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Analysis of Perturbed Volterra Integral Equations on Time Scales

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Abstract: This paper describes the effect of perturbation of the kernel on the solutions of linear Volterra integral equations on time scales and proposes a new perspective for the stability analysis of numerical methods.

Keywords: Volterra integral equations; perturbation; stability; time scales

1. Introduction

In this paper, we consider Volterra Integral Equations (VIEs) on time scales of the type

$$x(t) = g(t) + \int_{t_0}^{t} k(t,s)x(s)\Delta s,\ t \in [t_0, +\infty)_{\mathbb{T}} = [t_0, +\infty) \cap \mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is a time scale that is a nonempty, closed subset of $\mathbb{R}$ in Equation (1), $t_0 \in \mathbb{T}$, and the integral sign is intended as a delta-integral (see Definition 4 in Section 2). We assume that the given real-valued functions $g(t)$ and $k(t,s)$ are defined in $[t_0, +\infty)_{\mathbb{T}}$ and $[t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}}$, respectively.

The theory of Volterra equations on time scales goes back to 2008 when, for the first time in [1], qualitative and quantitative results on the solutions were given. This laid the foundations for fruitful research and served as tools for continued works on VIEs. In addition, we refer to the book by Adivar et al. [2] and the references therein for a complete and extensive studies on recent results on the subject.

The study of integral equations in general stems from the study of existence and uniqueness of solutions of nonlinear differential equations. To see this, we consider the Δ-differential equation on time scale $\mathbb{T}$ as described above,

$$x^{\Delta}(t) = a(t)x(t) + g(t, x(t)),$$

where a and g are continuous on their respective domains. Integrating the above equation from t_0 to t yields the VIE on time scales

$$x(t) = x(t_0) + \int_{t_0}^{t} [a(s)x(s) + g(s, x(s))]\Delta s,\ t \in [t_0, +\infty)_{\mathbb{T}}.$$

For another important application to VIE, we look at the totally nonlinear delay dynamic equation

$$x^{\Delta}(t) = -a(t)g(x(\eta(t)))\eta^{\Delta}(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{2}$$

on a time scale $\mathbb{T}$, such that $\sup \mathbb{T} = \infty$. The delay function $\eta : [t_0, \infty)_{\mathbb{T}} \to [\eta(t_0), \infty)_{\mathbb{T}}$ is invertible, strictly increasing and Δ–differentiable, such that $t > \eta(t)$, $|\eta^{\Delta}(t)|$ is bounded for $t \in \mathbb{T}$, and $\eta(t_0) \in \mathbb{T}$. In addition, the functions a and g are *rd*-continuous (see Definition 5 in Section 2). Since the solution depends on a given initial function, we assume the existence of a *rd*-continuous function $\psi := [\eta(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}$, then $x(t) := x(t; t_0, \psi)$ is the solution of (2) if $x(t) = \psi(t)$ on $[\eta(t_0), t_0]_{\mathbb{T}}$ and satisfies (2) for all $t \geq t_0$. We notice that, under suitable conditions on the relevant coefficients, (see [2]), Equation (2) can be put in the form

$$x^{\Delta}(t) = -a\left(\eta^{-1}(t)\right) g(x(t)) + \left(\int_{\eta(t)}^{t} a\left(\eta^{-1}(s)\right) g(x(s))\Delta s\right)^{\Delta}. \tag{3}$$

If x is a solution of (3), then we have the VIE

$$\begin{aligned} x(t) = {} & e_{-a(\eta^{-1})}(t, t_0)\psi(t_0) + \int_{\eta(t)}^{t} a\left(\eta^{-1}(s)\right) g(x(s))\Delta s \\ & -e_{-a(\eta^{-1})}(t, t_0) \int_{\eta(t_0)}^{t_0} a\left(\eta^{-1}(s)\right) g(\psi(s))\Delta s \\ & - \int_{t_0}^{t} a\left(\eta^{-1}(s)\right) e_{-a(\eta^{-1})}(t, \sigma(s)) \left(\int_{\eta(s)}^{s} a\left(\eta^{-1}(u)\right) g(x(u))\Delta u\right) \Delta s \\ & + \int_{t_0}^{t} a\left(\eta^{-1}(s)\right) e_{-a(\eta^{-1})}(t, \sigma(s)) \left[x(s) - g(x(s))\right] \Delta s, \end{aligned} \tag{4}$$

where $e_{\alpha}(t, t_0)$ is the exponential function on time scales (see Definition 6 in Section 2). Note that (2) is totally nonlinear and the integral equation form of its solution given by (4) allows us to use fixed point theory and analyze the boundedness of solutions and the stability of its zero solution. A slight variation of (4) permits us to show the existence of a periodic solution. For more on the above discussion, we refer to [2].

From now on, we assume that, in Equation (1), the kernel $k : [t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ is continuous and the forcing function g is continuous on $[t_0, +\infty)_{\mathbb{T}}$. This research utilizes the asymptotic response of the solution of (1) and obtains results concerning the asymptotic response when the kernel is perturbed.

Since time scales calculus is now a well-established theory, we refer to the classical literature [3–6] for a comprehensive review. Moreover, in Section 2, we state some background material that is useful in this paper. The rest of the paper is organized as follows. In Section 3, we obtain some results on the asymptotic behavior of the solution to (1), which are essentially the generalization to time scales of two theorems proved in [7] for summation equations. In Section 4, the perturbed solution is written in terms of the unperturbed one through a new equation where x acts as a forcing term, so the error analysis is carried out by the definition of the resolvent related to the kernel of the new equation.

The use of time scales as a representation domain of mathematical problems allows unification of continuous and discrete domains plus other time sets on which some phenomena can be more realistically represented or defined. In this paper, our primary interest is to construct a single environment for a consistent analysis of the stability of (1) and, at the same time, of numerical methods for its resolution. Such numerical methods can be seen as integral Volterra equations on the time scale $h\mathbb{Z}$, where the new forcing function and kernel are still related to the known terms in (1). The connection between the problem on $\mathbb{R}$ and the problem on $h\mathbb{Z}$, will be emphasized in Section 5, while, in Section 6, a general overview on the applications of the theory proposed in this paper is discussed. Finally, some conclusions are drawn in Section 7.

2. Background Material

In this section, we recall some definitions and theorems that have been used in the paper (see [3,4,6] and the bibliography therein).

A time scale $\mathbb{T}$ is defined as an arbitrary closed and nonempty subset of $\mathbb{R}$. We assume here that $\mathbb{T}$ inherits the standard topology in $\mathbb{R}$.

Definition 1. *For all $t \in \mathbb{T}$ and $t < \sup \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by*

$$\sigma(t) = \inf \{\tau > t : \tau \in \mathbb{T}\},$$

and for $t > \inf \mathbb{T}$ the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) = \sup \{\tau < t : \tau \in \mathbb{T}\}.$$

The point $t \in \mathbb{T}$ is said to be right-scattered (resp. left-scattered) if $\sigma(t) > t$ (resp. $\rho(t) < t$). Furthermore, the point $t \in \mathbb{T}$ is said to be right-dense (resp. left-dense) if $\sigma(t) = t$ (resp. $\rho(t) = t$). A point $t \in \mathbb{T}$ that is simultaneously right and left-scattered is called isolated. The function $\mu : \mathbb{T} \to [0, +\infty)$, defined by $\mu(t) = \sigma(t) - t$, is the graininess of the time scale $\mathbb{T}$.

For the trivial examples of time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, we have that $\sigma(t) = t$, $\mu(t) = 0$, and $\sigma(t) = t + 1$, $\mu(t) = 1$, respectively.

Definition 2. *[8] A function $f : \mathbb{T} \to \mathbb{R}$ has a limit L at $t_0 \in \mathbb{T}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $t \in [t_0 - \delta, t_0 + \delta]$, then*

$$|f(t) - L| < \epsilon.$$

If t_0 is an isolated point, then $L = f(t_0)$. If the limit exists, we write

$$\lim_{t \to t_0} f(t) = L.$$

If $\mathbb{T}$ has left-scattered maximum, then we define $\mathbb{T}^k = \mathbb{T} - \max \mathbb{T}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 3. *Consider a function $f : \mathbb{T} \to \mathbb{R}$, and $t \in \mathbb{T}^k$. Then, define $f^\Delta(t)$ to be the number (if it exists) such that, given any $\epsilon > 0$, there is a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. $f^\Delta(t)$ is called the delta-derivative of f at t.

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$, the usual derivative, and, if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$, the forward difference operator.

Definition 4. *If $F^\Delta(t) = f(t)$, and $t,\ t_0 \in \mathbb{T}$, we define the delta-integral by*

$$\int_{t_0}^{t} f(s)\Delta s = F(t) - F(t_0).$$

If $\mathbb{T} = \mathbb{R}$, then $\int_{t_0}^{t} f(s)\Delta s$ corresponds to the Cauchy integral $\int_{t_0}^{t} f(s)ds$ and, if $\mathbb{T} = \mathbb{Z}$, then $\int_{t_0}^{t} f(s)\Delta s = \sum_{s=t_0}^{t-1} f(s)$.

Definition 5. *A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense (rd) continuous ($f \in C_{rd}(\mathbb{T}, \mathbb{R})$) if it is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists for every left-dense point $t \in \mathbb{T}$. Similarly, a function*

$f : \mathbb{T} \to \mathbb{R}$ is left-dense (ld) continuous ($f \in C_{ld}(\mathbb{T}, \mathbb{R})$) if it is continuous at every left-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^+} f(s)$ exists for every right-dense point $t \in \mathbb{T}$.

We remark that every *rd*-continuous function on $\mathbb{T}$ is delta-integrable on $\mathbb{T}$ (see, for example [9]) and that every continuous function on $\mathbb{T}$ is also *rd* and *ld*-continuous on $\mathbb{T}$.

Define the set of regressive functions as

$$\mathcal{R} = \{\alpha \in C_{rd} \text{ and } 1 + \alpha(t)\mu(t) \neq 0,\ \forall t \in \mathbb{T}\}.$$

Definition 6. *For $\alpha \in \mathcal{R}$, the exponential function $e_\alpha(t, t_0)$, $t \in \mathbb{T}$ is defined as the unique solution of the initial value problem*

$$x^\Delta = \alpha(t)x,\ x(t_0) = 1. \tag{5}$$

The explicit form of $e_\alpha(t, t_0)$ is given by

$$e_\alpha(t, t_0) = \begin{cases} \exp(\int_{t_0}^t \alpha(s)ds, & \text{if } \mu = 0 \\ \exp\left(\int_{t_0}^t \frac{\ln(1 + \mu(s)\alpha(s))}{\mu(s)} \Delta s\right), & \text{if } \mu > 0. \end{cases} \tag{6}$$

3. Asymptotics for Linear Equations

Consider Equation (1). The resolvent kernel $r_k(t, s)$ associated with $k(t, s)$ is defined as the solution of the following equation:

$$r_k(t, s) = k(t, s) + \int_{\sigma(s)}^t r_k(t, \tau)k(\tau, s)\Delta\tau, \tag{7}$$

where $\sigma(t)$ is the forward jump operator (see Definition 1 in Section 2). Then, the solution of the linear Equation (1) may be written in terms of g as follows:

$$x(t) = g(t) + \int_{t_0}^t r_k(t, s)g(s)\Delta s. \tag{8}$$

The resolvent $r_k(t, s)$ for the kernel $k(t, s)$ may be defined equivalently as the solution of the equation

$$r_k(t, s) = k(t, s) + \int_{\sigma(s)}^t k(t, \tau)r_k(\tau, s)\Delta\tau. \tag{9}$$

We refer the reader to [10] for sufficient conditions on the existence of the resolvent $r_k(t, s)$ when $k(t, s)$ is continuous on $[t_0, +\infty)_\mathbb{T} \times [t_0, +\infty)_\mathbb{T}$. For weaker conditions, we refer the reader to [11], in which existence is proven by asking just *rd*-continuity in both t and s.

Theorem 1. *Considering the linear integral Equation (1), let $k : [t_0, +\infty)_\mathbb{T} \times [t_0, +\infty)_\mathbb{T} \to \mathbb{R}$ be continuous in both variables and $g : [t_0, +\infty)_\mathbb{T} \to \mathbb{R}$ be continuous. Furthermore, assume that there exists $\bar{t} \in [t_0, +\infty)_\mathbb{T}$ and $R > 0$ such that*

$$\sup_{t \in [\bar{t}, +\infty)_\mathbb{T}} \int_{t_0}^t |r_k(t, s)|\Delta s < R, \tag{10}$$

then:
(i). if there exists a constant $\bar{g} > 0$ such that $\sup_{t \in [t_0, +\infty)_\mathbb{T}} |g(t)| \leq \bar{g}$, then there exists a constant $\bar{x} > 0$ such that

$$\sup_{t \in [t_0, +\infty)_\mathbb{T}} |x(t)| < \bar{x},$$

(ii). if $\lim_{t\to+\infty} g(t) = 0$*, and*

$$\lim_{t\to+\infty} r_k(t,s) = 0, \textit{ for all } s \in [t_0,t]_{\mathbb{T}}, \tag{11}$$

then

$$\lim_{t\to+\infty} x(t) = 0.$$

Proof.
(i). if there exists a constant $\bar{g} > 0$ such that $\sup_{t\in[t_0,+\infty)_{\mathbb{T}}} |g(t)| \leq \bar{g}$, then, from (8), we have

$$|x(t)| \leq \bar{g}\left(1 + \int_{t_0}^{t} |r_k(t,s)|\Delta s\right) \leq \bar{g}(1+R),$$

where the last inequality holds for any $t \in [\bar{t},+\infty)_{\mathbb{T}}$. Since $k(t,s)$ is continuous in t and s, then it is bounded for $(t,s) \in [t_0,\bar{t}]_{\mathbb{T}} \times [t_0,\bar{t}]_{\mathbb{T}}$, and therefore $x(t)$ is also bounded (see [1,12]) by a positive constant $\bar{\bar{x}}$, then $|x(t)| \leq \bar{x} = \max\{\bar{\bar{x}}, \bar{g}(1+R)\}$.
(ii). Since $g(t)$ vanishes at infinity, let $\epsilon > 0$ and $T > \bar{t}$ such that $|g(t)| < \frac{\epsilon}{2(1+R)}$ for each $t > T$. Then, we write (8) as

$$x(t) = g(t) + \int_{t_0}^{T} r_k(t,s)g(s)\Delta s + \int_{T}^{t} r_k(t,s)g(s)\Delta s,$$

and hence

$$|x(t)| \leq \frac{\epsilon}{2} + \int_{t_0}^{T} |r_k(t,s)||g(s)|\Delta s.$$

Because of (11), there exists $\nu > 0$ such that $|r_k(t,s)| < \frac{\epsilon}{2\bar{g}T}$, for $t > \nu$. Thus, consider $t > \max\{\bar{t},\nu\}$, and the result follows straightforwardly. □

Some assumptions on $k(t,s)$, which assure that $r_k(t,s)$ satisfies (10) and/or (11), are given, for example, in [13,14] when $\mathbb{T} = \mathbb{Z}$, and in [13,15] for $\mathbb{T} = \mathbb{R}$. A general result on time scales can be found in [12].

When $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ and the kernel k of Equation (1) is of convolution type, assumption (10) states the summability of the resolvent r_k. In this case, for any bounded function ψ on $[t_0,+\infty)_{\mathbb{T}}$ and for any $T \in [t_0,+\infty)_{\mathbb{T}}$, we have that $\int_{t_0}^{T} |r_k(\delta_-(t,\sigma(s))||\psi(s)|\Delta s$ vanishes (see Section 3.2 for the definition of shift operator δ_-). This implies that assumption (11) is not necessary anymore to prove that $x(t)$ vanishes.

4. Linear Perturbed Equations

In this section, we investigate stability of the solution of Equation (1). Assume a continuous perturbation $p(t,s)$ of the kernel $k(t,s)$, and then consider the perturbed equation

$$\tilde{x}(t) = g(t) + \int_{t_0}^{t} (k(t,s) + p(t,s))\tilde{x}(s)\Delta s. \tag{12}$$

Then, the solution can be rewritten as

$$\tilde{x}(t) = x(t) + \int_{t_0}^{t} a(t,s)\tilde{x}(s)\Delta s, \tag{13}$$

with

$$a(t,s) = p(t,s) + \int_{\sigma(s)}^{t} r_k(t,\tau)p(\tau,s)\Delta\tau. \tag{14}$$

Hence,

$$\tilde{x}(t) = x(t) + \int_{t_0}^{t} \tilde{r}(t,s)x(s)\Delta s,$$

with $\tilde{r}(t,s)$ satisfying

$$\tilde{r}(t,s) = a(t,s) + \int_{\sigma(s)}^{t} \tilde{r}(t,\tau)a(\tau,s)\Delta\tau. \tag{15}$$

Thus, $\tilde{r}$ is the resolvent corresponding to the kernel a of Equation (13). In order to prove (13), consider Equation (12) written in the form

$$\tilde{x}(t) = G(t) + \int_{t_0}^{t} k(t,s)\tilde{x}(s)\Delta s,$$

with $G(t) = g(t) + \int_{t_0}^{t} p(t,s)\tilde{x}(t)\Delta s$. Since $r_k(t,s)$ is the resolvent corresponding to the kernel $k(t,s)$, we write the solution $\tilde{x}(t)$ of the equation above in terms of $G(t)$ as

$$\begin{aligned}
\tilde{x}(t) =& G(t) + \int_{t_0}^{t} r_k(t,s)G(s)\Delta s \\
=& g(t) + \int_{t_0}^{t} p(t,s)\tilde{x}(s)\Delta s + \int_{t_0}^{t} r_k(t,s)\left(g(s) + \int_{t_0}^{s} p(s,u)\tilde{x}(u)\Delta u\right)\Delta s \\
=& g(t) + \int_{t_0}^{t} r_k(t,s)g(s)\Delta s + \int_{t_0}^{t} p(t,s)\tilde{x}(s)\Delta s \\
&+ \int_{t_0}^{t}\left(r_k(t,s)\int_{t_0}^{s} p(s,u)\tilde{x}(u)\Delta u\right)\Delta s.
\end{aligned}$$

From (8), we obtain (13), which relates the perturbed solution of Equation (1) to the unperturbed one.

The resolvent $\tilde{r}(t,s)$ associated with the kernel $a(t,s)$ defined in (14) satisfies (15). For more on this, we refer the reader to [10], and its relation with the kernel k, the resolvent r_k and the perturbation p is evident by the following equation:

$$\begin{aligned}
\tilde{r}(t,s) = p(t,s) &+ \int_{\sigma(s)}^{t} r_k(t,\tau)p(\tau,s)\Delta\tau \\
&+ \int_{\sigma(s)}^{t}\left(p(t,\tau) + \int_{\sigma(\tau)}^{t} r_k(t,u)p(u,s)\Delta u\right)\tilde{r}(\tau,s)\Delta\tau.
\end{aligned} \tag{16}$$

The dependence of the perturbed equation on $x(t)$, highlighted in (13)–(14), suggests that, in order to obtain a coherent and reasonable behavior of the two solutions, it is necessary to make some hypotheses on the known function $p(t,s)$ and the resolvent $r_k(t,s)$ related to the unperturbed Equation (1).

4.1. Stability

Theorem 2. *Consider the linear integral Equation (1), let $k : [t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous in both variables and $g : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous. Furthermore, assume that, for the resolvent $r_k(t,s)$, defined by Equation (7), hypotheses (10) and (11) hold and that*

$$\int_{t_0}^{t} |p(t,s)|\Delta s \le P(t), \text{ with } \lim_{t\to+\infty} P(t) = 0, \tag{17}$$

where $P(t)$ is a continuous function on $[t_0, +\infty)_{\mathbb{T}}$. Then, for the perturbed solution defined by Equations (13) and (14), it holds:
(i). if there exists a constant $\bar{x}$ such that $\sup_{t\in[t_0,+\infty)_{\mathbb{T}}} |x(t)| \le \bar{x}$, then there exists $X >$ such that

$$\sup_{t\in[t_0,+\infty)_{\mathbb{T}}} |\tilde{x}(t)| < X,$$

(ii). if $\lim_{t\to+\infty} x(t) = 0$, then

$$\lim_{t\to+\infty} \tilde{x}(t) = 0.$$

Proof.
(i). From (14), by changing the order of integration (see for example ([10] Lem.2.1)), it is obvious that

$$\int_{t_0}^{t} |a(t,s)|\Delta s \le |P(t)| + \int_{t_0}^{t} |r_k(t,s)||P(s)|\Delta s.$$

Thus, for $t > \bar{t}$,

$$\int_{t_0}^{t} |a(t,s)|\Delta s \le |P(t)| + \int_{t_0}^{\bar{t}} |r_k(t,s)||P(s)|\Delta s + \int_{\bar{t}}^{t} |r_k(t,s)||P(s)|\Delta s.$$

Since $P(t)$ vanishes for $t \to +\infty$, then, let $\epsilon > 0$, there exists $\nu_1 > 0$ such that $|P(t)| < \epsilon$, for $t > \nu_1$. Furthermore, because of assumption (11), which is $r_k(t,s) \to 0$, for $t \to +\infty$, it holds that for all $s \in [t_0, \bar{t})_{\mathbb{T}}$ there exists $\nu_2 > 0$ such that $|r_k(t,s)| < \epsilon$, for $t > \nu_2$. Considering $t > \bar{\bar{t}} = \max\{\bar{t}, \nu_1, \nu_2\}$ then

$$\int_{t_0}^{t} |a(t,s)|\Delta s \le \epsilon + P\epsilon + \epsilon R,$$

where $P = \max_{t\in[t_0,\bar{t}]_{\mathbb{T}}} P(t)$, which exists because $P(t)$ is continuous (see for example [16]). We arbitrarily choose $\epsilon = \frac{\alpha}{1+R+P}$, with $0 < \alpha < 1$. the boundedness of $\tilde{x}$ is implied by Theorem 7 in [12].
(ii). From part (i) of the proof we have that $\tilde{x}(t)$ is bounded and

$$\lim_{t\to+\infty} \int_{t_0}^{t} |a(t,s)|\Delta s = 0, \tag{18}$$

and thus, letting $\epsilon > 0$, there exists a constant $T > 0$ such that $\int_{t_0}^{t} |a(t,s)|\Delta s \le \epsilon < 1$, for $t > T$. From (12),

$$|\tilde{x}(t)| \le |x(t)| + \int_{t_0}^{T} |a(t,s)||\tilde{x}(s)|\Delta s + \int_{T}^{t} |a(t,s)||\tilde{x}(s)|\Delta s.$$

Then,

$$\begin{aligned} \limsup_{t\to+\infty} |\tilde{x}(t)| \le \lim_{t\to+\infty} |x(t)| + \sup_{t_0\le t\le T} |\tilde{x}(t)| \lim_{t\to+\infty} \int_{t_0}^{T} |a(t,s)| \\ + \sup_{t\ge T} |\tilde{x}(t)| \sup_{t\ge T} \int_{t_0}^{t} |a(t,s)|\Delta s. \end{aligned} \tag{19}$$

For $t > T$, $\int_{t_0}^{T} |a(t,s)|\Delta s < \int_{t_0}^{t} |a(t,s)|\Delta s$, and then $\lim_{t\to+\infty} \int_{t_0}^{T} |a(t,s)|\Delta s = 0$.
Passing to the limit as $T \to +\infty$, in (19), we arrive at

$$\limsup_{t\to+\infty} |\tilde{x}(t)| \le \frac{1}{1-\epsilon} \lim_{t\to+\infty} |x(t)|.$$

By the assumption that $x(t)$ tends to zero, we have completed this proof. □

Part (ii) of Theorem 2 extends to the case $\lim_{t\to+\infty} x(t) = x_\infty$, as follows:

Corollary 1. *Considering the linear integral Equation (1), let $k : [t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous in both variables and $g : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous. Furthermore, assume that, for the resolvent $r_k(t,s)$, associated with Equation (1), assumptions (10) and (11) hold. Let*

$$\lim_{t\to+\infty} x(t) = x_\infty. \tag{20}$$

Then, for the perturbed solution defined by Equations (13) and (14), with $P(t)$ satisfying (17), we have that

$$\lim_{t\to+\infty} \tilde{x}(t) = \tilde{x}_\infty = x_\infty.$$

Proof. This behavior is clear by applying Theorem 2 to Equation (13), which, in view of (18) and (20), can be rewritten as

$$\tilde{v}(t) = v(t) + \int_{t_0}^{t} a(t,s)\tilde{v}(s)\Delta s,$$

with $\tilde{v}(t) = \tilde{x}(t) - \tilde{x}_\infty$ and $v(t) = x(t) - x_\infty + x_\infty \int_{t_0}^{t} a(t,s)\Delta s$. □

Remark 1. *As we remarked in Section 3, when $\mathbb{T} = \mathbb{R}$ and when $\mathbb{T} = \mathbb{Z}$ and the kernel k of Equation (1) is of convolution type, assumption (11) is not necessary for the convergence of the perturbed solution. Thus, condition (10) states the summability of the resolvent corresponding to the convolution part of the kernel controls the stability of the system, a necessary and sufficient condition for $r_k(\delta_-(t,\sigma(s)))$ to be summable whenever $k(t)$ is summable is given by the Paley–Wiener results [17].*

4.2. Summability

Theorem 3. *Considering the linear integral Equation (1), let $k : [t_0, +\infty)_{\mathbb{T}} \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous in both variables and $g : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ be continuous. Assume that, for the resolvent $r_k(t,s)$, associated with Equation (1),*

$$\sup_{s\in[t_0,+\infty)_{\mathbb{T}}} \int_{s}^{+\infty} |r_k(t,s)|\Delta t \leq \bar{\bar{R}}. \tag{21}$$

In addition, let

$$\int_{\sigma(s)}^{+\infty} |p(t,s)|\Delta t \leq P(s), \text{ with } \lim_{s\to+\infty} P(s) = 0, \tag{22}$$

where $P(s)$ is a continuous function on $[t_0, +\infty)_{\mathbb{T}}$. Then, if $\int_{t_0}^{+\infty} |x(t)|\Delta t < +\infty$, the perturbed solution defined by Equations (13) and (14), satisfies

$$\int_{t_0}^{+\infty} |\tilde{x}(t)|\Delta t < +\infty.$$

Proof. For the kernel $a(t,s)$ of Equation (13) given in (14), we get

$$\int_{\sigma(s)}^{+\infty} |a(t,s)|\Delta t \leq \int_{\sigma(s)}^{+\infty} |p(t,s)\Delta t + \int_{\sigma(s)}^{+\infty} \int_{\sigma(s)}^{t} |r_k(t,\tau)||p(\tau,s)|\Delta\tau\Delta t,$$

and thus, interchanging the order of integration, for $s \geq \bar{s}$ sufficiently large, it results in

$$\int_{\sigma(s)}^{+\infty} |a(t,s)|\Delta t \leq \int_{\sigma(s)}^{+\infty} |p(t,s)|\Delta t + \int_{\sigma(s)}^{+\infty} \int_{\sigma(\tau)}^{+\infty} |r_k(t,\tau)||p(\tau,s)|\Delta t\Delta\tau \leq (1+\bar{\bar{R}})P(s) < \epsilon < 1.$$

By Equation (13), again interchanging the order of integrations, one has

$$\int_{t_0}^{+\infty} |\tilde{x}(t)|\Delta t \leq \int_{t_0}^{+\infty} |x(t)|\Delta t + \int_{t_0}^{\sigma(\bar{s})} \int_{\sigma(s)}^{+\infty} |a(t,s)||\tilde{x}(s)|\Delta t\Delta s + \int_{\sigma(\bar{s})}^{+\infty} \int_{\sigma(s)}^{+\infty} |a(t,s)||\tilde{x}(s)|\Delta t\Delta s.$$

Thus,

$$(1-\epsilon)\int_{t_0}^{+\infty} |\tilde{x}(t)|\Delta t \leq \int_{t_0}^{+\infty} |x(t)|\Delta t + C(\bar{s}),$$

with $C(\bar{s})$ being a positive constant bound for $\int_{t_0}^{\sigma(\bar{s})} \int_{\sigma(s)}^{+\infty} |a(t,s)||\tilde{x}(s)|\Delta t\Delta s$, which exists since, with $a(t,s)$ given by (14), assumptions (21) and (22) hold. □

5. Time Scale and Stability of Numerical Methods

One of the main advantages of time scale is that continuous and discrete problems can be analyzed within the same theoretical framework. This responds well to the needs of numerical analysis when addressing the problem of numerical stability. As a matter of fact, in these cases, one may want to identify a class of test equations and study the conditions for the analytical and numerical problems under which some characteristics of the solutions are preserved.

The continuous ($\mathbb{T} = \mathbb{R}$) version of problem (1) is the following Volterra integral equation:

$$x(t) = g(t) + \int_{t_0}^{t} k(t,s)x(s)ds, \quad t \geq t_0, \tag{23}$$

and we refer to [18] and the bibliography therein for a comprehensive account of theory development and applications. For $\mathbb{T} = \mathbb{Z}$, the resulting Volterra summation equation reads

$$x(t) = g(t) + \sum_{s=t_0}^{t-1} k(t,s)x(s), \quad t \geq t_0, \tag{24}$$

whose analysis has been the subject of great interest over the years (see, for example [7,14,19–21]) due to its importance in some epidemic models (see e.g. [19]), in some engineering applications (see, for example [22]) and, above all, for its direct connection with numerical methods for (23).

Let $x_0 = x(t_0)$, and $t_n = t_0 + nh$, for $n = 0, 1, \ldots$, be the time step with mesh size $h > 0$. Then, a n_0-step ($n_0 \geq 1$) (ρ, σ)-method for the approximation of (23) reads

$$x_n = g(t_n) + h\sum_{j=0}^{n_0-1} w_{nj}k(t_n,t_j)x_j + h\sum_{j=n_0}^{n} \omega_{n-j}k(t_n,t_j)x_j, \tag{25}$$

where $x_1, ..., x_{n_0-1}$ are given starting values and $x_n \approx x(t_n)$ for $n \geq n_0$. Regarding the weights w_{nj} and ω_n, we refer to [23] Section 2.6.

Equation (25) can be written as

$$X(t) = G(t) + h\sum_{s=t_0}^{t-1} K(t,s)X(s), \quad t \in [t_0, +\infty] \cap h\mathbb{Z}. \tag{26}$$

Here,

$$\begin{aligned} G(t) &= \frac{g(t) + h\sum_{j=0}^{n_0-1} w_{nj}k(t,t_j)x_j}{1 - h\omega_0 k(t,t)}, \\ K(t,s) &= \frac{\omega_{n-j}k(t,s)}{1 - h\omega_0 k(t,t)}, \end{aligned} \tag{27}$$

where $n = t/h$ and $n - j = (t - s)/h$. Thus, a numerical (ρ, σ) method for (23) corresponds to Equation (26) on the time scale $h\mathbb{Z}$, where the forcing G and the kernel K are linked to their continuous counterparts by (27).

All the results of this paper allow us a contextual discussion on the asymptotic properties of the analytical solution to Volterra integral equations and of the approximate one, supposing that the characteristics of the known terms k and g of (23) are inherited by K and G in (26). This is not obvious and is of course one of the main concerns when dealing with the stability of numerical methods. Some results, under additional assumptions on the regularity of the kernel and on the properties of the weights, can be found, for example, in [24,25]. Here, it is proved that, if

(a) $\sup_{t \geq s} |k(t,s)| < \infty, \forall t \geq s \geq 0,$

(b) $\sup\limits_{t \geq 0} \int_0^t |k(t,s)| ds \leq \alpha < 1$, and

(c) $\sup\limits_{t \geq 0} \int_0^t \left| \frac{\partial k(t,s)}{\partial s} \right| ds < +\infty,$

then, there exists a constant $A > 0$ such that

$$\sup_{n \geq n_0} h \sum_{j=n_0}^{n} \omega_{n-j} |k_{nj}| \leq \alpha + Ah.$$

Referring to the kernel in (27), we have that

$$h \sum_{j=n_0}^{n} |K(t_n, t_j)| = \frac{h \sum_{j=n_0}^{n-1} \omega_{n-j} |k_{nj}|}{|1 - h\omega_0 k_{nn}|},$$

which is still less than one. Indeed,

$$h \sum_{j=n_0}^{n-1} \omega_{n-j} |k_{nj}| = \sum_{j=n_0}^{n} \omega_{n-j} |k_{nj}| - h\omega_0 |k_{nn}| \leq \alpha + hA - h\omega_0 |k_{nn}|.$$

Then, when considering positive weights and a sufficiently small stepsize h, it is $h\omega_0 k_{nn} < 1$ and $|1 - h\omega_0 k_{nn}| \geq |1 - h\omega_0 |k_{nn}|| = 1 - h\omega_0 |k_{nn}|$, thus

$$\begin{aligned} \frac{h \sum_{j=n_0}^{n-1} \omega_{n-j} |k_{nj}|}{|1 - h\omega_0 k_{nn}|} &\leq \frac{\alpha + hA - h\omega_0 |k_{nn}|}{1 - h\omega_0 |k_{nn}|} \\ &= \frac{\alpha - h\omega_0 |k_{nn}|}{1 - h\omega_0 |k_{nn}|} + h \frac{A}{1 - h\omega_0 |k_{nn}|}. \end{aligned}$$

The first term on the right side is less than 1 since $\alpha < 1$.

6. Applications

The theorems reported in Section 4 give theoretical instruments to analyze the stability of VIEs on time scales. As already mentioned, an interesting case arises when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and $k(t,s)$ in (1) is of convolution type i.e., (i.e., $k(t,s) = k(t-s)$ and $k(t,s) = k(t-s-1)$, respectively) so Theorem 2 represents a perturbative approach whose aim is to obtain global results on non-convolution equations through perturbation of convolution ones (see [15] Sections 9 and 10) for $\mathbb{T} = \mathbb{R}$). Another interesting application consists of describing asymptotic properties of quasi–convolution equations characterized by integral terms consisting of a convolution product plus a non-convolution one. In these cases, the analysis of (13) in Theorem 2 serves to relate the behavior of the solution to the one of a convolution equation, governed by the resolvent r_k, corresponding to the convolution part of the kernel k, and thus described by the Paley–Wiener results. These equations have been treated in [12,25–27] for $\mathbb{T} = \mathbb{R}$,

$\mathbb{T} = \mathbb{Z}$ and for numerical methods and have received particular attention since they arise in linearised models of cell migration and collective motion, as described in [26,28,29]. For this reason, they will be the subject of future studies.

Another advantage in studying the stability of (12) by splitting the kernel into two parts is that Theorem 2 states the stability of the solution in weaker hypotheses than the ones in literature. Among these, a typical one (see, for example [1]) is

$$\int_{t_0}^{t} |k(t,s) + p(t,s)|\Delta s < 1.$$

In our case, it is sufficient to ask that $\int_{t_0}^{t} |k(t,s)|\Delta s < 1$, in order to obtain (10) and (11) on $r_k(t,s)$, while $p(t,s)$ enjoys greater freedom, although being subject to (17).

7. Open Problem

In this article, we made use of the known characteristics of VIEs on time scales to analytically and numerically analyze solutions of a perturbed Volterra Integral equation. As for an open problem, we consider the Volterra integro-dynamical equations on time scales

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\Delta s, \; t \in [t_0, \infty)_{\mathbb{T}}, \tag{28}$$

where A is an $n \times n$ matrix function that is continuous on $[t_0, \infty)_{\mathbb{T}}$, B is an $n \times n$ matrix function that is continuous on

$$\Omega := \{(t,u) \in \mathbb{T} \times \mathbb{T} : \; t_0 \le u \le t < \infty\}.$$

It was shown in [2] that the resolvent matrix solution $R(t,s)$ of (28) is the unique solution of

$$R^{\Delta_s}(t,s) = -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))B(u,s)\Delta u, \quad R(t,t) = I, \tag{29}$$

where I is the $n \times n$ identity matrix. To properly describe the solution of (28), we let $\varphi(t)$ be a given bounded and initial function. We say that $x(t,\tau_0,\varphi)$ is a solution of (28) if $x(t) = \varphi(t)$ for $t_0 \le t \le \tau_0$ and $x(t,\tau_0,\varphi)$ satisfies (28) for $t \ge \tau_0$. Then, one can refer to [2] to show that, if φ is a given bounded and continuous initial function defined on $t_0 \le t \le \tau_0$, then $x(t)$ is a solution of (28) if and only if

$$x(t) = R(t,\tau_0)\varphi(\tau_0) + \int_{\tau_0}^{t} R(t,\sigma(s)) \int_{t_0}^{\tau_0} B(s,u)\varphi(u)\Delta u \Delta s. \tag{30}$$

The application of the study of this paper to (28) and its perturbed counterpart, using (29) and (30), may be the subject of future work.

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