + ad + du & cu + (f-d)v + bd + dv \\ (g-d)u + vc + ad - gu - cv & -2au + a^2 + au & -bu - av + ab + av \\ cu + (f-d)v + bd - cu - fv & -bu - av + ab + bu & -2bv + b^2 + bv \end{pmatrix} \\ &= \begin{pmatrix} a(u-a) + b(v-b) & g(u-a) + c(v-b) & c(u-a) + f(v-b) \\ -d(u-a) & -a(u-a) & -b(u-a) \\ -d(v-b) & -a(v-b) & -b(v-b) \end{pmatrix} \\ &= \begin{pmatrix} a\hat{u} + b\hat{v} & g\hat{u} + c\hat{v} & c\hat{u} + f\hat{v} \\ -d\hat{u} & -a\hat{u} & -b\hat{u} \\ -d\hat{v} & -a\hat{v} & -b\hat{v} \end{pmatrix} = \hat{C}A. \end{aligned}$$

Hence, Equation (18) is true, and then $\hat{U}T = T_x + TU$ is valid.

In the second place, we prove $\hat{V}T = T_t + TV$, where $\hat{V} = V|_{u=\hat{u},v=\hat{v}}$. Denote

$$V_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & -u & v \\ u & 0 & w \\ -v & -w & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} -u^2+v^2 & vw-u_x & -uw+v_x \\ vw-u_x & u^2+\frac{1}{2}w^2 & r \\ -uw+v_x & r & -v^2-\frac{1}{2}w^2 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & -P & -Q \\ P & 0 & q \\ Q & -q & 0 \end{pmatrix},$$

so that $T = \lambda E + A$, and $V = \lambda^3 V_1 + \lambda^2 V_2 + \lambda V_3 + V_4$. Denote $\hat{V}_2 = V_2|_{u=\hat{u},v=\hat{v}}$, $\hat{V}_3 = V_3|_{u=\hat{u},v=\hat{v}}$, and $\hat{V}_4 = V_4|_{u=\hat{u},v=\hat{v}}$, then $\hat{V} = \lambda^3 V_1 + \lambda^2 \hat{V}_2 + \lambda \hat{V}_3 + \hat{V}_4$. Comparing the coefficients of λ^j in $\hat{V}T = T_t + TV$, we have

$$\hat{V}_2 + V_1 A = V_2 + AV_1, \tag{23}$$
$$\hat{V}_3 + \hat{V}_2 A = V_3 + AV_2, \tag{24}$$
$$\hat{V}_4 + \hat{V}_3 A = V_4 + AV_3, \tag{25}$$
$$\hat{V}_4 A = A_t + AV_4. \tag{26}$$

So we prove that expressions (23)–(26) are true.
It is easy to see that

$$\begin{aligned} \hat{V}_2 + V_1 A &= \begin{pmatrix} 0 & -\hat{u} & \hat{v} \\ \hat{u}+a & g & \hat{w}+c \\ -\hat{v}-b & -\hat{w}-c & -f \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(u-a) & v-b \\ (u-a)+a & g & (w-2c)+c \\ -(v-b)-b & -(w-2c)-c & -f \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u+a & v-b \\ u & g & w-c \\ -v & -w+c & -f \end{pmatrix} = V_2 + AV_1. \end{aligned}$$

So expression (23) holds.
A direct computation shows that

$$\hat{V}_3 + \hat{V}_2 A = \begin{pmatrix} -\hat{u}^2+\hat{v}^2-a\hat{u}+b\hat{v} & \hat{v}\hat{w}-\hat{u}_x-g\hat{u}+c\hat{v} & -\hat{u}\hat{w}+\hat{v}_x-c\hat{u}+f\hat{v} \\ \hat{v}\hat{w}-\hat{u}_x+d\hat{u}+b\hat{w} & \hat{u}^2+\frac{1}{2}\hat{w}^2+a\hat{u}+c\hat{w} & b\hat{u}+f\hat{w}+\hat{r} \\ -\hat{u}\hat{w}+\hat{v}_x-d\hat{v}-a\hat{w} & -a\hat{v}-g\hat{w}+\hat{r} & -\hat{v}^2-\frac{1}{2}\hat{w}^2-b\hat{v}-c\hat{w} \end{pmatrix}. \tag{27}$$

Substituting (16) and (22) into (27), we obtain

$$\begin{aligned} \hat{V}_3 + \hat{V}_2 A &= \begin{pmatrix} -u^2+v^2+au-bv & vw-u_x-du-bw & -uw+v_x+dv+aw \\ vw-u_x+gu-cv & u^2+\frac{1}{2}w^2-au-cw & av+gw+r \\ -uw+v_x+cu-fv & -bu-fw+r & -v^2-\frac{1}{2}w^2+bv+cw \end{pmatrix} \\ &= V_3 + AV_2. \end{aligned}$$

Therefore, the expression (24) is true.
From Equation (19), it is immediate that

$$a^2+d^2-f^2-c^2=0, \quad a^2+g^2-b^2-f^2=0, \quad c^2+g^2-b^2-d^2=0. \tag{28}$$

And the following expressions can be derived from Equation (6):

$$
\begin{aligned}
\hat{P} =& 2u^3 - 2au^2 + av^2 + uv^2 + \tfrac{1}{2}(u-a)w^2 + (g-d)wv - cwu + rv - br \\
&+ (ac - bg)w + (cd - ab)v + (2a^2 + d^2 + b^2 - gd)u + (d-g)u_x + (c-w)v_x \\
&+ u_{xx} - a(a^2 + b^2 + d^2), \\
\hat{Q} =& -2v^3 - u^2 v - bu^2 + 2bv^2 + \tfrac{1}{2}(b-v)w^2 + (d-f)wu + cwv + (u-a)r \\
&+ (af - bc)w + (fd - a^2 - 2b^2 - d^2)v + (ab - cd)u + (w-c)u_x + (f-d)v_x \\
&- v_{xx} + b(a^2 + b^2 + d^2).
\end{aligned} \tag{29}
$$

Because

$$
\hat{V}_4 + \hat{V}_3 A = \begin{pmatrix} (-\hat{u}^2 + \hat{v}^2)d + (\hat{v}\hat{w} - \hat{u}_x)a + (-\hat{u}\hat{w} + \hat{v}_x)b & (-\hat{u}^2 + \hat{v}^2)a + (\hat{v}\hat{w} - \hat{u}_x)g + (-\hat{u}\hat{w} + \hat{v}_x)c - \hat{P} & (-\hat{u}^2 + \hat{v}^2)b + (\hat{v}\hat{w} - \hat{u}_x)c + (-\hat{u}\hat{w} + \hat{v}_x)f - \hat{Q} \\ (\hat{v}\hat{w} - \hat{u}_x)d + (\hat{u}^2 + \frac{1}{2}\hat{w}^2)a + \hat{r}b + \hat{P} & (\hat{v}\hat{w} - \hat{u}_x)a + (\hat{u}^2 + \frac{1}{2}\hat{w}^2)g + \hat{r}c & (\hat{v}\hat{w} - \hat{u}_x)b + (\hat{u}^2 + \frac{1}{2}\hat{w}^2)c + \hat{r}f + \hat{q} \\ (-\hat{u}\hat{w} + \hat{v}_x)d + (-\hat{v}^2 - \frac{1}{2}\hat{w}^2)b + \hat{r}a + \hat{Q} & (-\hat{u}\hat{w} + \hat{v}_x)a + (-\hat{v}^2 - \frac{1}{2}\hat{w}^2)c + \hat{r}g - \hat{q} & (-\hat{u}\hat{w} + \hat{v}_x)b + (-\hat{v}^2 - \frac{1}{2}\hat{w}^2)f + \hat{r}c \end{pmatrix}, \tag{30}
$$

inserting (16), (22) and (29) into (30), we have

$$
\begin{aligned}
\hat{V}_4 + \hat{V}_3 A =& \begin{pmatrix} (-u^2 + v^2)d + (vw - u_x)a + (-uw + v_x)b & (vw - u_x)d + (u^2 + \frac{1}{2}w^2)a + br - P & (-uw + v_x)d + (-v^2 - \frac{1}{2}w^2)b + ar - Q \\ (-u^2 + v^2)a + (vw - u_x)g + (-uw + v_x)c + P & (vw - u_x)a + (u^2 + \frac{1}{2}w^2)g + cr & (-uw + v_x)a + (-v^2 - \frac{1}{2}w^2)c + q + rg \\ (-u^2 + v^2)b + (vw - u_x)c + (-uw + v_x)f + Q & (vw - u_x)b + (u^2 + \frac{1}{2}w^2)c + fr - q & (-uw + v_x)b + (-v^2 - \frac{1}{2}w^2)f + cr \end{pmatrix} \\
=& V_4 + AV_3.
\end{aligned}
$$

So the expression (25) is valid.

In what follows, we show that expression (26) holds. From direct calculations we get

$$
A_t + AV_4 = \begin{pmatrix} d_t + aP + bQ & a_t - dP - bq & b_t - dQ + aq \\ a_t + gP + cQ & g_t - aP - cq & c_t - aQ + gq \\ b_t + cP + fQ & c_t - bP - fq & f_t - bQ + cq \end{pmatrix}, \tag{31}
$$

$$
\hat{V}_4 A = \begin{pmatrix} -\hat{P}a - \hat{Q}b & -\hat{P}g - \hat{Q}c & -\hat{P}c - \hat{Q}f \\ \hat{P}d + \hat{q}b & \hat{P}a + \hat{q}c & \hat{P}b + \hat{q}f \\ \hat{Q}d - \hat{q}a & \hat{Q}a - \hat{q}g & \hat{Q}b - \hat{q}c \end{pmatrix}. \tag{32}
$$

From the equation $\Phi_{1,t} = (V|_{\lambda=\lambda_1})\Phi_1$, we arrive at

$$
\begin{aligned}
\beta_{1,t} =& \lambda_1^2 u + \lambda_1(vw - u_x) + P + [\lambda_1^3 + \lambda_1(2u^2 - v^2 + \tfrac{1}{2}w^2)]\beta_1 \\
&+ (\lambda_1^2 w + \lambda_1 r + q)\gamma_1 - [\lambda_1^2 v + \lambda_1(-uw + v_x) - Q]\beta_1\gamma_1 \\
&+ [\lambda_1^2 u - \lambda_1(vw - u_x) + P]\beta_1^2, \\
\gamma_{1,t} =& -\lambda_1^2 v + \lambda_1(-uw + v_x) + Q + (-\lambda_1^2 w + \lambda_1 r - q)\beta_1 \\
&+ [-\lambda_1^3 - \lambda_1(2v^2 - u^2 + \tfrac{1}{2}w^2)]\gamma_1 + [\lambda_1^2 u - \lambda_1(vw - u_x) + P]\beta_1\gamma_1 \\
&- [\lambda_1^2 v + \lambda_1(-uw + v_x) - Q]\gamma_1^2.
\end{aligned} \tag{33}
$$

From (14) and (15) and the equation above, we obtain the first derivatives of a, b, c, d, g, f with respect to t:

$$
\begin{aligned}
a_t =& 2(d-g)u^3 + 2cv^3 + (ag-ad)u^2 + (ad-bc)v^2 + \tfrac{1}{2}(d-g)uw^2 + \tfrac{1}{2}cvw^2 \\
&+ \tfrac{1}{2}(ag-bc)w^2 + (d-g)uv^2 + cu^2v - (ab+dc)wu + (dg-g^2-c^2)wv \\
&+ bq + (d-g)vr - cur + (bg+ac)r + b(a^2+c^2+g^2)w + (cb^2+cd^2-abf-abd)v \\
&+ (d-g)(a^2+c^2+g^2)u + (g^2+c^2-dg-cw)u_x + (ab+cd+gw-dw)v_x \\
&+ (d-g)u_{xx} + cv_{xx} + (ag-bc)(a^2+b^2+d^2), \\
b_t =& -2cu^3 + 2(f-d)v^3 + (ac-bd)u^2 + (bd-bf)v^2 - \tfrac{1}{2}cuw^2 + \tfrac{1}{2}(f-d)vw^2 \\
&+ \tfrac{1}{2}(ac-bf)w^2 + (f-d)u^2v - cuv^2 + (a^2+d^2-fb)wu + (ab+cd)wv \\
&- aq + (d-f)ur - cvr + (bc+af)r - a(b^2+c^2+f^2)w - (a^2+b^2+d^2)cu \\
&+ (f-d)(a^2+b^2+d^2)v + (wd-wf-ab-cd)u_x + (cw+fd-d^2-a^2)v_x \\
&- cu_{xx} + (f-d)v_{xx} + (ac-bf)(a^2+b^2+d^2), \\
c_t =& 2bu^3 - 2av^3 + (gc-ab)u^2 + (ab-cf)v^2 + \tfrac{1}{2}buw^2 - \tfrac{1}{2}avw^2 + \tfrac{1}{2}(gc-cf)w^2 \\
&- au^2v + buv^2 - (bc+af)wu + (ac+bg)wv + aru + brv + (gf-f^2-b^2)r \\
&+ (f-g)q + (f-g)(a^2+c^2+g^2)w - a(a^2+c^2+g^2)v + b(a^2+c^2+g^2)u \\
&+ (wa-ac-bg)u_x + (bc+af-bw)v_x + bu_{xx} - av_{xx} + (gc-cf)(a^2+b^2+d^2), \\
d_t =& -2aP - 2bQ + 2abr + 2b(a^2+b^2+d^2)v - 2a(a^2+b^2+d^2)u - 2bdwu + 2adwv \\
&- 2adu_x + 2bdv_x + (2a^2+b^2)u^2 - (a^2+2b^2)v^2 + \tfrac{1}{2}(a^2-b^2)w^2 \\
&+ (a^2-b^2)(a^2+b^2+d^2), \\
g_t =& 2aP + 2cq + 2cgr + 2a(a^2+c^2+g^2)u + 2c(a^2+c^2+g^2)w + 2agwv - 2agu_x \\
&+ 2acv_x - (2a^2+c^2)u^2 - (\tfrac{1}{2}a^2+c^2)w^2 + (a^2-c^2)v^2 - (a^2+2c^2)(a^2+c^2+g^2), \\
f_t =& 2bQ - 2cq + 2fcr - 2c(a^2+c^2+g^2)w - 2b(a^2+b^2+d^2)v - 2bfwu + 2bcwv \\
&+ 2bfv_x - 2bcu_x + (c^2-b^2)u^2 + (2b^2+c^2)v^2 + (c^2+\tfrac{1}{2}b^2)w^2 \\
&+ (b^2+2c^2)(a^2+b^2+d^2).
\end{aligned}
\tag{34}
$$

By calculating the Formulas (14) and (15), we can also deduce

$$
a^2+b^2+d^2 = a^2+c^2+g^2 = b^2+c^2+f^2 = \lambda_1^2. \tag{35}
$$

Substituting (34) and (6) into (31), and inserting (29) and (16) into (32), we derive $\hat{V}_4 A = A_t + AV_4$. Theorem 1 is proved. □

3. Exact Solutions

In this section, we derive some examples of exact solutions to (1) by using Darboux transform (16).
(**i**) Let the seed solution be $u = v = w = r = q = 0$, and we have

$$
U = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^3 & 0 \\ 0 & 0 & -\lambda^3 \end{pmatrix}.
$$

As a result, $\Phi_x = U\Phi$ and $\Phi_t = V\Phi$ have the general solution:

$$
\Phi = \alpha_1 \begin{pmatrix} e^{\lambda x} \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ e^{\lambda^3 t} \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ e^{-\lambda^3 t} \end{pmatrix},
$$

which means

$$\beta_1 = \frac{\alpha_2 e^{\lambda_1^3 t}}{\alpha_1 e^{\lambda_1 x}} = \frac{\alpha_2}{\alpha_1} e^{\lambda_1^3 t - \lambda_1 x}, \tag{36}$$

$$\gamma_1 = \frac{\alpha_3 e^{-\lambda_1^3 t}}{\alpha_1 e^{\lambda_1 x}} = \frac{\alpha_3}{\alpha_1} e^{-\lambda_1^3 t - \lambda_1 x}. \tag{37}$$

Without loss of generality, we set $\alpha_1 = 1$. Then Equations (14) and (15) imply

$$a = \frac{-2\alpha_2\lambda_1 e^{\lambda_1^3 t - \lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{38}$$

$$b = \frac{-2\alpha_3\lambda_1 e^{-\lambda_1^3 t - \lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{39}$$

$$c = \frac{-2\alpha_2\alpha_3\lambda_1 e^{-2\lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{40}$$

$$d = \lambda_1 + \frac{-2\lambda_1}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{41}$$

$$g = \lambda_1 + \frac{-2\alpha_2^2\lambda_1 e^{2\lambda_1^3 t - 2\lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{42}$$

$$f = \lambda_1 + \frac{-2\alpha_3^2\lambda_1 e^{-2\lambda_1^3 t - 2\lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}. \tag{43}$$

From Darboux transformation (16) we obtain an explicit solution $(\hat{u}, \hat{v}, \hat{w}, \hat{r}, \hat{q})$ for (1),

$$\hat{u} = \frac{2\alpha_2\lambda_1 e^{\lambda_1^3 t - \lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{44}$$

$$\hat{v} = \frac{2\alpha_3\lambda_1 e^{-\lambda_1^3 t - \lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{45}$$

$$\hat{w} = \frac{4\alpha_2\alpha_3\lambda_1 e^{-2\lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}, \tag{46}$$

$$\hat{r} = \frac{4\alpha_2\alpha_3\lambda_1^2 e^{-4\lambda_1 x}(\alpha_3^2 e^{-2\lambda_1^3 t} - \alpha_2^2 e^{2\lambda_1^3 t})}{(1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x})^2}, \tag{47}$$

$$\hat{q} = \frac{4\alpha_2\alpha_3\lambda_1^3 e^{-2\lambda_1 x}}{1 + \alpha_2^2 e^{2\lambda_1^3 t - 2\lambda_1 x} + \alpha_3^2 e^{-2\lambda_1^3 t - 2\lambda_1 x}}. \tag{48}$$

When $\lambda_1 = -1$, $\alpha_2 = e^{-1}$, $\alpha_3 = e$, the solution is illustrated in Figure 1.
(ii) Let $u = v = 0$, $w = 2$, $r = 0$, $q = 1$ and $\lambda_1 = 1$. When $\lambda = \lambda_1$ we have

$$U|_{\lambda=\lambda_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V|_{\lambda=\lambda_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{pmatrix}.$$

Then $\Phi_{1,x} = (U|_{\lambda=\lambda_1})\Phi_1$ and $\Phi_{1,t} = (V|_{\lambda=\lambda_1})\Phi_1$ have the general solution:

$$\Phi_1 = \alpha_1 \begin{pmatrix} e^x \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1+3t \\ -3t \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 3t \\ 1-3t \end{pmatrix}.$$

Letting $\alpha_1 = 1$, we have

$$\begin{aligned} \beta_1 &= \alpha_2(1+3t)e^{-x} + 3\alpha_3 t e^{-x}, \\ \gamma_1 &= -3\alpha_2 t e^{-x} + \alpha_3(1-3t)e^{-x}. \end{aligned} \tag{49}$$

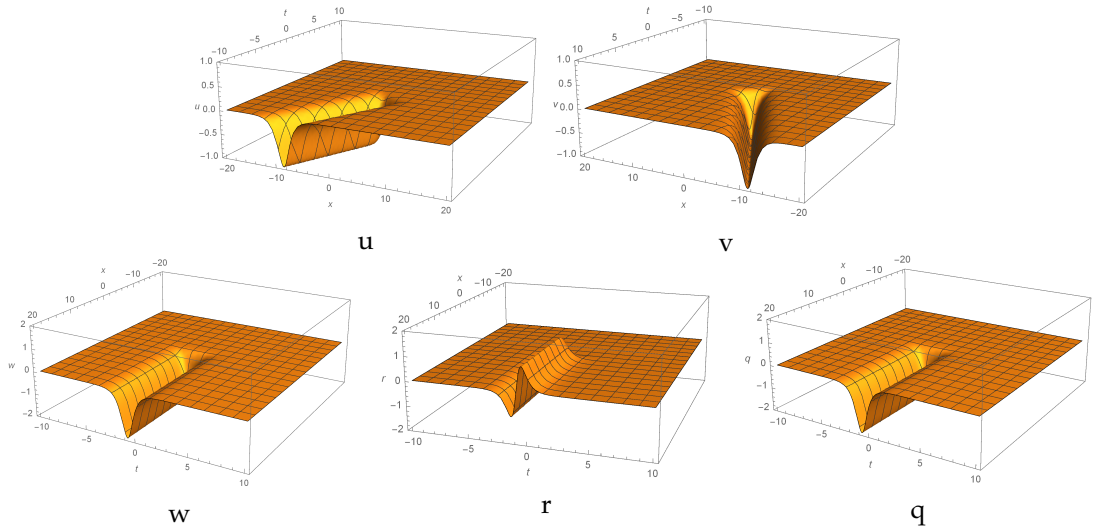

Figure 1. A soliton-like solution.

An explicit solutions of the soliton Equation (1) can be derived by solving Equations (14), (15) and Darboux transformation (16). The expressions are as follows:

$$\hat{u} = \frac{[2\alpha_2(1+3t) + 6\alpha_3 t]e^x}{e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2}, \tag{50}$$

$$\hat{v} = \frac{[-6\alpha_2 t + 2\alpha_3(1-3t)]e^x}{e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2}, \tag{51}$$

$$\hat{w} = 2 + \frac{-12\alpha_2^2(t+3t^2) + 4\alpha_2\alpha_3(1-18t^2) + 12\alpha_3^2(t-3t^2)}{e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2}, \tag{52}$$

$$\hat{r} = \frac{-4\alpha_2^2(1+6t) - 48\alpha_2\alpha_3 t - 4\alpha_3^2(6t-1)}{e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2} \tag{53}$$

$$+ \Big(\frac{12\alpha_2^2(t+3t^2) - 4\alpha_2\alpha_3(1-18t^2) - 12\alpha_3^2(t-3t^2)}{[e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2]^2} \tag{54}$$

$$\times [\alpha_2^2(1+6t) + 12\alpha_2\alpha_3 t + \alpha_3^2(6t-1)]\Big), \tag{55}$$

$$\hat{q} = 1 + \frac{4\alpha_2^2(1-3t-9t^2) + 12\alpha_2\alpha_3(1-6t^2) + 4\alpha_3^2(1+3t-9t^2)}{e^{2x} + \alpha_2^2(1+6t+18t^2) + \alpha_3^2(1-6t+18t^2) + 36\alpha_2\alpha_3 t^2}. \tag{56}$$

When $\alpha_2 = \alpha_3 = 1$, the solution is illustrated in Figure 2.

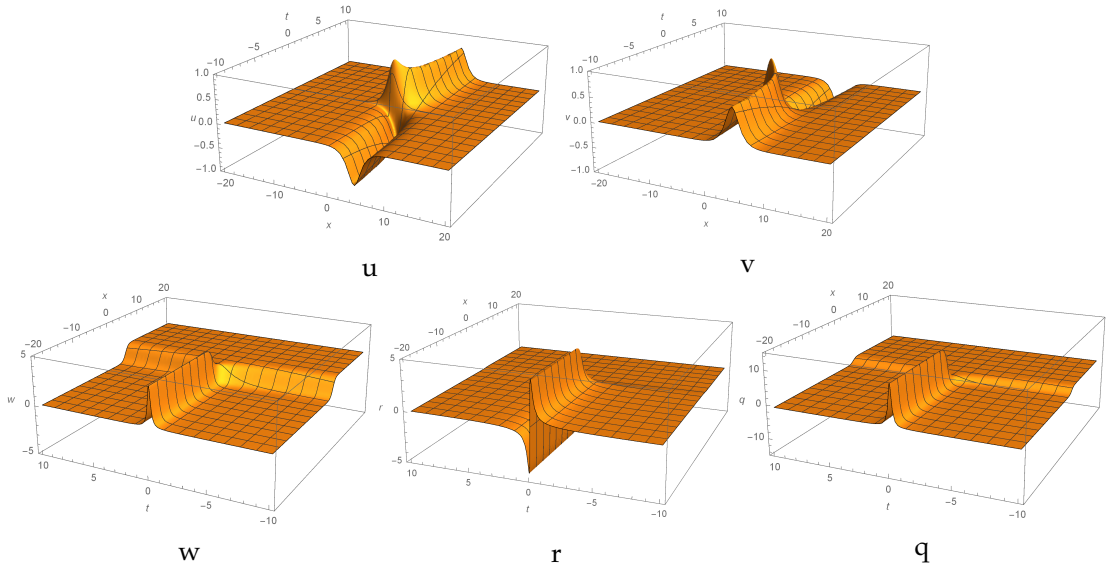

Figure 2. A solution that is a rational function of e^x and t.

4. Conclusions

In this paper, we propose a new five-component nonlinear integrable equation associated with a 3×3 spectral problem. This five-component equation has two different mKdV equations a reduction. By means of Darboux transformations between Lax pairs, we constructed a Darboux transformation for this equations. Finally, as applications of the Darboux transformations, we obtained exact solutions for (1) including a solution that consists of rational functions of e^x and t.

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Reference

1. Gu, C.H. *Soliton Theory and Its Application*; Zhejiang Publishing House of Science and Technology: Hangzhou, China, 1990.
2. Ablowitz, M.J.; Clarkson, P.A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*; Cambridge University Press: Cambridge, UK, 1991.
3. Zarkhrov, V.E.; Shabat, A.B. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering transform. *Funct. Anal. Appl.* **1974**, *8*, 226–235. [CrossRef]
4. Li, X.M.; Geng, X.G. Lax matrix and a generalized coupled KdV hierarchy. *Phys. Lett. A* **2003**, *327*, 357–370. [CrossRef]
5. Tu, G.Z. A new hierarchy of integrable systems and its Hamiltonian structures. *Sci. China Ser. A* **1989**, *2*, 142–153.
6. Tu, G.Z.; Meng, D.Z. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems(II). *Acta Math. Appl. Sin. (Engl. Ser.)* **1989**, *1*, 89–96. [CrossRef]
7. Tu, G.Z. A trace identity and its applications to the theory of discrete integrable systems. *J. Phys. A Math. Gen.* **1990**, *23*, 3903–3922.
8. Hirota, R.; Satsuma, J. Soliton solutions of a coupled Korteweg-de Vries equation. *Phys. Lett. A* **1981**, *85*, 407–408. [CrossRef]
9. Li, Y.S.; Ma, W.X.; Zhang, J.E. Darboux transformations of classical Boussinesq system and its new solutions. *Phys. Lett. A* **2000**, *275*, 60–66. [CrossRef]

10. Li, Y.S.; Zhang, J.E. Bidirectional soliton solutions of the classical Boussinesq system and AKNS system. *Chaos Solitons Fractals* **2003**, *16*, 271–277. [CrossRef]
11. Gu, C.H.; Hu, H.S. A unified explicit form of Bäcklund transformations for generalized hierarchies of KdV equations. *Lett. Math. Phys.* **1986**, *11*, 325–335.
12. Wang, M.L.; Wang, Y.M. A new Bäcklund transformation and multi-solutions to the KdV equation with general variable coefficients. *Phys. Lett. A* **2001**, *287*, 211–216. [CrossRef]
13. Rogers, C.; Schief, W.K. *Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2002.
14. Yu, J.; Chen, S.T.; Han, J.W.; Ma, W.X. N-fold Darboux Transformation for Integrable Couplings of AKNS Equations. *Commun. Theor. Phys.* **2018**, *69*, 367–374. [CrossRef]
15. Hirota, R. *The Direct Method in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2004.
16. Wei, J.; Geng, X.G.; Zeng, X. The Riemann theta function solutions for the hierarchy of Bogoyavlensky lattices. *Trans. Am. Math. Soc.* **2019**, *371*, 1483–1507. [CrossRef]
17. Geng, X.G.; Liu, H. The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation. *J. Nonlinear Sci.* **2018**, *28*, 739–763. [CrossRef]
18. Geng, X.G.; Tam, H.W. Darboux transformation and soliton solutions for generalized nonlinear Schrödinger equations. *J. Phys. Soc. Jpn.* **1999**, *68*, 1508–1512. [CrossRef]
19. Zhao, X.J.; Guo, R.; Hao, H.Q. N-fold Darboux transformation and discrete soliton solutions for the discrete Hirota equation. *Appl. Math. Lett.* **2018**, *75*, 114–120. [CrossRef]
20. Xue, B.; Wu, C.M. Conversation laws and Darboux transformation for Sharma-Tasso-Olver equation. *Commun. Theor. Phys.* **2012**, *58*, 317–322. [CrossRef]
21. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: New York, NY, USA, 1993.
22. Geng, X.G.; Wu, L.H. Darboux transformation and explicit solutions for Drinfel'd-Sokolov-Wilson equation. *Theor. Phys.* **2010**, *53*, 1090–1096.
23. Cao, J.N. Darboux transformations for differential-difference principal chiral equation and its continuous limits. *J. Math. Phys.* **2000**, *41*, 4687–4694.
24. Xue, B.; Wang, X. Darboux transformation and new explicit solutions for Belov-Chaltikion lattice. *Chin. Phys. Lett.* **2012**, *29*, 100201. [CrossRef]
25. Li, Y.S.; Zhang, J.E. Darboux transformations of classical Boussinesq system and its multi-soliton solutions. *Phys. Lett. A* **2001**, *284*, 253–258. [CrossRef]
26. Geng, X.G.; Li, R.M.; Xue, B. A vector general nonlinear Schrödinger equation with $(m+n)$-components. *J. Nonlinear Sci.* **2020** *30*, 991–1013. [CrossRef]
27. Li, R.M.; Geng, X.G. On a vector long wave-short wave-type model. *Stud. Appl. Math.* **2020** *144*, 164–184. [CrossRef]
28. Li, R.M.; Geng, X.G. Rogue periodic waves of the sine-Gordon equation. *Appl. Math. Lett.* **2020**, *102*, 106147. [CrossRef]
29. Chen, J.B.; Pelinovsky, D.E. Rogue periodic waves of the modified KdV equation. *Nonlinearity* **2017**, *31*, 1955–1980. [CrossRef]
30. Chen, J.B.; Pelinovsky, D.E. Rogue periodic waves of the focusing nonlinear Schrödinger equation. *Proc. R. Soc. Math. Phys. Eng. Sci.* **2018** *474*, 20170814. [CrossRef] [PubMed]
31. Triki, H.; Ismail, M.S. Solitary wave solutions for a coupled pair of mKdV equations. *Appl. Math. Comput.* **2010**, *217*, 1540–1548. [CrossRef]
32. Geng, X.G.; Zhai, Y.Y.; Dai, H.H. Algebro-geometric solutions of the coupled modified Korteweg-de Vries hierarchy. *Adv. Math.* **2014**, *263*, 123–153. [CrossRef]
33. Salkuyeh, D.K.; Bastani, M. Solution of the complex modified Korteweg-de Vries equation by the projected differential transform method. *Appl. Math. Comput.* **2013**, *219*, 5105–5112. [CrossRef]

34. Degasperis, A.; Lombardo, S.; Sommacal, M. Rogue Wave Type Solutions and Spectra of Coupled Nonlinear Schrödinger Equations. *Fluids* **2019**, *4*, 4010057. [CrossRef]
35. Ling, L.M.; Zhao, L.C.; Guo, B.L. Darboux transformation and classification of solution for mixed coupled nonlinear Schrödinger equations. *Commun. Nonlinear Sci.* **2016**, *32*, 285–304. [CrossRef]

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