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λ-Symmetry and μ-Symmetry Reductions and Invariant Solutions of Four Nonlinear Differential Equations

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Abstract: On one hand, we construct λ-symmetries and their corresponding integrating factors and invariant solutions for two kinds of ordinary differential equations. On the other hand, we present μ-symmetries for a (2+1)-dimensional diffusion equation and derive group-reductions of a first-order partial differential equation. A few specific group invariant solutions of those two partial differential equations are constructed.

Keywords: λ-symmetries; μ-symmetries; integrating factors; invariant solutions

1. Introduction

Lie symmetry method is a powerful technique which can be used to solve nonlinear differential equations algorithmically, and there are many such examples in mathematics, physics and engineering [1,2]. If an nth-order ordinary differential equation (ODE) that admits an n-dimensional solvable Lie algebra of symmetries, then a solution of the ODE, involving n arbitrary constants, can be constructed successfully by quadrature. If a partial differential equation (PDE) admits a Lie point symmetry, then its dimension can be reduced by one, and further its group invariant solution can be systematically constructed. However, there exist some kinds of differential equations which have trivial Lie point symmetries or have no symmetry, and Lie symmetry method cannot be applied directly. It is also known that the existence of nontrivial Lie point symmetries is not necessary for guaranteeing the integrability by quadrature for differential equations [3,4].

In 2001, a new kind of symmetries, called λ-symmetries, was introduced by Muriel and Romero [3]. Indeed, ODEs which have trivial Lie point symmetries or no symmetry but possess λ-symmetries can be integrated by means of the λ-symmetry approach. λ-symmetries can also be used to construct first integrals and integrating factors of such equations [5,6]. Gaeta and Morando considered the case of PDEs, and extended λ-symmetries to μ-symmetries [7,8]. It was proved that μ-symmetries are as useful as standard symmetries in respect to symmetry reductions, and the determination of invariant solutions by using μ-symmetries is completely similar to the standard one in the Lie symmetry method (see, for example, [9–16] for many other interesting applications and theoretical developments about λ- and μ-symmetries).

Both λ-symmetries and μ-symmetries are generalizations of Lie point symmetries, which could be viewed as Lie point symmetries of integrable couplings [17], and provide new insights into the development of the Lie symmetry theory. The determination of both symmetries depends on the prolongation formula that generalizes the standard Lie symmetry prolongation of vector fields. The most outstanding factor is that the determining equations are nonlinear, and so calculations are much more complicated. In this paper, we use the package of the differential characteristic set method [18,19] and symbolic computing systems to determine the existence of generalized symmetries and to simplify the corresponding determining equations. The differential characteristic set method, developed by Wentsun Wu [20] in the 1970s, is a fundamental algorithmic method, together with the Gröbner base algorithm. The method is very effective in calculating both classical and non-classical symmetries (for further applications, please refer to [21]).

This paper is structured as follows. In Section 2, we calculate λ-symmetries of two kinds of second-order ODEs and construct their integrating factors and invariant solutions by using the obtained λ-symmetries. In Section 3, we generate μ-symmetries of two different PDEs and construct some invariant solutions of the equations through applying the obtained μ-symmetries. In Section 4, we are devoted to providing some concluding remarks.

2. λ-Symmetries of Ordinary Differential Equations

2.1. The Basic Concept of λ-symmetries

Consider an n-th order ordinary differential equation (ODE)

$$\Delta(x, u^{(n)}) = 0, \tag{1}$$

where $(x, u^{(k)}) = (x, u, u_1, \cdots, u_k)$ and for $i = 1, ..., k$, u_i denotes the derivative of order i of the dependent variable u with respect to the independent variable x. The canonical form of this equation reads as follows

$$u^{(n)} = \Psi(x, u^{(n-1)}). \tag{2}$$

Recall [3] that if $v = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$ is a vector field on M, where M is an open subset of the independent and dependent variables, and λ is an arbitrary smooth function defined on the jet space $C^\infty(M^{(k)})$, then the λ-prolongation of order n of v, denoted by $v^{[\lambda,(n)]}$, is the vector field defined on $M^{(n)}$ by

$$v^{[\lambda,(n)]} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x, u^{(i)})\frac{\partial}{\partial u_i}, \tag{3}$$

where $\eta^{[\lambda,(0)]} = \eta(x,u)$ and

$$\begin{aligned} \eta^{[\lambda,(i)]}(x, u^{(i)}) = {} & D_x(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)})) - D_x(\xi(x,u))u_i \\ & + \lambda(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)}) - \xi(x,u)u_i), \end{aligned} \tag{4}$$

for $1 \le i \le n$, where total derivative $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \cdots$.

If there exists a function $\lambda \in C^\infty(M^{(k)})$ such that

$$v^{[\lambda,(n)]}(\triangle(x, u^{(n)}))|_{\triangle(x,u^{(n)})=0} = 0, \tag{5}$$

we will say that a vector field v, defined on M, is a λ-symmetry of the Equation (1). Obviously, if $\lambda = 0$, the λ-prolongation of order n of v is exactly the classical nth prolongation of v [1].

2.2. Applications of λ-Symmetries

2.2.1. λ-Symmetries Reductions and Integrating Factors without Using Lie Symmetries

Consider the following ordinary differential equation

$$u_{xx} = \frac{u_x^2}{u} + u_x(\frac{x}{u^3} + \frac{1}{x}) + \alpha x u, \alpha \in R. \tag{6}$$

We can use the differential characteristic set method [18] to determine that this equation has no Lie point symmetries easily.

Assume that λ-symmetry generator of Equation (6) is

$$v = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u},$$

and the second prolongation formula is of the form

$$v^{[\lambda,(2)]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta^{[\lambda,(1)]}\frac{\partial}{\partial u_x} + \eta^{[\lambda,(2)]}\frac{\partial}{\partial u_{xx}}.$$

From Equations (5), we know that v satisfies the following λ-symmetry condition:

$$v^{[\lambda,(2)]}[u_{xx} - \frac{u_x^2}{u} - u_x(\frac{x}{u^3} + \frac{1}{x}) - \alpha x u] \mid_{u_{xx} - \frac{u_x^2}{u} - u_x(\frac{x}{u^3} + \frac{1}{x}) - \alpha x u = 0} = 0. \tag{7}$$

The determining equation of (6) is

$$\eta^{[\lambda,(2)]} + \eta^{[\lambda,(1)]}(-\frac{2u_x}{u} - \frac{1}{x} - \frac{x}{u^3}) + \eta(\frac{u_x^2}{u^2} + \frac{3xu_x}{u^4} - \alpha x) + \xi(-u_x(\frac{1}{u^3} - \frac{1}{x^2}) - \alpha x) = 0, \tag{8}$$

where

$$\begin{aligned}
\eta^{[\lambda,(1)]} &= \eta_x + (\eta_u - \xi_x - \lambda\xi)u_x - \xi_u(u_x)^2 + \lambda\eta, \\
\eta^{[\lambda,(2)]} &= \eta_{xx} + \lambda_x\eta + 2\lambda\eta_x + \lambda^2\eta + (2\eta_{xu} - \xi_{xx} - 2\lambda\xi_x - \lambda_x\xi + \lambda_u\eta \\
&\quad + 2\lambda\eta_u - \lambda^2\xi)u_x + (\eta_u - 2\xi_x - 2\lambda\xi + \lambda_{u_x}\eta)u_{xx} + (\eta_{uu} - 2\xi_{xu} \\
&\quad - \lambda_u\xi - 2\lambda\xi_u)(u_x)^2 - (3\xi_u + \lambda_{u_x}\xi)u_x u_{xx} - \xi_{uu}(u_x)^3.
\end{aligned}$$

Substituting the above $\eta^{[\lambda,(1)]}, \eta^{[\lambda,(2)]}$ into the Equation (8), one can get a set of over-determined homogeneous differential equations for ξ, η

$$\left\{\begin{array}{l}
-\frac{\xi_u}{u} - \xi_{uu} - \frac{\xi\lambda_{u_x}}{u} = 0, \\
\frac{\eta}{u^2} - \frac{\eta_u}{u} - \frac{2\xi_u}{x} - \frac{2x\xi_u}{u^3} - 2\lambda\xi_u + \eta_{uu} - 2\xi_{xu} + \frac{\eta\lambda_{u_x}}{u} - \frac{\xi\lambda_{u_x}}{x} - \frac{x\xi\lambda_{u_x}}{u^3} - \xi\lambda_u = 0, \\
\frac{3x\eta}{u^4} - \frac{2\lambda\eta}{u} - \frac{\xi}{u^3} + \frac{\xi}{x^2} - \frac{\lambda\xi}{x} - \frac{x\lambda\xi}{u^3} - \lambda^2\xi + 2\lambda\eta_u - 3\alpha x u\xi_u - \frac{2\eta_x}{u} - \frac{\xi_x}{x} - \frac{x\xi_x}{u^3} \\
-2\lambda\xi_x + 2\eta_{xu} - \xi_{xx} + \frac{\eta\lambda_{u_x}}{x} + \frac{x\eta\lambda_{u_x}}{u^3} - \alpha x u\xi\lambda_{u_x} + \eta\lambda_u - \xi\lambda_x = 0, \\
-\alpha x\eta - \frac{\lambda\eta}{x} - \frac{x\lambda\eta}{u^3} + \lambda^2\eta - \alpha u\xi - 2\alpha x u\lambda\xi + \alpha x u\eta_u - \frac{\eta_x}{x} - \frac{x\eta_x}{u^3} + 2\lambda\eta_x \\
-2\alpha x u\xi_x + \eta_{xx} + \alpha x u\eta\lambda_{u_x} + \lambda_x\eta = 0.
\end{array}\right.$$

It can be checked that these equations, whose unknowns are ξ, η and λ, admit the solution $\xi = 0, \eta = u, \lambda = \frac{x}{u^3}$. Hence, if $\lambda = \frac{x}{u^3}$, the vector field $v = u\frac{\partial}{\partial u}$ is a λ-symmetry of Equation (6).

Now, we use the prolongation formula (4) to construct invariant solutions. We can determine $v^{[\lambda,(2)]}$ with $\lambda = \frac{x}{u^3}$ and obtain

$$v^{[\lambda,(2)]} = u\frac{\partial}{\partial u} + (u_x + \frac{x}{u^2})\frac{\partial}{\partial u_x} + (\frac{x^2}{u^5} + \frac{1}{u^2} + \frac{xu_x}{u^3} + u_{xx})\frac{\partial}{\partial u_{xx}}.$$

It can be checked that

$$y = x,\ w = \frac{u_x}{u} + \frac{x}{3u^3},$$

are two functionally independent invariants for $v^{[\lambda,(1)]}$.

Upon calculating an additional invariant by derivation [1]

$$w_y = \frac{D_x w}{D_x y} = \frac{u - 3xu_x - 3u^2u_x^2 + 3u^3u_{xx}}{3u^4},$$

Equation (6) can be reduced to the equation of y, w, w_y,

$$w_y - \frac{w}{y} - \alpha y = 0. \tag{9}$$

Solving (9), one can get

$$w = \alpha y^2 + c_1 y, c_1 \in R.$$

We recover the invariant solution of Equation (6) by solving the auxiliary first-order differential equation

$$3u^2u_x + x - 3\alpha x^2u^3 - 3c_1xu^3 = 0. \tag{10}$$

Let $\tilde{u} = u^3$. The equation of (10) turns into

$$\tilde{u}_x + x - 3\alpha x^2\tilde{u} - 3c_1x\tilde{u} = 0, \tag{11}$$

and by integrating this equation, we get the invariant solution of the Equation (6):

$$u = [\exp(\frac{3}{2}c_1x^2 + \alpha x^3)(c_2 - \int_1^x \exp(-\frac{3}{2}c_1p^2 - \alpha p^3)pdp)]^{\frac{1}{3}},$$

where c_1, c_2 are arbitrary constants.

Now we calculate first integrals of the Equation (6) by using method given in [5]. According to [5], if the equation admits a λ-symmetry: $v = \frac{\partial}{\partial u}$, then we can construct an integrating factor. From (4) in Section 2, we have

$$\begin{aligned} \eta^{[\lambda,(1)]} &= \lambda, \\ \eta^{[\lambda,(2)]} &= \lambda_x + \lambda^2 + \lambda_u u_x + \lambda_{u_x} u_{xx}. \end{aligned}$$

Substituting $\eta^{[\lambda,(1)]}, \eta^{[\lambda,(2)]}$ into the Equation (8), we have

$$\begin{aligned} &(u^3u_x^2 + \frac{u^4u_x}{x} + xuu_x + \alpha xu^5)\lambda_{u_x} + u^4u_x\lambda_u + u^4\lambda_x + u^4\lambda^2 \\ &- (2u^3u_x + \frac{u^4}{x} + xu)\lambda + u^2u_x^2 + 3xu_x - \alpha xu^4 = 0. \end{aligned} \tag{12}$$

For the sake of simplicity, the solution of λ is assumed to be $\lambda = \lambda_1(x,u)u_x + \lambda_2(x,u)$, and then the Equation (12) turns into

$$\begin{cases} \lambda_{1u}u^4 + \lambda_1^2u^4 - \lambda_1u^3 + u^2 = 0, \\ \lambda_{1x}u^4 + \lambda_{2u}u^4 + 2\lambda_1\lambda_2u^4 - 2\lambda_2u^3 + 3x = 0, \\ \lambda_{2x}u^4 + \lambda_2^2u^4 - \lambda_2xu - \frac{\lambda_2u^4}{x} + \alpha x\lambda_1u^5 - \alpha xu^4 = 0. \end{cases} \tag{13}$$

From the first equation of the system (13), we get a special solution of $\lambda_1(x,u) = \frac{1}{u}$, and the other equation becomes

$$\begin{cases} 3x + \lambda_{2u}u^4 = 0, \\ \lambda_{2x}u^4 + \lambda_2^2 u^4 - x\lambda_2 u - \frac{\lambda_2 u^4}{x} = 0. \end{cases} \tag{14}$$

From the first equation of (14), we get $\lambda_2(x,u) = \frac{x}{u^3} + c_1(x)$, and substituting it to the second equation, we get

$$(c_1'(x) + c_1^2(x) - \frac{c_x}{x})u^4 + xuc_1(x) = 0.$$

Taking $c_1(x) = 0$, we find that $v = \frac{\partial}{\partial u}$ is λ-symmetry for $\lambda = \frac{u_x}{u} + \frac{x}{u^3}$, and substituting $\lambda = \frac{u_x}{u} + \frac{x}{u^3}$ to $w_u + \lambda w_{u_x} = 0$, we get

$$w_u + (\frac{u_x}{u} + \frac{x}{u^3})w_{u_x} = 0. \tag{15}$$

Then the corresponding characteristic equation of the Equation (15) is

$$\frac{du}{1} = \frac{du_x}{\frac{u_x}{u} + \frac{x}{u^3}}.$$

So one can get a first integral of $v^{[\lambda,1]}$

$$w(x,u,u_x) = \frac{3u^2 u_x + x}{3u^3}. \tag{16}$$

Then upon calculating function $D[w]$

$$\begin{aligned} D[w] &= \frac{\partial w}{\partial x} + u_x \frac{\partial w}{u} + u_{xx}\frac{\partial w}{\partial u_x} \\ &= \frac{\partial w}{\partial x} + u_x \frac{\partial w}{u} + (\frac{u_x^2}{u} + u_x(\frac{x}{u^3} + \frac{1}{x}) + \alpha x u)\frac{\partial w}{\partial u_x}, \end{aligned} \tag{17}$$

substituting (16) into (17), and simplifying, the result turns into

$$F(x,w) = D[w] = \alpha x + \frac{1}{3u^3} + \frac{u_x}{x}\frac{1}{u} = \alpha x + \frac{w}{x}.$$

Now we calculate the first-order partial differential equation

$$G_x + (\alpha x + \frac{w}{x})G_w = 0. \tag{18}$$

Solving the corresponding characteristic equation of (18), we get a special solution

$$G(x,w) = \frac{w - \alpha x^2}{x}. \tag{19}$$

Substituting (16) into (19), we get the first integral

$$I(x,u,u_x) = \frac{u_x}{xu} + \frac{1}{3u^3} - \alpha x.$$

Therefore, from Theorem 1 in [5], the integrating factor of the Equation (6) is

$$\mu(x,u,u_x) = I_{u_x}(x,u,u_x) = \frac{1}{xu}.$$

2.2.2. λ-Symmetry Reductions and Integrating Factors Using Lie Symmetry

Consider the following ordinary differential equation

$$xu_{xx} - Au_x^3 + \frac{1}{2}u_x = 0, \tag{20}$$

where A is an arbitrary constant. The Lie symmetries of Equation (20) are

$$P_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \;\; P_2 = 2xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}, \;\; P_3 = \frac{\partial}{\partial u}. \tag{21}$$

Now we use the relationship between Lie point symmetries and λ-symmetres given in [3] to get λ-symmetries of Equation (20).

Let us consider P_1. Then we have

$$\xi_1 = x, \eta_1 = u,$$

and the characteristic function of P_1

$$Q_1 = \eta_1 - \xi_1 u_x = u - xu_x,$$

and the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + (-\frac{u_x}{2x} + \frac{Au_x^3}{x})\frac{\partial}{\partial u_x}.$$

The symmetry $v_1 = \frac{\partial}{\partial u}$ is the λ-symmetry [5] when

$$\lambda_1 = \frac{A(Q_1)}{Q_1} = \frac{u_x - 2Au_x^3}{2u - 2xu_x}.$$

Similarly, we consider P_2 and obtain

$$\lambda_2 = \frac{u_x(u - 2xu_x - 2Auu_x^2)}{u^2 - 2uxu_x}.$$

The above-mentioned (v_1, λ_1) and (v_2, λ_2) are not equivalent, owing to

$$\begin{vmatrix} 1 & u_x & \phi \\ \xi_1 & \eta_1 & (A+\lambda_1)(Q_1) + \xi_1\phi \\ \xi_2 & \eta_2 & (A+\lambda_2)(Q_2) + \xi_2\phi \end{vmatrix} = Q_1(A+\lambda_2)(Q_2) - Q_2(A+\lambda_1)(Q_1) \neq 0,$$

where $\phi = u_{xx} = -\frac{u_x}{2x} + \frac{Au_x^3}{x}$ and $Q_i = \eta_i - \xi_i u_x, i = 1, 2$.

Now we calculate a first integral from λ_1.

Firstly, substituting $\lambda_1 = \frac{u_x - 2Au_x^3}{2u - 2xu_x}$ into $w_u + \lambda w_{u_x} = 0$, we get

$$w_u + \frac{u_x - 2Au_x^3}{2u - 2xu_x} w_{u_x} = 0. \tag{22}$$

Integrating the characteristic equation of (22)

$$\frac{du}{1} = \frac{(2u - 2xu_x)du_x}{u_x - 2Au_x^3},$$

we get a special solution

$$w(x,u,u_x) = \frac{-u+2xu_x+2Auu_x^2}{u_x^2}. \tag{23}$$

Secondly, calculating function $D[w]$, one can get

$$\begin{aligned} D[w] &= \frac{\partial w}{\partial x} + u_x\frac{\partial w}{u} + (-\frac{u_x}{2x} + \frac{Au_x^3}{x})\frac{\partial w}{\partial u_x} \\ &= \frac{2Au}{x} - \frac{u}{xu_x^2} + \frac{2}{u_x} \\ &= \frac{w}{x} = F(x,w). \end{aligned}$$

Next, calculating the first-order partial differential equation

$$G_x + \frac{w}{x}G_w = 0$$

and solving the corresponding characteristic equation, we get a special solution

$$G(x,w) = \frac{w}{x}. \tag{24}$$

Finally, substituting (23) into (24), we get the first integral

$$I_1(x,u,u_x) = \frac{-u+2xu_x+2Auu_x^2}{xu_x^2}.$$

Similarly, we get a first integral from λ_2

$$I_2(x,u,u_x) = \frac{-2Au^2u_x^2+(u-2xu_x)^2}{2xu_x^2}.$$

In the following, we calculate an integrating factor from λ_1.

According to [5], we get

$$\mu_u + (\frac{u_x-2Au_x^3}{2u-2xu_x}\mu)_{u_x} = 0. \tag{25}$$

The corresponding characteristic equation is

$$\frac{du}{1} = \frac{du_x}{\frac{u_x-2Au_x^3}{2u-2xu_x}} = \frac{d\mu}{\frac{(u-6Auu_x^2+4Axu_x^3)\mu}{2(u-xu_x)^2}}.$$

So we get a special solution of the Equation (25)

$$\mu_1 = \frac{2u}{xu_x^3} - \frac{2}{u_x^2}.$$

So the above formula provides an integrating factor of the Equation (20).

Using the same procedure as above, we get another integrating factor from λ_2

$$\mu_2 = -\frac{u^2}{xu_x^3} + \frac{2u}{u_x^2}.$$

By using both of the first integrals I_1 and I_2, the invariant solution of the Equation (20) can be obtained. The resulting solution is

$$u(x) = \frac{2}{8A - I_1^2}(2I_1 I_2 \pm \sqrt{2(-8AxI_2 + xI_1^2 I_2 + 4AI_2^2}). \tag{26}$$

3. μ-Symmetries of Partial Differential Equations

3.1. The Basic Concept of μ-Symmetries

Let us consider the kth-order partial differential equation (PDE)

$$\Delta : F(x, u, u^{(1)}, u^{(2)}, \cdots u^{(k)}) = 0,$$

where $u = u(x) = u(x_1, x_2, \cdots, x_p)$ and $u^{(k)}$ represents all kth order derivatives of u with respect to x. We recall that M is vector space with the coordinates x and u, and M can be prolonged to the k-th jet bundle $(J^{(k)}M, \pi_k, B)$, with $J^0 M \equiv M$. We equip $(J^{(1)}M, \pi, B)$ with a distinguished semi-basic one-form μ [16],

$$\mu = \lambda_i dx_i.$$

We require that μ is compatible with the contact structure defined in $J^{(k)}M$, for $k \geq 2$, in the sense that

$$d\mu \in J(\varepsilon), \tag{27}$$

where $J(\varepsilon)$ is the Cartan ideal generated by ε. According to [16], condition (27) is equivalent to

$$D_i \lambda_j - D_j \lambda_i = 0. \tag{28}$$

Lemma 1 ([16]). *Let Y be a vector field on the jet space $J^{(k)}M$, written in coordinates as*

$$Y = X + \sum_{|J|=1}^{k} \psi_J \frac{\partial}{\partial u_J},$$

where $X = \xi^i \frac{\partial}{\partial x^i} + \varphi \frac{\partial}{\partial u}$ is a vector field on M. Let ε be the standard contact structure in $J^{(k)}M$, and $\mu = \lambda_i dx_i$ a semi-basic one-form on $(J^{(1)}M, \pi, B)$, compatible with ε. Then Y is the μ-prolongation of X if and only if its coefficients (with $\psi_0 = \varphi$) satisfy the μ-prolongation formula

$$\psi_{J,i} = (D_i + \lambda_i)\psi_j - u_{J,m}(D_i + \lambda_i)\xi^m.$$

Furthermore, if $Y : S \to TS, T \subset J^{(k-1)}M$, we say that X is a μ-symmetry for Δ, where $S \subset J^{(k)}M$ is the solution manifold for Δ. If Y leaves invariant each level manifold for Δ, we say that X is a strong μ-symmetry for Δ.

3.2. Applications of μ-Symmetries

3.2.1. An Example of $(2+1)$-Dimensional Equation

Let us consider the diffusion equation

$$u_t - u_x u_y - 2uu_{xy} = 0. \tag{29}$$

The Lie point symmetry of Equation (29) is

$$P = (-\frac{c_1}{2}x - c_2 t + c_4)\frac{\partial}{\partial x} + (-c_1 t + c_3)\frac{\partial}{\partial t} + (\frac{c_1}{2}y + c_2)\frac{\partial}{\partial y} + c_1 u \frac{\partial}{\partial u}.$$

Assume that a μ-symmetry of Equation (29) is

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial u}, \tag{30}$$

where ξ, τ, η, φ are function of x, y, t, u. With the ansatz $\mu = f(y)dy + g(t)dt + h(x)dx$, this guarantees that the compatibility condition (28) is satisfied. For the convenience of calculation, we take $h(x) = 0$.

The second prolongation is of the form

$$Y = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial u} + \psi^x\frac{\partial}{\partial u_x} + \psi^y\frac{\partial}{\partial u_y} + \psi^t\frac{\partial}{\partial u_t} + \psi^{xy}\frac{\partial}{\partial u_{xy}}.$$

Y satisfies the following μ-symmetry condition:

$$-2u_{xy}\varphi - u_y\psi^x - u_x\psi^y + \psi^t - 2u\psi^{xy}|_{u_t - u_x u_y - 2uu_{xy} = 0} = 0, \tag{31}$$

where

$$\begin{cases} \psi^x = (D_x + h(x))\varphi - u_x(D_x + h(x))\xi - u_y(D_x + h(x))\tau - u_t(D_x + h(x))\eta, \\ \psi^y = (D_y + f(y))\varphi - u_x(D_y + f(y))\xi - u_y(D_y + f(y))\tau - u_t(D_y + f(y))\eta, \\ \psi^t = (D_t + g(t))\varphi - u_x(D_t + g(t))\xi - u_y(D_t + g(t))\tau - u_t(D_t + g(t))\eta, \\ \psi^{xy} = (D_y + f(y))\psi^x - u_{xx}(D_y + f(y))\xi - u_{xy}(D_y + f(y))\tau - u_{xt}(D_y + f(y))\eta. \end{cases} \tag{32}$$

From (31) and (32), one can get an over-determined system for ξ, τ, η, φ:

$$\begin{cases} \xi_u = 0, \quad \xi_y + f(y)\xi = 0, \\ \tau_x = 0, \quad \tau_u = 0, \\ \eta_x = 0, \quad \eta_u = 0, \\ \eta_y + f(y)\eta = 0, \\ 2\varphi_u - \xi_x - \tau_y - f(y)\tau + 2u\varphi_{uu} = 0, \\ \tau_t + g(t)\tau + \varphi_x - 2u\varphi_{xu} = 0, \\ \xi_t + g(t)\xi + \varphi_y + f(y)\varphi + 2u\varphi_{uy} - 2u\xi_{xy} + 2uf(y)\varphi_u - 2uf(y)\xi_x = 0, \\ -\varphi_u + \eta_t + g(t)\eta = 0, \\ -\varphi_t - g(t)\varphi + 2u\varphi_{xy} + 2uf(y)\varphi_x = 0, \\ 2\varphi + 2u\varphi_u - 2u\xi_x - 2u\tau_y - 2uf(y)\tau = 0. \end{cases} \tag{33}$$

Calculating (33), we have

$$\begin{cases} \xi = [-f(y)e^{\int_1^t -g(k_1)dk_1}c_2 y - e^{\int_1^t -g(k_1)dk_1}c_2 \\ \qquad +2g(t)e^{\int_1^y -f(k_1)dk_1}c_1 t + 2e^{\int_1^y -f(k_1)dk_1}c_1]x + H(y,t), \\ \tau = e^{\int_1^t -g(k_1)dk_1}c_2 y, \\ \eta = e^{\int_1^y -f(k_1)dk_1}c_1 t, \\ \varphi = g(t)ue^{\int_1^y -f(k_1)dk_1}c_1 t + uc_1 e^{\int_1^y -f(k_1)dk_1}, \end{cases}$$

where c_1, c_2 are arbitrary constants, and $g(t), f(y), H(y,t)$ are arbitrary function which satisfy

$$\begin{cases} g^2(t)t + 2g(t) + g'(t)t = 0, \\ g(t)H(y,t) + H_t(y,t) = 0, \\ c_2 x e^{\int_1^t -g(k_1)dk_1}(f^2(y)y + 2f(y) + f'(y)y) + f(y)H(y,t) + H_y(y,t) = 0. \end{cases}$$

Let $\mu = -\frac{1}{y}dy$. When $f(y) = -\frac{1}{y}, g(t) = 0, H(y,t) = c_3 y$, then

$$X = (2c_1xy + c_3y)\frac{\partial}{\partial x} + c_2y\frac{\partial}{\partial y} + c_1yt\frac{\partial}{\partial t} + c_1yu\frac{\partial}{\partial u}$$

is a μ-symmetry. Specifically, letting $c_1 = 0, c_2 = c_3 = 1$, we have $X = y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Then the characteristic equation of X is

$$\frac{dx}{y} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{0}.$$

Solving the above system, we get the invariant $\theta = y - x$, $u = F(\theta)$. After substituting u into Equation (29), the original equation can be reduced to the ordinary differential equation as follows

$$F'^2 + 2FF'' = 0.$$

We obtain $F = (3\theta - 2a_1)^{\frac{2}{3}}a_2$ ($\theta = y - x$, a_1, a_2 are arbitrary constant). Therefore we have invariant solution $u = (3(y-x) - 2a_1)^{\frac{2}{3}}a_2$ (a_1, a_2 are arbitrary constant).

Let $c_1 = c_3 = 0, c_2 = 1$. Then $X = y\frac{\partial}{\partial y}$ and we have the invariant solution $u = p(x)$ ($p(x)$ is an arbitrary function of x).

Similarly, $\mu = -\frac{1}{t}dt$ when $g(t) = -\frac{1}{t}, f(y) = 0, H(y,t) = c_3 t$, then we have $X = (-c_2tx + c_3t)\frac{\partial}{\partial x} + c_2ty\frac{\partial}{\partial y} + c_1t\frac{\partial}{\partial t}$. Let $c_1 = c_3 = 0, c_2 = 1$. We obtain $X = -tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}$. The characteristic equation of X is

$$\frac{dx}{-tx} = \frac{dy}{ty} = \frac{dt}{0} = \frac{du}{0}.$$

We have the invariant $\theta = xy$, $u = F(\theta)$. Upon substituting u into Equation (29), then the original equation can be reduced to the ordinary differential equation as follows

$$\theta F'^2 + 2\theta FF'' + 2FF' = 0.$$

Solving this euqatin, we obtain $F = (3\log\theta + 2a_1)^{\frac{2}{3}}a_2$ ($\theta = xy$, a_1, a_2 are arbitrary constant). Therefore we have the invariant solution $u = (3\log(xy) + 2a_1)^{\frac{2}{3}}a_2$ (a_1, a_2 are arbitrary constant).

Let $c_1 = c_2 = 1, c_3 = 0$. We have $X = -tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}$, Thus, we obtain the invariant solution $u = F(\theta_1, \theta_2)$ ($\theta_1 = xy, \theta_2 = t - \ln y$), where F satisfies

$$F_{\theta_2} + F_{\theta_1}F_{\theta_2} - \theta_1 F_{\theta_1}^2 - 2\theta_1 FF_{\theta_1\theta_1} + 2FF_{\theta_1\theta_2} - 2FF_{\theta_1} = 0.$$

Solving this equation one finds that $F(\theta_1, \theta_2) = \frac{\theta_1}{3LambertW(\frac{1}{3}e^{-\theta_2 - d_1})}$, where d_1 is a constant and *LambertW* is a MAPLE function. Then we have

$$u = \frac{xy}{3LambertW(\frac{1}{3}e^{-t+\ln y - d_1})}.$$

$X = c_3yt\frac{\partial}{\partial x} + c_2ty\frac{\partial}{\partial y} + c_1yt\frac{\partial}{\partial t}$ is a μ-symmetry of $\mu = -\frac{1}{t}dt - \frac{1}{y}dy$ ($f(y) = -\frac{1}{y}, g(t) = -\frac{1}{t}$, $H(y,t) = c_3yt$). Specifically, when $c_1 = c_2 = c_3 = 1$, we have the invariant solution $u = F(\theta_1, \theta_2)$ ($\theta_1 = y - x, \theta_2 = t - y$), where F satisfies

$$F_{\theta_2} + F_{\theta_1}^2 - F_{\theta_1}F_{\theta_2} + 2FF_{\theta_1\theta_1} - 2FF_{\theta_1\theta_2} = 0.$$

The solution of this equation is $F(\theta_1, \theta_2) = \frac{(\theta_1+\theta_2)(d_2)^2 + (-\theta_1 + d_1)d_2 - d_1}{d_2 - 1}$, where d_1, d_2 are arbitrary constants. Then the invariant solution of Equation (29) is

$$u = \frac{(t-x)(d_2)^2 + (x - y + d_1)d_2 - d_1}{d_2 - 1}.$$

3.2.2. An Example of (1+1)-Dimensional Equation

We consider the following equation in (1+1)-dimensions:

$$2tu_t + xu_x - u = 0. \tag{34}$$

Assume that μ-symmetry of Equation (34) is

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial u}, \tag{35}$$

where ξ, τ, φ are function of x, t, u. With the ansatz $\alpha = \alpha(x,t,u), \beta = \beta(x,t,u)$, this should be complemented with the requirement that $D_x\beta = D_t\alpha$. Specially the ansatz $\mu = f(t)dt$ guarantees that the compatibility condition (28) is satisfied.

Proceeding as mentioned above, the determining equation for μ-symmetries of Equation (34) is split into the following system

$$\begin{cases} \tau_u = 0, \quad \xi_u = 0, \\ -\varphi + x\varphi_x + 2t\varphi_t + 2tf(t)\varphi = 0, \\ \xi + x\varphi_u - x\xi_x - 2t\xi_t - 2t\xi f(t) = 0, \\ 2\tau - x\tau_x + 2t\varphi_u - 2t\tau_t - 2tf(t)\tau = 0. \end{cases} \tag{36}$$

Specially, when $f(t) = \frac{1}{2t}$, we get several sets of solutions of the above system:

$$\begin{cases} \xi_1 = x, \;\; \tau_1 = 2t, \;\; \varphi_1 = u, \\ \xi_2 = \frac{x^2}{t}, \;\; \tau_2 = x \;\; \varphi_2 = \frac{x^2}{t}, \\ \xi_3 = \phi(\frac{x^2}{t}), \;\; \tau_3 = x, \;\; \varphi_3 = \phi(\frac{x^2}{t}), \\ \xi_4 = \frac{x^2}{t}, \;\; \tau_4 = \frac{x^3}{t}, \;\; \varphi_4 = \frac{x^2}{t}, \\ \xi_5 = \phi(\frac{x^2}{t}), \;\; \tau_5 = x\phi(\frac{x^2}{t}), \;\; \varphi_5 = \phi(\frac{x^2}{t}), \\ \xi_6 = \frac{t}{x^2}, \;\; \tau_6 = x, \;\; \varphi_6 = \frac{t}{x^2}, \\ \xi_7 = \phi(\frac{t}{x^2}), \;\; \tau_7 = x, \;\; \varphi_7 = \phi(\frac{t}{x^2}), \\ \xi_8 = \frac{t}{x^2}, \;\; \tau_8 = \frac{t}{x}, \;\; \varphi_8 = \frac{t}{x^2}, \\ \xi_9 = \phi(\frac{t}{x^2}), \;\; \tau_9 = x\phi(\frac{t}{x^2}), \;\; \varphi_9 = \phi(\frac{t}{x^2}). \end{cases}$$

Now we get invariant solutions by using μ-symmetries.

Consider $X = \frac{x^2}{t}\frac{\partial}{\partial x} + \frac{x^3}{t}\frac{\partial}{\partial t} + \frac{x^2}{t}\frac{\partial}{\partial u}$, and then the characteristic equation of X is

$$\frac{dx}{\frac{x^2}{t}} = \frac{dt}{\frac{x^3}{t}} = \frac{du}{\frac{x^2}{t}}.$$

From $\frac{dx}{\frac{x^2}{t}} = \frac{dt}{\frac{x^3}{t}}$, we get the invariant $\theta = x^2 - 2t$ (θ is a constant of integration). According to $\frac{dt}{\frac{x^3}{t}} = \frac{du}{\frac{x^2}{t}}$, we obtain $u = \frac{t}{x} + C$ (C is a constant of integration). Let $C = F(\theta) = F(x^2 - 2t)$. Then $u = \frac{t}{x} + F(x^2 - 2t)$. Substituting this u into Equation (34), we see that the original equation can be reduced to the ordinary differential equation as follows

$$2\theta F' - F = 0. \tag{37}$$

Solving (37), we obtain $F = c_1\theta^{\frac{1}{2}}$ ($\theta = x^2 - 2t$). Substituting $F = c_1\theta^{\frac{1}{2}}$ into $u = \frac{t}{x} + F(\theta)$, we finally obtain $u = \frac{t}{x} + c_1(x^2 - 2t)^{\frac{1}{2}}$ (c_1 is arbitrary constant).

For $X = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$, we have $u = c_2 t^{-\frac{1}{2}}x^2$ (c_2 is an arbitrary constant).

For $X = \frac{t}{x^2}\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + \frac{t}{x^2}\frac{\partial}{\partial u}$, we gain $u = \frac{t^2}{2x^3} + c_3(x^4 - 2t^2)^{\frac{1}{4}}$ (c_3 is an arbitrary constant).

4. Conclusions

λ-symmetries and μ-symmetries are both useful in establishing effective alternative methods to analyze nonlinear differential equations without using Lie point symmetries. In this paper, we presented four examples to illustrate the efficiency of λ-symmetries and μ-symmetries for analyzing nonlinear differential equations. The integrating factors and invariant solutions of two kinds of nonlinear ordinary differential equations were constructed by using λ-symmetries and different techniques. And using μ-symmetries, we found many satisfactory new invariant solutions of two types of nonlinear partial differential equations.

The main obstacle to determining λ-symmetries and μ-symmetries is to solve the nonlinear determining equations. At present, there is no general algorithm and package to solve this problem directly. Therefore, it is difficult to determine the general form of λ and μ. However, appropriate assumptions of λ and μ can simplify the difficult calculation, so that the existing algorithms and programs can be used and satisfactory results can be obtained. In this paper, we used the package of the differential characteristic set method and symbolic computing systems to determine the complicated work of existence of generalized symmetries and to reduce the corresponding determining equations. It is an open question to improve the efficiency of symmetry computations and any alternative advanced algorithm for computing μ-symmetry needs to be investigated. It is also interesting to see if μ-symmetries can be used to generate lump solutions, particularly with higher-order dispersion relations [22], or in the case of linear partial differential equations (see, e.g., [23]).

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References

1. Olver, P. J. *Applications of Lie Groups to Differential Equations*; Springer: Berlin, Germany, 1986.
2. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
3. Muriel, C.; Romero, J.L. New methods of reduction for ordinary differential equations. *IMA J. Appl. Math.* **2001**, *66*, 111–125. [CrossRef]
4. Muriel, C.; Romero, J.L.; Ruiz, A. λ-symmetries and integrability by quadratures. *IMA J. Appl. Math.* **2017**, *82*, 1061–1087. [CrossRef]
5. Muriel, C.; Romero, J.L. First integrals, integrating factors and λ-Symmetries of second-order differential equations. *J. Phys. A Math. Theor.* **2009**, *42*, 365207. [CrossRef]
6. Guha, P.; Choudhury, A.G.; Khanra, B. λ-symmetries, isochronicity, and integrating factors of nonlinear ordinary differential equations. *J. Eng. Math.*, **2013**, *82*, 85–99. [CrossRef]
7. Cicogna, G.; Gaeta, G.; Morando, P. On the relation between standard and μ-symmetries for PDEs. *J. Phys. A Math. Gen.* **2004**, *37*, 9467. [CrossRef]
8. Gaeta, G.; Morando, P. On the geometry of λ-symmetries and PDE reduction. *J. Phys. A Math. Gen.* **2004**, *37*, 6955. [CrossRef]
9. Ferraioli, D.C. Nonlocal aspects of λ-symmetries and ODEs reduction. *J. Phys. A Math. Theor.* **2007**, *40*, 5479. [CrossRef]
10. Muriela, C.; Romeroa, J.L.; Olver, P.J. Variational C^{∞}-symmetries and Euler–Lagrange equations. *J. Differ. Equ.* **2006**, *222*, 164 – 184. [CrossRef]
11. Muriel, C.; Romero, J.L. Integrating Factors and λ-symmetries. *J. Nonlinear Math. Phys.* **2008**, *15* (Suppl. 3), 300–309. [CrossRef]

12. Levi, D.; Rodriguez, M.A. λ-symmetries for discrete equations. *J. Phys. A Math. Theor.* **2010**, *43*, 1859–1860. [CrossRef]
13. Bhuvaneswari, A.; Kraenkel, R.A.; Senthilvelan, M. Application of the λ-symmetries approach and time independent integral of the modified Emden equation. *Nonlinear Anal. Real World Appl.* **2012**, *13*, 1102–1114. [CrossRef]
14. Ruiz, A.; Muriel, C. Applications of C^{∞}-symmetries in the construction of solvable structures. In *Trends in Differential Equations and Applications*; Springer: Cham, Switzerland, 2016; pp. 387–403.
15. Cicogna, G.; Gaet, G. Noether theorem for μ-symmetries. *J. Phys. A Math. Theor.* **2007**, *40*, 11899 [CrossRef]
16. Gaeta, G.A. gauge-theoretic description of μ-prolongations, and μ-symmetries of differential equations. *J. Geom. Phys.* 2009, 59 519–539. [CrossRef]
17. Ma, W.X. Integrable couplings and matrix loop algebras. In *Nonlinear and Modern Mathematical Physics*; Ma, W.X., Kaup, D., Eds.; American Institute of Physics: Melville, NY, USA, 2013; Volume 1562, pp. 105–122,
18. Temuer, C.; Bai Y. New algorithm for classical and nonclassical symmetry of a PDE based on Wus method. *Sci. Sin. Math.* **2010**, *40*, 331–348. (In Chinese)
19. Bluman, G.W. Conservation laws for nonlinear telegraph equations. *J. Math. Anal. Appl.* **2005**, *310*, 459–476. [CrossRef]
20. Wu, W.T. On the foundation of algebraic differential geometry. *J. Syst. Sci. Complex.* **1989**, *2*, 289–312.
21. Gao, X.S.; Wang, D.K.; Liao, Q.; Yang, H. *Equation Solving and Machine Proving C Problem Solving with MMP*; Science Press: Beijing, China, 2006. (In Chinese).
22. Ma, W.X.; Zhang, L.Q. Lump solutions with higher-order rational dispersion relations. *Pramana—J. Phys.* **2020**, *94*, 43. [CrossRef]
23. Ma, W.X. Lump and interaction solutions to linear PDEs in 2+1 dimensions via symbolic computation. *Mod. Phys. Lett. B* **2019**, *33*, 1950457. [CrossRef]