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Three-Way Decisions Making Using Covering Based Fractional Orthotriple Fuzzy Rough Set Model

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Abstract: On the basis of decision-theoretical rough sets (DTRSs), the three-way decisions give new model of decision approach for deal with the problem of decision. This proposed model of decision method is based on the loss function of DTRSs. First, the concept of fractional orthotriple fuzzy β-covering (FOF β-covering) and fractional orthotriple fuzzy β-neighborhood (FOF β-neighborhood) was introduced. We combined loss feature of DTRSs with covering-based fractional orthotriple fuzzy rough sets (CFOFSs) under the fractional orthotriple fuzzy condition. Secondly, we proposed a new FOF-covering decision-theoretical rough sets model (FOFCDTRSs) and developed related properties. Then, based on the grade of positive, neutral and negative membership of fractional orthotriple fuzzy numbers (FOFNs), five methods are established for addressing the expected loss expressed in the form of FOFNs and the corresponding three-way decisions are also derived. Based on this, we presented a FOFCDTRS-based algorithm for multi-criteria decision making (MCDM). Then, an example verifies the feasibility of the five methods for solving the MCDM problem. Finally, by comparing the results of the decisions of five methods with different loss functions.

Keywords: covering-based fractional orthotriple fuzzy rough sets; fractional orthotriple fuzzy β-covering decision-theoretic rough sets; fractional orthotriple fuzzy β-neighborhood; multi-attribute decision making; decision-theoretic rough sets

1. Introduction

Multi-criteria decision making analysis is also used in different contexts [1,2]. Intuitionistic fuzzy set (IFS) [3], a vital extension of fuzzy set (FS) [4], is considered as suitable tool to handle these information. An IFS contains two membership grades $\rho_{\vartheta}(\hbar) \in [0,1]$ and $\check{n}_{\vartheta}(\hbar) \in [0,1]$ in a finite universe of discourse $\daleth$ with $\rho_{\vartheta}(\hbar) + \check{n}_{\vartheta}(\hbar) \leq 1$, for each $\hbar \in \daleth$. Since the introduction of IFS, the theories and applications of IFS have been studied comprehensively, including its' applications in decision making problems (DMPs). These researches are very appropriate to tackle DMPs under IFS environment only owing to the condition $0 \leq \rho_{\vartheta} + \check{n}_{\vartheta} \leq 1$. However, in practical DMPs, the experts provide evaluation-value in the form of $(\rho_{\vartheta}, \check{n}_{\vartheta})$, but it may be not satisfy the condition $\rho_{\vartheta}(\hbar) + \check{n}_{\vartheta}(\hbar) \leq 1$ and beyond the upper bound 1.

As IFSs have only two kinds of responses, i.e., "yes" and "no" but there is some issue with three types of reply in the case of election, e.g., "yes", "no" and "refusal", and the ambitious answer is "refusal". In order to overcome this defect, Cuong [5,6] developed the idea of picture fuzzy set (PFS), which dignified the positives, neutral and negative membership grades in three different functions. Cuong [7] addressed some PFSs characteristics and also accepted distance measurements. Cuong & Hai [8] defined fuzzy logic operators and specify basic operations in the picture fuzzy logic for fuzzy derivation types. Cuong et al. [9] analyzed the features of the blurry t-norm and t-conorm picture. Phong et al. [10] discussed some configuration of picture fuzzy relationships. Wei et al. [11–13] have identified several procedures for calculating the closeness between picture fuzzy sets. Many authors have currently built more models in the condition of PF sets: Singing [14] proposes the correlation coefficient of PFS and apply it to the clustering analysis. Son et al. [15,16] give time and temperature estimates based on the PF sets domain. Son [17,18] describes PF as isolation, distance and association measurements, often combined with the condition of PFSs. Van Viet and Van Hai [19] described a novel PFS fluid derivation structure and improved a classic fluid inference technique. Thong et al. [20,21] using the PF clustering technique for the optimization of complex & particle clumps. Wei [22] defined some basic leadership methodology using the PF weighted cross-entropy principle and used this method to rate the alternative. Yang et al. [23] described flexible soft matrix of decision making using PFSs. In [24], Garg feature aggregation of MCDM problems with PFSs. Peng et al. introduced the PFSs solution in [25] and apply in decision making. For the PF-set, readers see also [26–28]. Ashraf et al. [29] extend cubic set structure to PFSs.

Three-way decisions are one of the important ways in solving the decision making problems under uncertainty. Their key strategy is to consider a decision making problem as a ternary classification one labeled by three decision actions of acceptance, rejection and non-commitment in practice. In general, many theories can be utilized for inducing three-way decisions such as shadowed sets [30,31], modal logic [32] and orthopairs [33]. The essential idea of three-way decisions is to divide a universal set into three pairwise disjoint regions named as the positive, negative and boundary regions. The three regions are then processed to make different decisions with accept, reject and deferment [34]. The general framework of three-way decisions was outlined by Yao [35,36].

Zakowski's [37] Covering-based fuzzy rough sets (CRS) is a variant of the classical rough sets (RS) generalization. It is an extension of Pawlak RS partition to RS cover. Two rough approximation operators are built on this basis, and several conclusions are drawn. Many scholars then studied several types of RS models based on reporting from different angles. In 2003, Zhu & Wang [38] introduced the generalized rough set cover model, and studied the model's reduction and axiomatic properties. They then introduced three different types of CRS models based on the known models and identified several important features. Safari et al. [39] introduced twelve types of coverage approximation operators in 2016, and studied the structural properties and interrelations of these twelve CRS models. In addition, Ma [40] substitutes for the classical equivalence relationship with the general binary relationship (neighborhood relationship), thus generalizing the CRS. Many scholars have applied the classical CRS to the fuzzy world in recent years. The rough fuzzy set (RFS) and the fuzzy rough set (FRS) were introduced in Dubios et al. [41]. Researchers have done some researches on CFRS. The generalized CFRS structure was introduced by Ma [42] Deer et al. [43,44] introduced the fuzzy β-neighborhoods and fuzzy neighborhoods definition. Hussain [45] introduced the q-rung orthopair fuzzy TOPSIS method for the MCDM problem which depends on the Cq-ROFRS model. Quek et al. [46] defined the concept of Plithogenic set is an extension of the crisp set, fuzzy set, intuitionistic fuzzy set, and neutrosophic sets, whose elements are characterized by one or more attributes, and each attribute can assume many values. Zeng [47] proposed a framework for solving MADM problem based on complex Spherical fuzzy rough set (CSFRS) models and created a TOPSIS method for dealing with MADM problem.

In recent years, research on decision-theoretical rough sets (DTRSs) has made great progress. Many scholars have studied this theory. The key directions for research include the reduction of attributes, loss feature and some new extended models focused on DTRSs [48–59]. Yao suggested three-way decisions which are modern DTRS theories. Three-way decisions divide the universal set into three disjoint parts: positive area, boundary area and negative region. The Three-way decisions are a combination of DTRSs and Bayesian decision process, which has solved several classification problems successfully. The theory of three-way decisions has been extended to many specific areas, such as cluster analysis [60,61], risk decision taking by the government [62], medical evaluation [63], investment decision making [64], multi-attribute community decision making (MAGDM) [65], etc. Current work has concentrated on conditional likelihood and loss function to extend the idea of three-way decisions. Yao & Zhou [66] determined the conditional probability on the basis of Bayes' theorem and the naive probabilistic independence. Liu [67] calculated the conditional probability through logistic regression. In order to address the problem that it is difficult to calculate the loss accurately in a specific situation, there is a trend towards reducing the precision of loss calculation by some kind of fluid method. Liang & Liu [68] have developed a new model of three-way decisions that calculates the loss function by using hesitant fuzzy sets. Liang & Liu [69] also considered IFSs as a new framework for evaluating the loss feature in Three-way decisions and then developed a new Three-way decision model. Mandal & Ranadive [70] introduced PFNs into the loss function and developed three methods with Pythagorean fuzzy decision-theoretical rough sets (PFDTRSs) to extract Three-way decisions. These studies have encouraged widespread application of DTRS and Three-way Decisions. While Mandal & Ranadive [70] introduced PFNs into the loss function and proposed the concept of PFDTRSs. The factional orthotriple fuzzy set is new generalized tool to describe the uncertainty and Pythagorean fuzzy set (PyFS) and q-rung orthopair fuzzy set is particulars cases. In case, we have $f = \frac{p}{q} = 2$, then the fractional orthotriple fuzzy set is reduced a Pythagorean fuzzy set, and if $f = \frac{p}{q} = p$ and $q = 1$, then the fractional orthotriple fuzzy set is reduced to q-rung orthopair fuzzy set. The model of [70] did not implement on the fractional orthotriple fuzzy environment. To fill this research space, this paper tries to study the model of (fractional orthotriple fuzzy covering-based decision-theoretical rough sets (FOFCDTRSs) through fractional orthotriple fuzzy (FOF) β-neighborhood structures and Three-way decisions. Using the positive, neutral and negative characteristics of FOFNs, we develop five methods to resolve fractional orthotriple fuzzy numbers (FOFNs) and deduce appropriate Three-way decisions. We focus on the determination of loss functions, using the opinions of multiple experts. We compare the five approaches (Methods), summarize their advantages and drawbacks and establish a corresponding algorithm for deriving FOF β-covering Three-way decisions with DTRSs. In real life, the FOFCDTRS model is a critical instrument for coping with ambiguity and confusion. In addition, by adjusting the value of $0 \leq \rho_{\vartheta}(\hbar)^2, \check{n}_{\vartheta}(\hbar)^2, \nu_{\vartheta}(\hbar)^2 \leq 1$, it is found that FOFCDTRSs is an important extension of covering-based Spherical fuzzy decision-theoretic rough sets (CSFDTRSs). And by adjusting $0 \leq \rho_{\vartheta}(\hbar), \check{n}_{\vartheta}(\hbar), \nu_{\vartheta}(\hbar) \leq 1$, it is an important extension of covering-based picture fuzzy decision-theoretic rough sets (CPFDTRSs). This shows that the FOFCDTRS model is more capable of dealing with uncertainty than the CPFDTRSs and CSFDTRSs.

The factional orthotriple fuzzy set is new generalized tool to describe the uncertainty and Pythagorean fuzzy set and q-rung orthopair fuzzy set is particulars cases. In case, we have $f = p/q = 2$, then the fractional orthotriple fuzzy set is reduced a Pythagorean fuzzy set, and if $f = p/q = p$ *and* $q = 1$, then the fractional orthotriple fuzzy set is reduced to q-rung orthopair fuzzy set. The model of [70] did not implement on the fractional orthotriple fuzzy environment.

The role of the fractional orthotriple fuzzy sets (FOFSs) in the decision making problem is very important among the other extension of fuzzy sets. In the FOFS, the opinion is not only restricted to yes or no, also having some sort of refusal or abstinence. The best example for representing the FOFS as, voting systems, in voting systems, there are four type of voters, i.e vote in favor, or against vote, refuse to vote, or neutral for vote. In FOFS, the MD is used for vote in favor, NMD is used for against vote, ND is used for neutral for vote and RD is used for refuse to vote. In many cases of real life, we have exist situation where the experts plans for best decision by using more accurate tools. The FOFS is a very important tool to describe the object with no uncertainty, and in other tool the information diverse and having uncertainty. For example, we consider a country want build or start a project for the medical treatment or health care center. The government party will give high favor for his project, Govt assigned MD 0.8, while the opposition party will show it, the same project is not good, they will highly against. The opposition party will assigned NMD 0.75. The other small party will remain neutral and they will assigned NM is 0.2, in case of picture fuzzy set, $0.8 + 0.75 + 0.2 = 1.75 > 1$, in this case the picture fuzzy set failed to explain such information. Now consider SFS, $(0.8)^2 + (0.75)^2 + (0.2)^2 = 1.243 > 1$, also in this case the SFS failed to explain the such information, In case of FOFS, $(0.8)^f + (0.75)^f + (0.2)^f \leq 1$, where $f \in Q^+$.

In order to handle such problem of uncertainty, we need a comprehensive tool to describe such type of problem during the decision making process.

The rest of this paper is arranged as follows: the basic concepts of FOFSs and their generalization are introduced in Section 2. In Section 3, the concept of CFOFRSs based on FOF β-neighborhoods is proposed along with the corresponding axiomatic system. Apart from these, the method of obtaining conditional probability is discussed in this section. In Section 4, we propose the FOFCDTRSs model and give the minimum cost decision rules under FOF environment, and further study the decision rules $(P_1) - (N_1)$ according to different comparison methods of FOFNs, and propose five methods to deduce Three-way decisions with FOFCDTRS. Then, an application algorithm based on FOFCDTRSs model to solve MCDM is designed in Section 5, and also an example shows the implementation of the latest three-way decisions, and contrasts and analyzes the five approaches proposed. Section 6, concludes the paper and discusses future research.

2. Preliminaries

The fuzzy set theory was first time defined by Zadeh [4], which contribute a fruitful scheme for representing and manipulating uncertainty in the form of gradualness. In 1986, Atanassov update the FS into IFS [3], by developing the notion of negative membership grade along with a positive membership grade.

This section presented the briefly remembrance the rudiments of IFS, PyFS, PFS and SFS.

Definition 1 ([3])**.** *Let $\beth \neq \phi$ are the genral set. An intuitionistic fuzzy set ϑ is described as;*

$$\beth = \{(\hbar, \rho_\vartheta(\hbar), \nu_\vartheta(\hbar) | \hbar \in \beth\}. \tag{1}$$

where the functions $\rho_\vartheta(\hbar) : \beth \to [0,1]$ and $\nu_\vartheta(\hbar) : \mathbb{R} \to [0,1]$ represent the grade of positive and negative membership of each number, with $0 \leq \rho_\vartheta(\hbar) + \nu_\vartheta(\hbar) \leq 1$ for all $\hbar \in \mathbb{R}$.

Definition 2 ([71])**.** *For any fixed set $\gimel$. A PyFS ϑ on $\gimel$ is described with the pair of mappings $\rho_\vartheta : \gimel \to [0,1]$ and $\check{n}_\vartheta : \gimel \to [0,1]$ where each $\hbar \in \gimel$, $\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar)$ and $\nu_\vartheta(\hbar)$ are said to be positive and negative membership grades of $\hbar$, respectively, and $\rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) \leq 1$. That is,*

$$\vartheta = \{(\hbar, \rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar))\}.$$

Conventionally, $\pi_\vartheta(\hbar) = \sqrt{1 - s_\vartheta^2(\hbar)}$, where $s^2(\hbar) = \rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) + \nu_\vartheta^2(\hbar)$ is said to be the hesitancy grade of $\hbar$, and $\rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) \leq 1$ for each $\hbar \in \gimel$.

Definition 3 ([72])**.** *For any fixed set $\beth$. A q-rung orthopair fuzzy set (q-ROFS) ϑ on $\beth$ is described with the pair of mappings $\rho_\vartheta : \beth \to [0,1]$, $\check{n}_\vartheta : \beth \to [0,1]$ and $\nu_\vartheta : \beth \to [0,1]$, where each $\hbar \in \beth$, $\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar)$ and $\nu_\vartheta(\hbar)$ are said to be positive and negative grades of $\hbar$, correspondingly, and $0 \leq \rho_\vartheta(\hbar)^f + \nu_\vartheta(\hbar)^f \leq 1, (f \geq 1)$. That is*

$$\vartheta = \left\{(\hbar, \rho_\vartheta(\hbar), \nu_\vartheta(\hbar)) \; : \rho_\vartheta(\hbar)^f + \nu_\vartheta(\hbar)^f \leq 1 \text{ for each } \hbar \in \beth\right\}$$

Conventionally, $\pi_\vartheta(\hbar) = \left(1 - \rho_\vartheta(\hbar)^f - \nu_\vartheta(\hbar)^f\right)^{\frac{1}{f}}$ is said to be the indeterminacy membership grade of $\hbar$.

Definition 4 ([5])**.** *For any fixed set* $\beth$*. An picture fuzzy set (PFS)* ϑ *on* $\beth$ *is described with the pair of mappings* $\rho_\vartheta : \beth \rightarrow [0,1]$, $\check{n}_\vartheta : \beth \rightarrow [0,1]$ *and* $\nu_\vartheta : \beth \rightarrow [0,1]$ *, where each* $\hbar \in \beth$, $\rho_\vartheta(\hbar)$, $\check{n}_\vartheta(\hbar)$ *and* $\nu_\vartheta(\hbar)$ *are said to be positive, neutral and negative membership grades of* $\hbar$*, respectively, and* $\rho_\vartheta(\hbar) + \check{n}_\vartheta(\hbar) + \nu_\vartheta(\hbar) \leq 1$*. That is*

$$\vartheta = \{(\hbar, \rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar)) \;\; : \rho_\vartheta(\hbar) + \check{n}_\vartheta(\hbar) + \nu_\vartheta(\hbar) \leq 1 \text{ for each } \hbar \in \beth\}.$$

Conventionally, $\pi_\vartheta(\hbar) = 1 - \rho_\vartheta(\hbar) - \check{n}_\vartheta(\hbar) - \nu_\vartheta(\hbar)$ *is said to be the indeterminacy membership grade of* $\hbar$*. We note that a standard membership grade is a special case of an picture positive membership grade where* $\check{n}_\vartheta(\hbar) = 1 - \rho_\vartheta(\hbar) - \nu_\vartheta(\hbar)$*. Also, standard membership grade has* $\pi_\vartheta(\hbar) = 0$*.*

Definition 5 ([73])**.** *For any fixed set* $\beth$*. A Spherical fuzzy set (SFS)* ϑ *on* $\beth$ *is described with the pair of mappings* $\rho_\vartheta : \beth \rightarrow [0,1]$, $\check{n}_\vartheta : \beth \rightarrow [0,1]$ *and* $\nu_\vartheta : \beth \rightarrow [0,1]$ *where each* $\hbar \in \beth$, $\rho_\vartheta(\hbar)$, $\check{n}_\vartheta(\hbar)$ *and* $\nu_\vartheta(\hbar)$ *are said to be positive, neutral and negative membership grades of* $\hbar$*, respectively, and* $\rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) + \nu_\vartheta^2(\hbar) \leq 1$*. That is*

$$\vartheta = \left\{(\hbar, \rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar)) \;\; : \rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) + \nu_\vartheta^2(\hbar) \leq 1 \text{ for each } \hbar \in \beth\right\}.$$

Conventionally, $\pi_\vartheta(\hbar) = \sqrt{1 - s_\vartheta^2(\hbar)}$ *, where* $s^2(\hbar) = \rho_\vartheta^2(\hbar) + \check{n}_\vartheta^2(\hbar) + \nu_\vartheta^2(\hbar)$ *is said to be the hesitancy grade of* $\hbar$*.*

Definition 6 ([74])**.** *A fuzzy-rough set is the pair of lower and upper approximations of a fuzzy set* ϑ *in a universe* $\beth$ *on which a fuzzy relation* R *is defined. The fuzzy-rough model is obtained by fuzzifying the definitions of the crisp lower and upper approximation. Recall that the condition for an element to belong to the crisp lower approximation is*

$$\forall y \in \beth(x,y) \in R \rightarrow y \in \vartheta$$

The equivalence relation R *is now a fuzzy relation, and* ϑ *is a fuzzy set. The values* $R(x,y)$ *and* $\vartheta(y)$ *are connected by a fuzzy implication* Γ*, so* $\Gamma(R(x,y), \vartheta(y))$ *expresses to what extent elements that are similar* x *to belong to* ϑ*. The membership value of an element* $x \in \beth$ *to the lower approximation is high if these values* $\Gamma(R(x,y), \vartheta(y))$ *are high for all* $y \in \vartheta$*:*

$$\begin{aligned} \forall y &\in \beth(R \downarrow \vartheta)(x) = \min_{y \in \beth} \Gamma(R(x,y), \vartheta(y)) \\ \forall y &\in \beth(R \uparrow \vartheta)(x) = \underset{y \in \beth}{mix} \Gamma(R(x,y), \vartheta(y)) \end{aligned}$$

This upper approximation expresses to what extent there exist instances that are similar to x *and belong to* ϑ*.*

Definition 7 ([47])**.** *(1) Consider a universal set* $\beth$ *and* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, ...\mathcal{S}_n\}$ *and each* $\mathcal{S}_i \in SFS(\beth)$*. Then,* $\mathcal{S}$ *is called spherical* β*-covering(SF* β*-covering) of* $\beth$*, if there is another SFS* β *of* $\beth$ *such that* $\left(\bigcup_{i=1}^{n} \mathcal{S}_i\right)(x) \succcurlyeq \beta$ *for all* $x \in \beth$*. Thus, the pair* $(\beth, \mathcal{S})$ *is said to a be SFCAS.*

(2) Consider $(\beth, \mathcal{S})$ *be a SFCAS, for* β *and SF* β*-covering* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, ...\mathcal{S}_n\}$ *of* $\beth$*. Then*

$$\mathbb{N}_{\mathcal{S}(x)}^{\beta} = \bigcap \{\mathcal{S}_i \in \mathcal{S} / \mathcal{S}_i(x)\} \succcurlyeq \beta, i = 1, 2, 3, ..., n$$

is called an SF β*-covering neighborhood of* $\beth$*.*

(3) Consider $\mathbb{N}_{\mathcal{S}}^{\beta} = \left\{\mathbb{N}_{\mathcal{S}(x)}^{\beta} / x \in \beth\right\}$ *denote an SF* β*-covering neighborhood system induced by an SF* β*-covering* $\mathcal{S}$*. Further, we have the representation of SF* β*-neighborhood systems as following;*

$$M_{\mathcal{S}}^{\beta} = \left[\mathbb{N}_{\mathcal{S}(x)}^{\beta}(x)\right]_{(x_1, x_2) \in \beth \times \beth}$$

Definition 8 ([75]). *Decision-theoretic rough set models are a probabilistic extension of the algebraic rough set model. The required parameters for defining probabilistic lower and upper approximations are calculated based on more familiar notions of costs (risks) through the well-known Bayesian decision procedure.*

3. Fractional Orthotriple Fuzzy Set

Definition 9. *For any fixed set $\beth$. A fractional orthotriple fuzzy set (FOFS) ϑ on $\beth$ is described with the triple of mappings $\rho_\vartheta : \beth \to [0,1]$, $\check{n}_\vartheta : \beth \to [0,1]$ and $\nu_\vartheta : \beth \to [0,1]$, where each $\hbar \in \beth$, $\rho_\vartheta(\hbar)$, $\check{n}_\vartheta(\hbar)$ and $\nu_\vartheta(\hbar)$ are said to be positive, neutral and negative grades of $\hbar$, correspondingly, and $0 \le \rho_\vartheta(\hbar)^f + \check{n}_\vartheta(\hbar)^f + \nu_\vartheta(\hbar)^f \le 1, (f \ge 1)$. That is*

$$\vartheta = \left\{ (\hbar, \rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar)) \; : \rho_\vartheta(\hbar)^f + \check{n}_\vartheta(\hbar)^f + \nu_\vartheta(\hbar)^f \le 1 \text{ for each } \hbar \in \beth \right\}. \tag{2}$$

Conventionally, $\pi_\vartheta(\hbar) = \left(1 - \rho_\vartheta(\hbar)^f - \check{n}_\vartheta(\hbar)^f - \nu_\vartheta(\hbar)^f\right)^{\frac{1}{f}}$ is said to be the indeterminacy membership grade of $\hbar$.

For convenience, fractional orthotriple fuzzy number (FOFN) is denoted as $(\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar))$ for all $\hbar \in \beth$, and the collection of all FOFSs on $\beth$ is written by FOF($\beth$).

Definition 10. *Suppose $\vartheta_1(\hbar) = \left(\rho_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \nu_{\vartheta_1}(\hbar)\right)$ and $\vartheta_2(\hbar) = \left(\rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_2}(\hbar)\right)$ are two FOFNs. Then, one has the following properties;*

1. $\vartheta_1(\hbar) \subseteq \vartheta_2(\hbar)$ if $\rho_{\vartheta_1}(\hbar) \le \rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_1}(\hbar) \ge \check{n}_{\vartheta_2}(\hbar)$ and $\nu_{\vartheta_1}(\hbar) \ge \nu_{\vartheta_2}(\hbar)$;
2. $\vartheta_1(\hbar) = \vartheta_2(\hbar)$ if $\rho_{\vartheta_1} = \rho_{\vartheta_2}, \check{n}_{\vartheta_1} = \check{n}_{\vartheta_2}$ and $\nu_{\vartheta_1} = \nu_{\vartheta_2}$;
3. $\vartheta_1(\hbar) \cap \vartheta_2(\hbar) = \left\{\min\left(\rho_{\vartheta_1}(\hbar), \rho_{\vartheta_2}(\hbar)\right), \max\left(\check{n}_{\vartheta_1}(\hbar), \check{n}_{\vartheta_2}(\hbar)\right), \max\left(\nu_{\vartheta_1}(\hbar), \nu_{\vartheta_2}(\hbar)\right)\right\}$;
4. $\vartheta_1(\hbar) \cup \vartheta_2(\hbar) = \left\{\max\left(\rho_{\vartheta_1}(\hbar), \rho_{\vartheta_2}(\hbar)\right), \min\left(\check{n}_{\vartheta_1}(\hbar), \check{n}_{\vartheta_2}(\hbar)\right), \min\left(\nu_{\vartheta_1}(\hbar), \nu_{\vartheta_2}(\hbar)\right)\right\}$;
5. $\vartheta_1^c(\hbar) = \left(\nu_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \rho_{\vartheta_1}(\hbar)\right)$;
6. $\vartheta_1(\hbar) \oplus \vartheta_2(\hbar) = \left\{ \begin{array}{c} \left(\rho_{\vartheta_1}(\hbar)^f + \rho_{\vartheta_2}(\hbar)^f - \rho_{\vartheta_1}(\hbar)^f \cdot \rho_{\vartheta_2}(\hbar)^f\right)^{\frac{1}{f}}, \\ \check{n}_{\vartheta_1}(\hbar) \cdot \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_1}(\hbar) \cdot \nu_{\vartheta_2}(\hbar) \end{array} \right\}$;
7. $\vartheta_1(\hbar) \otimes \vartheta_2(\hbar) = \left\{ \begin{array}{c} \rho_{\vartheta_1}(\hbar) \cdot \rho_{\vartheta_2}(\hbar), \left(\check{n}_{\vartheta_1}(\hbar)^f + \check{n}_{\vartheta_2}(\hbar)^f - \check{n}_{\vartheta_1}(\hbar)^f \cdot \check{n}_{\vartheta_2}(\hbar)^f\right)^{\frac{1}{f}}, \\ \left(\nu_{\vartheta_1}(\hbar)^f + \nu_{\vartheta_2}(\hbar)^f - \nu_{\vartheta_1}(\hbar)^f \cdot \nu_{\vartheta_2}(\hbar)^f\right)^{\frac{1}{f}} \end{array} \right\}$;
8. $\Psi\vartheta_1(\hbar) = \left\{ \left(1 - \left(1 - \rho_{\vartheta_1}(\hbar)^f\right)^\Psi\right)^{\frac{1}{f}}, \check{n}_{\vartheta_1}(\hbar)^\Psi, \nu_{\vartheta_1}(\hbar)^\Psi \right\}, \Psi > 0$;
9. $(\vartheta_1(\hbar))^\Psi = \left\{ \left(\rho_{\vartheta_1}(\hbar)\right)^\Psi, \left(1 - \left(1 - \check{n}_{\vartheta_1}(\hbar)^f\right)^\Psi\right)^{\frac{1}{f}}, \left(1 - \left(1 - \nu_{\vartheta_1}(\hbar)^f\right)^\Psi\right)^{\frac{1}{f}} \right\}$.

Definition 11. *Consider two FOFNs $\vartheta_1(\hbar) = \left(\rho_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \nu_{\vartheta_1}(\hbar)\right)$ and $\vartheta_2(\hbar) = \left(\rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_2}(\hbar)\right)$. Then, there are a natural quasi-ordering on the FOFNs is defined as follows;*

$$\vartheta_1(\hbar) \ge \vartheta_2(\hbar) \Leftrightarrow \rho_{\vartheta_1}(\hbar) \ge \rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_1}(\hbar) \le \check{n}_{\vartheta_2}(\hbar) \text{ and } \nu_{\vartheta_1}(\hbar) \le \nu_{\vartheta_2}(\hbar) \tag{3}$$

Remark 1. *It is easy observed from Definition (11) that the FOFN $f^+ = (1,0,0)$ is the largest FOFN and the $f^- = (0,0,1)$ is the smallest FOFN, correspondingly. We called f^+, the positive ideal FOFN and f^-, the negative ideal FOFN.*

Definition 12. *Let $\vartheta(\hbar) = (\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar))$ be a FOFN, the score $S(\vartheta(\hbar))$ and the corresponding accuracy function $H(\vartheta(\hbar))$ are defined as follows;*

$$S(\vartheta(\hbar)) = \rho_\vartheta(\hbar)^f - \check{n}_\vartheta(\hbar)^f - \nu_\vartheta(\hbar)^f \tag{4}$$

and

$$H(\vartheta(\hbar)) = \rho_{\vartheta}(\hbar)^f + \check{n}_{\vartheta}(\hbar)^f + \nu_{\vartheta}(\hbar)^f \tag{5}$$

Obviously, $-1 \leq S(\vartheta(\hbar)) \leq 1$ *and* $0 \leq H(\vartheta(\hbar)) \leq 1$.

According to the Definition (12), the comparison rules for FOFNs as follows;

1. If $S(\vartheta_1(\hbar)) > S(\vartheta_2(\hbar))$, then $(\vartheta_1(\hbar)) > (\vartheta_2(\hbar))$;
2. If $S(\vartheta_1(\hbar)) < S(\vartheta_2(\hbar))$, then $(\vartheta_1(\hbar)) < (\vartheta_2(\hbar))$;
3. If $S(\vartheta_1(\hbar)) = S(\vartheta_2(\hbar))$, then;

   (a) If $H(\vartheta_1(\hbar)) > H(\vartheta_2(\hbar))$, then $(\vartheta_1(\hbar)) > (\vartheta_2(\hbar))$;

   (b) If $H(\vartheta_1(\hbar)) < H(\vartheta_2(\hbar))$, then $(\vartheta_1(\hbar)) < (\vartheta_2(\hbar))$;

   (c) If $H(\vartheta_1(\hbar)) = H(\vartheta_2(\hbar))$, then $(\vartheta_1(\hbar)) = (\vartheta_2(\hbar))$;

Definition 13. *Let* $\vartheta_1(\hbar) = \left(\rho_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \nu_{\vartheta_1}(\hbar)\right)$ *and* $\vartheta_2(\hbar) = \left(\rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_2}(\hbar)\right)$ *are two FOFNs, the generalized distance between* $\vartheta_1(\hbar)$ *and* $\vartheta_2(\hbar)$ *is defined as follows;*

$$d(\vartheta_1(\hbar), \vartheta_2(\hbar)) = \left\{ \begin{array}{c} \frac{1}{2}(1-p) \left(\begin{array}{c} \left|\rho_{\vartheta_1}(\hbar)^f - \rho_{\vartheta_2}(\hbar)^f\right|^{\lambda} \\ + \left|\check{n}_{\vartheta_1}(\hbar)^f - \check{n}_{\vartheta_2}(\hbar)^f\right|^{\lambda} \\ + \left|\nu_{\vartheta_1}(\hbar)^f - \nu_{\vartheta_2}(\hbar)^f\right|^{\lambda} \end{array} \right) \\ + p\left|\pi_{\vartheta_1}(\hbar)^f - \pi_{\vartheta_2}(\hbar)^f\right|^{\lambda} \end{array} \right\}^{\frac{1}{\lambda}} \tag{6}$$

where

$$\begin{aligned} \pi_{\vartheta_1}(\hbar) &= \left(1 - \rho_{\vartheta_1}(\hbar)^f - \check{n}_{\vartheta_1}(\hbar)^f - \nu_{\vartheta_1}(\hbar)^f\right)^{\frac{1}{q}}, \\ \pi_{\vartheta_2}(\hbar) &= \left(1 - \rho_{\vartheta_2}(\hbar)^f - \check{n}_{\vartheta_2}(\hbar)^f - \nu_{\vartheta_2}(\hbar)^f\right)^{\frac{1}{q}}, \end{aligned}$$

$\lambda > 0$ *and* $p \in [0,1]$. *When the parameters* λ *and* p *take different values, we will get some different distance measures.*

Case 1. When $\lambda = 1$ and $f = 2$, the distance will be reduced to Hamming-indeterminacy degree-preference distance.

$$d(\vartheta_1(\hbar), \vartheta_2(\hbar)) = \left\{ \frac{1}{2}(1-p) \left(\begin{array}{c} \left|\rho_{\vartheta_1}(\hbar)^2 - \rho_{\vartheta_2}(\hbar)^2\right| \\ + \left|\check{n}_{\vartheta_1}(\hbar)^2 - \check{n}_{\vartheta_2}(\hbar)^2\right| \\ + \left|\nu_{\vartheta_1}(\hbar)^2 - \nu_{\vartheta_2}(\hbar)^2\right| \end{array} \right) + p\left|\pi_{\vartheta_1}(\hbar)^2 - \pi_{\vartheta_2}(\hbar)^2\right| \right\}$$

In case 1, if $p = 0$, the effect of the indeterminacy grade is not considered. The distance will be reduced to metric distance.

$$d(\vartheta_1(\hbar), \vartheta_2(\hbar)) = \left\{ \frac{1}{2}(1-p) \left(\begin{array}{c} \left|\rho_{\vartheta_1}(\hbar)^2 - \rho_{\vartheta_2}(\hbar)^2\right| + \left|\check{n}_{\vartheta_1}(\hbar)^2 - \check{n}_{\vartheta_2}(\hbar)^2\right| \\ + \left|\nu_{\vartheta_1}(\hbar)^2 - \nu_{\vartheta_2}(\hbar)^2\right| \end{array} \right) \right\}$$

Case 2. When $\lambda = 2$ and $f = 2$, the distance will be reduced to Euclidean-indeterminacy grade-preference distance.

$$d\left(\vartheta_1(\hbar), \vartheta_2(\hbar)\right) = \left\{ \begin{array}{c} \frac{1}{2}(1-p) \left(\begin{array}{c} \left|\rho_{\vartheta_1}(\hbar)^2 - \rho_{\vartheta_2}(\hbar)^2\right|^2 \\ + \left|\check{n}_{\vartheta_1}(\hbar)^2 - \check{n}_{\vartheta_2}(\hbar)^2\right|^2 \\ + \left|\nu_{\vartheta_1}(\hbar)^2 - \nu_{\vartheta_2}(\hbar)^2\right|^2 \end{array} \right) \\ +p\left|\pi_{\vartheta_1}(\hbar)^2 - \pi_{\vartheta_2}(\hbar)^2\right|^{\frac{1}{2}} \end{array} \right\}$$

In the Case 2, if $p = 0$, the distance will be reduced to Euclidean distance.

$$d\left(\vartheta_1(\hbar), \vartheta_2(\hbar)\right) = \left\{ \frac{1}{2}(1-p) \left(\begin{array}{c} \left|\rho_{\vartheta_1}(\hbar)^2 - \rho_{\vartheta_2}(\hbar)^2\right|^2 \\ + \left|\check{n}_{\vartheta_1}(\hbar)^2 - \check{n}_{\vartheta_2}(\hbar)^2\right|^2 \\ + \left|\nu_{\vartheta_1}(\hbar)^2 - \nu_{\vartheta_2}(\hbar)^2\right|^2 \end{array} \right) \right\}^{\frac{1}{2}}$$

Definition 14. *Let* $\vartheta_1(\hbar) = \left(\rho_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \nu_{\vartheta_1}(\hbar)\right)$ *and* $\vartheta_2(\hbar) = \left(\rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_2}(\hbar)\right)$ *are two FOFNs, the distance d satisfied the following properties;*

1. $d\left(\vartheta_1(\hbar), \vartheta_2(\hbar)\right) \geq 0$;
2. $d\left(\vartheta_1(\hbar), \vartheta_2(\hbar)\right) = d\left(\vartheta_2(\hbar), \vartheta_1(\hbar)\right)$;
3. $d\left(\vartheta_1(\hbar), \vartheta_2(\hbar)\right) = 0 \Leftrightarrow d\left(\vartheta_1(\hbar)\right) = d\left(\vartheta_2(\hbar)\right)$.

According to the Definition (13), it is easy to find the distance of FOFN $\vartheta(\hbar) = (\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar))$ and the positive ideal FOFN $f^+ = (1, 0, 0)$ as follows;

$$d\left(\vartheta(\hbar), f^+\right) = \left\{ \frac{1}{2}(1-p)\left(\left|1 - \rho_\vartheta(\hbar)^f\right|^\lambda + \left|\check{n}_\vartheta(\hbar)^f\right|^\lambda + \left|\nu_\vartheta(\hbar)^f\right|^\lambda \right) \right\}^{\frac{1}{\lambda}} \tag{7}$$

and distance between the FOFN $\vartheta(\hbar) = (\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar))$ and the negative ideal FOFN $f^- = (0, 0, 1)$ as follows;

$$d\left(\vartheta(\hbar), f^-\right) = \left\{ \frac{1}{2}(1-p)\left(\left|\rho_\vartheta(\hbar)^f\right|^\lambda + \left|1 - \check{n}_\vartheta(\hbar)^f\right|^\lambda + \left|1 - \nu_\vartheta(\hbar)^f\right|^\lambda \right) \right\}^{\frac{1}{\lambda}} \tag{8}$$

Usually, the smaller the distance $d\left(\vartheta(\hbar), f^+\right)$ is the bigger the FOFN $\vartheta(\hbar)$ is; and on the contrary the larger the distance $d\left(\vartheta(\hbar), f^-\right)$ is, the bigger the FOFN $\vartheta(\hbar)$ is. Inspire by the concept of TOPSIS [76], we developed the idea of closeness index for the FOFN.

Definition 15. *Let* $\vartheta(\hbar) = (\rho_\vartheta(\hbar), \check{n}_\vartheta(\hbar), \nu_\vartheta(\hbar))$ *be a FOFN,* $f^+ = (1, 0, 0)$ *be the positive ideal FOFN and* $f^- = (0, 0, 1)$ *be the negative ideal FOFN, then the closeness index of* $\vartheta(\hbar)$ *is defined as following;*

$$\zeta\left(\vartheta(\hbar)\right) = \frac{d\left(\vartheta(\hbar), f^-\right)}{d\left(\vartheta(\hbar), f^-\right) + d\left(\vartheta(\hbar), f^+\right)} \tag{9}$$

Apparently, if $\vartheta(\hbar) = f^-$, then $\zeta\left(\vartheta(\hbar)\right) = 0$; if $\vartheta(\hbar) = f^+$, then $\zeta\left(\vartheta(\hbar)\right) = 1$. Meanwhile, it is easily noticed that the closeness index $\zeta\left(\vartheta(\hbar)\right) \in [0, 1]$.

And for two FOFNs $\vartheta_1(\hbar) = \left(\rho_{\vartheta_1}(\hbar), \check{n}_{\vartheta_1}(\hbar), \nu_{\vartheta_1}(\hbar)\right)$ and $\vartheta_2(\hbar) = \left(\rho_{\vartheta_2}(\hbar), \check{n}_{\vartheta_2}(\hbar), \nu_{\vartheta_2}(\hbar)\right)$, if $\zeta\left(\vartheta_1(\hbar)\right) \geq \zeta\left(\vartheta_2(\hbar)\right)$, then $\vartheta_1(\hbar) \geq \vartheta_2(\hbar)$.

4. Covering Based Fractional Orthotriple Fuzzy Rough Set

In this section, we defined some new concept of fractional orthotriple fuzzy β-covering (FOF β-covering), fractional orthotriple fuzzy covering approximation space (FOFCAS) and FOF β-neighborhood.

Definition 16.

1. *Assume $\beth$ is a universe set, $\widetilde{E} = \left(\widetilde{E}_1, ..., \widetilde{E}_n\right)$, where $\widetilde{E}_i \in FOF(\beth)$ and $k = 1, ..., n$. For any FOFN $\beta = \left(\rho_\beta(\hbar), \check{n}_\beta(\hbar), \nu_\beta(\hbar)\right)$, then $\widetilde{E}$ is called a FOF β-covering of $\beth$ if*

$$\left(\bigcup_{k=1}^{n} \widetilde{E}_k\right)(\hbar) \geq \beta \tag{10}$$

   for all $\hbar \in \beth$. The $\left(\beth, \widetilde{E}\right)$ is called a FOFCAS.

2. *Let $\left(\beth, \widetilde{E}\right)$ be a FOFCAS and $\widetilde{E} = \left(\widetilde{E}_1, ..., \widetilde{E}_n\right)$ be a FOF β-covering of $\beth$ for some $\beta = \left(\rho_\beta(\hbar), \check{n}_\beta(\hbar), \nu_\beta(\hbar)\right)$. Then,*

$$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)} = \left(\widetilde{E}_k \in \widetilde{E} | \widetilde{E}_k(\hbar) \geq \beta, k = 1, ..., n\right) \tag{11}$$

   is called a FOF β-neighborhood of $\hbar$ in $\beth$.

Based on the above FOF β-neighborhood $\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}$, a crisp set called fractional orthotriple β-neighborhood (FOF β-neighborhood) is introduced as follows;

Definition 17. *Given that $\widetilde{E} = \left(\widetilde{E}_1, ..., \widetilde{E}_n\right)$ is a FOF β-covering on $\beth$. $\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}$ is a FOF β-neighborhood of $\hbar$ in $\beth$. For a FOFN $\beta = \left(\rho_\beta(\hbar), \check{n}_\beta(\hbar), \nu_\beta(\hbar)\right)$, if each $\hbar \in \beth$, FO β-neighborhood $\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}$ of $\hbar$ is defined as;*

$$\widehat{N}^{\beta}_{\widetilde{E}(\hbar)} = \left\{ z \in \beth : \rho_{\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}}(z) \geq \rho_\beta(\hbar), \check{n}_{\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}}(z) \leq \check{n}_\beta(\hbar), \nu_{\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}}(z) \leq \nu_\beta(\hbar) \right\} \tag{12}$$

.

Definition 18. *Let $\left(\beth, \widetilde{E}\right)$ be a FOFCAS. The conditional probability in which the object $\hbar$ belongs to $\hat{H}$ with respect to $\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}$, denoted by $P_r\left(\hat{H} | \widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ for every $\hat{H} \subseteq \beth$, is defined as;*

$$P_r\left(\hat{H} | \widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \frac{\left|\hat{H} \cap \widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right|}{\left|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right|} \tag{13}$$

Clearly, for all $\hbar \in \beth, 0 \leq P_r\left(\hat{H} | \widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq 1$.

Example 1. *Suppose that* $(\vartheta, \sqsupset)$ *be a FOFCAS and* $\widetilde{E} = \left(\widetilde{E}_1, ..., \widetilde{E}_4\right)$ *is a set of FOFSs,* $f \geq 4$, *where* $\sqsupset = (\hbar_1, ..., \hbar_5)$, $\beta = (0.8.0.6.0.4)$. *Details are shown in Table 1.*

Table 1. FOF β—covering $\widetilde{E}$ in Example 1.

	$\widetilde{E}_1$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{E}_4$
$\hbar_1$	$(0.9, 0.1, 0.2)$	$(0.7, 0.3, 0.5)$	$(0.6, 0.3, 0.5)$	$(0.8, 0.5, 0.2)$
$\hbar_2$	$(0.5, 0.3, 0.4)$	$(0.9, 0.3, 0.1)$	$(0.8, 0.1, 0.4)$	$(0.7, 0.2, 0.4)$
$\hbar_3$	$(0.9, 0.4, 0.3)$	$(0.3, 0.2, 0.1)$	$(0.8, 0.2, 0.4)$	$(0.5, 0.4, 0.6)$
$\hbar_4$	$(0.8, 0.1, 0.4)$	$(0.5, 0.2, 0.1)$	$(0.9, 0.3, 0.4)$	$(0.8, 0.3, 0.1)$
$\hbar_5$	$(0.8, 0.4, 0.3)$	$(0.9, 0.4, 0.3)$	$(0.7, 0.3, 0.1)$	$(0.4, 0.2, 0.1)$

Therefore, $\widetilde{E}$ *is FOF* $\beta-$*covering of* $\sqsupset$. *Then,* $\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_1)} = \widetilde{E}_1 \cap \widetilde{E}_4, \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_2)} = \widetilde{E}_2 \cap \widetilde{E}_3,$ $\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_3)} = \widetilde{E}_1 \cap \widetilde{E}_3, \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_4)} = \widetilde{E}_3 \cap \widetilde{E}_4, \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_5)} = \widetilde{E}_1 \cap \widetilde{E}_2.$

By calculations, we have the FOF $\beta-$*neighborhood* $\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}}$ *as shown in Table 2.*

Table 2. FOF β—neighborhood $\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}}$ in Example 1.

	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$
$\hbar_1$	$(0.8, 0.1, 0.2)$	$(0.5, 0.2, 0.4)$	$(0.5, 0.4, 0.6)$	$(0.8, 0.1, 0.4)$	$(0.4, 0.2, 0.3)$
$\hbar_2$	$(0.6, 0.3, 0.5)$	$(0.8, 0.1, 0.4)$	$(0.3, 0.2, 0.4)$	$(0.5, 0.2, 0.4)$	$(0.7, 0.3, 0.3)$
$\hbar_3$	$(0.6, 0.1, 0.5)$	$(0.5, 0.1, 0.4)$	$(0.8, 0.2, 0.4)$	$(0.8, 0.1, 0.4)$	$(0.7, 0.3, 0.3)$
$\hbar_4$	$(0.6, 0.3, 0.5)$	$(0.7, 0.1, 0.4)$	$(0.5, 0.2, 0.6)$	$(0.8, 0.3, 0.4)$	$(0.4, 0.2, 0.1)$
$\hbar_5$	$(0.7, 0.1, 0.5)$	$(0.5, 0.3, 0.4)$	$(0.3, 0.2, 0.3)$	$(0.5, 0.1, 0.4)$	$(0.8, 0.4, 0.3)$

$\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_1)} = (\hbar_1), \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_2)} = (\hbar_2), \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_3)} = (\hbar_3), \widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_4)} = (\hbar_1, \hbar_3, \hbar_4),$ $\widetilde{N}^{(0.8.0.6.0.4)}_{\widetilde{E}(\hbar_5)} = (\hbar_5)$

Let the decision set $\hat{H} = (\hbar_1, \hbar_4, \hbar_5)$, *so we have the conditional probability as;*

$$
\begin{aligned}
P_r\left(\hat{H}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_1)}\right) &= \frac{|(\hbar_1) \cap (\hbar_1, \hbar_4, \hbar_5)|}{|(\hbar_1)|} = 1, \\
P_r\left(\hat{H}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_2)}\right) &= \frac{|(\hbar_2) \cap (\hbar_1, \hbar_4, \hbar_5)|}{|(\hbar_2)|} = 0, \\
P_r\left(\hat{H}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_3)}\right) &= \frac{|(\hbar_3) \cap (\hbar_1, \hbar_4, \hbar_5)|}{|(\hbar_3)|} = 1, \\
P_r\left(\hat{H}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_4)}\right) &= \frac{|(\hbar_1, \hbar_3, \hbar_4) \cap (\hbar_1, \hbar_4, \hbar_5)|}{|(\hbar_1, \hbar_3, \hbar_4)|} = \frac{2}{3}, \\
P_r\left(\hat{H}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_5)}\right) &= \frac{|(\hbar_5) \cap (\hbar_1, \hbar_4, \hbar_5)|}{|(\hbar_5)|} = 1.
\end{aligned}
$$

5. FOF β-Covering Decision-Theoretic Rough Set Model

In this section, we discuss the loss function of DTRS with FOFNs in view of the new uncertainty measurement of FOFSs, and construct a FOFCDTRS as per Bayesian decision procedure [53,68,69].

According to the results of Liang and Liu [69] and Bayesian decision procedure, the q- ROFCDTRS consists of two states and three actions. The family of states is denoted by $\Gamma = (D, \neg D)$, which means that an object is in the state D or not in the state D. And, the collection of three actions is denoted by $\mathbb{Z} = (b_P, b_B, b_N)$, in which b_P, b_B and b_N stand for the three actions in classifying an object $\hbar$, namely, deciding $\hbar \in POS(D)$, deciding $\hbar \in BND(D)$ and deciding $\hbar \in NEG(D)$, respectively. At the moment, $POS(D), BND(D)$ and $NEG(D)$ correspond the decision rules of three-way decisions. Using the idea Liang and Liu [69] and Bayesian decision procedure, under the fractional orthotriple fuzzy information, We create a loss function matrix for the risk or cost of behavior in the various states. The results are given in Table 3.

Table 3. The loss function matrix with FOFNs.

	D
b_P	$\vartheta(\lambda_{PP}) = (\rho_\vartheta(\lambda_{PP}), \check{n}_\vartheta(\lambda_{PP}), \nu_\vartheta(\lambda_{PP}))$
b_B	$\vartheta(\lambda_{BP}) = (\rho_\vartheta(\lambda_{BP}), \check{n}_\vartheta(\lambda_{BP}), \nu_\vartheta(\lambda_{BP}))$
b_N	$\vartheta(\lambda_{NP}) = (\rho_\vartheta(\lambda_{NP}), \check{n}_\vartheta(\lambda_{NP}), \nu_\vartheta(\lambda_{NP}))$
	$\neg$***D***
b_P	$\vartheta(\lambda_{PN}) = (\rho_\vartheta(\lambda_{PN}), \check{n}_\vartheta(\lambda_{PN}), \nu_\vartheta(\lambda_{PN}))$
b_B	$\vartheta(\lambda_{BN}) = (\rho_\vartheta(\lambda_{BN}), \check{n}_\vartheta(\lambda_{BN}), \nu_\vartheta(\lambda_{BN}))$
b_N	$\vartheta(\lambda_{NN}) = (\rho_\vartheta(\lambda_{NN}), \check{n}_\vartheta(\lambda_{NN}), \nu_\vartheta(\lambda_{NN}))$

In Table 3, the loss function $\vartheta(\lambda_{\cdot\cdot})$ is FOFN $(\cdot = P; B; N)$. When the object $\hbar$ is in the state D, its loss degrees with FOFNs are $\vartheta(\lambda_{PP}), \vartheta(\lambda_{BP})$ and $\vartheta(\lambda_{NP})$ incurred for taking actions of b_P, b_B and b_N, correspondingly. In the same way, when the object $\hbar$ does not belong to D, its loss degrees with FOFNs are $\vartheta(\lambda_{PP}), \vartheta(\lambda_{BP})$ and $\vartheta(\lambda_{NP})$ incurred for taking the same actions. Utilizing the property of FOFN and the semantics of three-way decisions, the loss functions of Table 3, have the following relationship:

$$\rho_\vartheta(\lambda_{PP}) \leq \rho_\vartheta(\lambda_{BP}) < \rho_\vartheta(\lambda_{NP}); \tag{14}$$

$$\check{n}_\vartheta(\lambda_{NP}) \leq \check{n}_\vartheta(\lambda_{BP}) < \check{n}_\vartheta(\lambda_{PP}); \tag{15}$$

$$\nu_\vartheta(\lambda_{NP}) \leq \nu_\vartheta(\lambda_{BP}) < \nu_\vartheta(\lambda_{PP}); \tag{16}$$

$$\rho_\vartheta(\lambda_{NN}) \leq \rho_\vartheta(\lambda_{BN}) < \rho_\vartheta(\lambda_{PN}); \tag{17}$$

$$\check{n}_\vartheta(\lambda_{PN}) \leq \check{n}_\vartheta(\lambda_{BN}) < \check{n}_\vartheta(\lambda_{NN}); \tag{18}$$

$$\nu_\vartheta(\lambda_{PN}) \leq \nu_\vartheta(\lambda_{BN}) < \nu_\vartheta(\lambda_{NN}). \tag{19}$$

Proposition 1. *Using the relationship of loss functions (14)–(19), we can obtain the following results;*

$$\vartheta(\lambda_{PP}) \leq \vartheta(\lambda_{BP}) < \vartheta(\lambda_{NP}) \tag{20}$$

$$\vartheta(\lambda_{NN}) \leq \vartheta(\lambda_{BN}) < \vartheta(\lambda_{PN}) \tag{21}$$

From Proposition (1), Equation (20) shows that the loss of classifying the object $\hbar$ belonging to D into the positive region $POS(D)$ is less than or equal to the loss of classifying it into the boundary region $BND(D)$, and both of them are less than the loss of classifying $\hbar$ into the negative region $NEG(D)$. The relationship (21) can be explained in the same way.

Assume that $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ is the conditional probability in which the object $\hbar$ belonging to D is described by its FO β-neighborhood $\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}$. Then, there exists a relationship $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) + P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = 1$. Now, for every $\hbar \in \beth$, the corresponding expected losses $R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ $(\cdot = P, B, N)$ can be shown as;

$$R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \vartheta\left(\lambda_{PP}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \oplus \vartheta\left(\lambda_{PN}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right); \tag{22}$$

$$R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \vartheta\left(\lambda_{BP}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \oplus \vartheta\left(\lambda_{BN}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right); \tag{23}$$

$$R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \vartheta\left(\lambda_{NP}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \oplus \vartheta\left(\lambda_{NN}\right) P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right); \tag{24}$$

Proposition 2. *According to* $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) + P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = 1$, *Equations (22)–(24), can be expressed as follows;*

$$R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \left\{\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{PP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}, \\ \check{n}_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}, \\ \nu_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \end{array}\right) \\ \oplus \left(\begin{array}{c} \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{PP}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}, \\ \check{n}_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}, \\ \nu_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \end{array}\right) \end{array}\right\} \tag{25}$$

$$R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = \left\{\begin{array}{c} \left(\begin{array}{c} \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{BP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}, \check{n}_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}, \\ \nu_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \end{array}\right) \\ \oplus \left(\begin{array}{c} \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{BP}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}, \\ \check{n}_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}, \\ \nu_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \end{array}\right) \end{array}\right\} \tag{26}$$

$$R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\left(1-\rho_{\vartheta}\left(\lambda_{NP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}},\\ \check{n}_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)},\\ \nu_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\end{array}\right)\\ \oplus\left(\begin{array}{c}\left(1-\left(1-\rho_{\vartheta}\left(\lambda_{NP}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}},\\ \check{n}_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)},\\ \nu_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\end{array}\right)\end{array}\right\} \tag{27}$$

Proposition 3. *The expected losses $R\left(b_{\cdot}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ $(\cdot = P, B, N)$ are expressed as follows;*

$$R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)=\left\{\begin{array}{c}\left(1-\left(1-\rho_{\vartheta}\left(\lambda_{PP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{PN}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}},\\ \left(\check{n}_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}\left(\lambda_{PN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right),\\ \left(\nu_{\vartheta}\left(\lambda_{PP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}\left(\lambda_{PN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)\end{array}\right\} \tag{28}$$

$$R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)=\left\{\begin{array}{c}\left(1-\left(1-\rho_{\vartheta}\left(\lambda_{BP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{BN}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}},\\ \left(\check{n}_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}\left(\lambda_{BN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right),\\ \left(\nu_{\vartheta}\left(\lambda_{BP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}\left(\lambda_{BN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)\end{array}\right\} \tag{29}$$

$$R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)=\left\{\begin{array}{c}\left(1-\left(1-\rho_{\vartheta}\left(\lambda_{PP}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{PN}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}},\\ \left(\check{n}_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}\left(\lambda_{NN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right),\\ \left(\nu_{\vartheta}\left(\lambda_{NP}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}\left(\lambda_{NN}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)\end{array}\right\}. \tag{30}$$

As can be seen from Proposition (3), the following results hold.

Proposition 4. *Based on (28)–(30), the expected losses* $R\left(b_{\cdot}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)(\cdot = P, B, N)$ *are calculated as follows;*

$$\begin{aligned} R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) &= \left\{ \begin{array}{c} \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{\cdot P}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{\cdot N}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}, \\ \left(\check{n}_{\vartheta}\left(\lambda_{\cdot P}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}\left(\lambda_{\cdot N}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right), \\ \left(\nu_{\vartheta}\left(\lambda_{\cdot P}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}\left(\lambda_{\cdot N}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) \end{array} \right\} \\ &= \left(\rho_{\cdot}, \check{n}_{\cdot}, \nu_{\cdot}\right)(\cdot = P, B, N) \end{aligned}$$

We give the following minimum cost decision rules under FOF environment as per the Bayesian decision-making process;

(P). *If* $R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ *and* $R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$, *decide* $\hbar \in POS(D)$;

(B). *If* $R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ *and* $R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$, *decide* $\hbar \in BND(D)$;

(N). *If* $R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ *and* $R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$, *decide* $\hbar \in NEG(D)$;

where $R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right), R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ *and* $R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ *are FOFNs. According to the above results, the researches on the decision rules* $(P)-(N)$ *are further conducted by using (28)–(30), as per the operations of FOFNs.*

5.1. Decision-Making Analysis of FOFCDTRS

In Section 4, we construct a FOFCDTRS model. At the same time,the decision rules $(P)-(N)$ are put forward. Since the expected losses of FOFCDTRS cannot be directly compared, we need to further investigate the decision rules $(P)-(N)$ as per the operations of FOFNs. A FOFN characterized both by positive, neutral and negative, gives a way to calculate the decision problem with the positive, neutral and the negative viewpoints. In this section, we defined five methods to deduce Three-way decisions with FOFCDTRS.

5.1.1. Method 1: A Positive Viewpoint

For decision rules $(P)-(N)$, the expected losses $R\left(b_{\cdot}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = (\rho_{\cdot}, \check{n}_{\cdot}, \nu_{\cdot})(\cdot = P, B, N)$ are FOFNs. With regard to the positive viewpoint, we directly utilize the positive degree of FOFNs to represent the expected losses. When we compare the expected losses, the positive degree of the expected losses keep in step with them. According to this scenario, decision rules $(P)-(N)$ can be re-expressed as;

(P_1). If $\rho_P^f \leq \rho_B^f$ and $\rho_P^f \leq \rho_N^f$, decide $\hbar \in POS(D)$;

(B_1). If $\rho_B^f \leq \rho_P^f$ and $\rho_B^f \leq \rho_N^f$, decide $\hbar \in BND(D)$;

(N_1). If $\rho_N^f \leq \rho_P^f$ and $\rho_N^f \leq \rho_B^f$, decide $\hbar \in NEG(D)$;

where, $\rho_{\cdot} = \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{\cdot P}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{\cdot N}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}(\cdot = P, B, N)$. With the conditions (14) and (17), we simplify the decision rules $(P_1)-(N_1)$. For the rule (P_1), the first condition is expressed as:

$$
\begin{aligned}
\rho_P^f \leq \rho_B^f &\Leftrightarrow 1-\left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\leq 1-\left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow \left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\geq \left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow \log\left(\left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\geq \log\left(\left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\Leftrightarrow \log\left(\left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right)+\left(\log\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\geq \log\left(\left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right)+\left(\log\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\log\left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)+P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\log\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right) \\
&\geq P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\log\left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)+P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\log\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right) \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right)/\log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f*1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f*1-\rho_\vartheta\left(\lambda_{BP}\right)^f}\right)
\end{aligned}
$$

Similarly, the second condition of rule (P_1) can be expressed as:

$$
\begin{aligned}
\rho_P^f \leq \rho_N^f &\Leftrightarrow 1-\left(1-\rho_\vartheta\left(\lambda_{PP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{PN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\leq 1-\left(1-\rho_\vartheta\left(\lambda_{NP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{NN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right)/\log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f*1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f*1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right)
\end{aligned}
$$

The first condition of rule (B_1) is the converse of the first condition of rule (P_1). It follows,

$$
\rho_B^f \leq \rho_P^f \Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)\geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right)/\log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f*1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f*1-\rho_\vartheta\left(\lambda_{BP}\right)^f}\right).
$$

For the second condition of rule (B_1), we have;

$$\begin{aligned}
\rho_B^f &\leq \rho_N^f \Leftrightarrow 1-\left(1-\rho_\vartheta\left(\lambda_{BP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{BN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\leq 1-\left(1-\rho_\vartheta\left(\lambda_{NP}\right)^f\right)^{P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_\vartheta\left(\lambda_{NN}\right)^f\right)^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BP}\right)^f * 1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f * 1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right)
\end{aligned}$$

The first condition of rule (N_1) is the converse of the second condition of rule (P_1) and the second condition of rule (N_1) is the converse of the second condition of rule (B_1). It follows,

$$\begin{aligned}
\rho_N^f &\leq \rho_P^f \Leftrightarrow \\
P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) &\geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f * 1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f * 1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right) \\
\rho_N^f &\leq \rho_B^f \Leftrightarrow \\
P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) &\geq \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BP}\right)^f * 1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f * 1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right)
\end{aligned}$$

On basis of the derivation of decision rules $(P_1)-(N_1)$, we denote the three expressions in these conditions by the following three thresholds;

$$\alpha_1 = \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f * 1-\rho_\vartheta\left(\lambda_{BN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f * 1-\rho_\vartheta\left(\lambda_{BP}\right)^f}\right) \tag{31}$$

$$\beta_1 = \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{BP}\right)^f * 1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{BN}\right)^f * 1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right) \tag{32}$$

$$\gamma_1 = \log\left(\frac{1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f}\right) / \log\left(\frac{1-\rho_\vartheta\left(\lambda_{PP}\right)^f * 1-\rho_\vartheta\left(\lambda_{NN}\right)^f}{1-\rho_\vartheta\left(\lambda_{PN}\right)^f * 1-\rho_\vartheta\left(\lambda_{NP}\right)^f}\right) \tag{33}$$

Then, the decision rules $(P_1)-(N_1)$, can be re-expressed concisely as;

(P_1) . If $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \alpha_1$ and $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \gamma_1$, decide $\hbar \in POS\left(D\right)$;
(B_1) . If $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \leq \alpha_1$ and $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \beta_1$, decide $\hbar \in BND\left(D\right)$;
(N_1) . If $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \leq \beta_1$ and $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \leq \gamma_1$, decide $\hbar \in NEG\left(D\right)$;

From the positive viewpoint, we finally determine the decision rule of the object $\hbar$ by comparing the conditional probability $P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)$ and the thresholds $(\alpha_1, \beta_1, \gamma_1)$.

5.1.2. Method 2: A Neutral Viewpoint

For decision rules $(P)-(N)$, the expected losses are $R\left(b_\cdot|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) = (\rho_\cdot, \check{n}_\cdot, \nu_\cdot)$ $(\cdot = P, B, N)$. With regard to the neutral viewpoint, we straightly adopt the neutral degree of FOFNs to analyze decision rules $(P)-(N)$. Under this situation, the neutral degree of the expected losse have opposite directions with the expected losses. Following this scenario decision rules $(P)-(N)$ can be expressed as:

(P_2) . If $\check{n}_P^f \leq \check{n}_B^f$ and $\check{n}_P^f \leq \check{n}_N^f$, decide $\hbar \in POS\left(D\right)$;
(B_2) . If $\check{n}_B^f \leq \check{n}_P^f$ and $\check{n}_B^f \leq \check{n}_N^f$, decide $\hbar \in BND\left(D\right)$;
(N_2) . If $\check{n}_N^f \leq \check{n}_P^f$ and $\check{n}_N^f \leq \check{n}_B^f$, decide $\hbar \in NEG\left(D\right)$;

where, $\check{n}_{\cdot} = \check{n}_{\vartheta}(\lambda_{\cdot P})^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{\cdot N})^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\ (\cdot = P, B, N)$. Under conditions of (15) and (18), we simplify the decision rules $(P_2) - (N_2)$. For the rule (P_2), the first condition is expressed as:

$$\begin{aligned}
\check{n}^f_P \quad \leq \quad & \check{n}^f_B \Leftrightarrow \check{n}_{\vartheta}(\lambda_{PP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{PN})\, f.^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\geq \quad & \check{n}_{\vartheta}(\lambda_{BP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{BN})\, f.^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\Leftrightarrow \quad & \check{n}_{\vartheta}(\lambda_{PP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{PN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\geq \quad & \check{n}_{\vartheta}(\lambda_{BP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{BN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\Leftrightarrow \quad & \log\left(\check{n}_{\vartheta}(\lambda_{PP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{PN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) \\
\geq \quad & \log\left(\check{n}_{\vartheta}(\lambda_{BP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{BN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) \\
\Leftrightarrow \quad & \log\left(\check{n}_{\vartheta}(\lambda_{PP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) + \log\left(\check{n}_{\vartheta}(\lambda_{PN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) \\
\geq \quad & \log\left(\check{n}_{\vartheta}(\lambda_{BP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) + \log\left(\check{n}_{\vartheta}(\lambda_{BN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right) \\
\Leftrightarrow \quad & f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\log\left(\check{n}_{\vartheta}(\lambda_{PP})\right) + f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\log\left(\check{n}_{\vartheta}(\lambda_{PN})\right) \\
\geq \quad & f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\log\left(\check{n}_{\vartheta}(\lambda_{BP})\right) + f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\log\left(\check{n}_{\vartheta}(\lambda_{BN})\right) \\
\Leftrightarrow \quad & P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\check{n}_{\vartheta}(\lambda_{BN})}{\check{n}_{\vartheta}(\lambda_{PN})}\right) / \log\left(\frac{\check{n}_{\vartheta}(\lambda_{PP}) * \check{n}_{\vartheta}(\lambda_{BN})}{\check{n}_{\vartheta}(\lambda_{PN}) * \check{n}_{\vartheta}(\lambda_{BP})}\right)
\end{aligned}$$

Similarly, the second condition of rule (P_2) can be expressed as:

$$\begin{aligned}
\check{n}^f_P \quad \leq \quad & \check{n}^f_N \Leftrightarrow \check{n}_{\vartheta}(\lambda_{PP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{PN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\leq \quad & \check{n}_{\vartheta}(\lambda_{NP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{NN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\Leftrightarrow \quad & P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\check{n}_{\vartheta}(\lambda_{NN})}{\check{n}_{\vartheta}(\lambda_{PN})}\right) / \log\left(\frac{\check{n}_{\vartheta}(\lambda_{PP}) * \check{n}_{\vartheta}(\lambda_{NN})}{\check{n}_{\vartheta}(\lambda_{PN}) * \check{n}_{\vartheta}(\lambda_{NP})}\right)
\end{aligned}$$

The first condition of rule (B_2) is the converse of the first condition of rule (P_2). It follows,

$$\check{n}^f_B \leq \check{n}^f_P \Leftrightarrow P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\check{n}_{\vartheta}(\lambda_{BN})}{\check{n}_{\vartheta}(\lambda_{PN})}\right) / \log\left(\frac{\check{n}_{\vartheta}(\lambda_{PP}) * \check{n}_{\vartheta}(\lambda_{BN})}{\check{n}_{\vartheta}(\lambda_{PN}) * \check{n}_{\vartheta}(\lambda_{BP})}\right).$$

For the second condition of rule (B_2), we have;

$$\begin{aligned}
\check{n}^f_B \quad \leq \quad & \check{n}^f_N \Leftrightarrow \check{n}_{\vartheta}(\lambda_{BP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{BN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\leq \quad & \check{n}_{\vartheta}(\lambda_{NP})^{f.P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \check{n}_{\vartheta}(\lambda_{NN})^{f.P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)} \\
\Leftrightarrow \quad & P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\check{n}_{\vartheta}(\lambda_{NN})}{\check{n}_{\vartheta}(\lambda_{BN})}\right) / \log\left(\frac{\check{n}_{\vartheta}(\lambda_{BP}) * \check{n}_{\vartheta}(\lambda_{NN})}{\check{n}_{\vartheta}(\lambda_{BN}) * \check{n}_{\vartheta}(\lambda_{NP})}\right)
\end{aligned}$$

The first condition of rule (N_2) is the converse of the second condition of rule (P_2) and the second condition of rule (N_2) is the converse of the second condition of rule (B_2). It follows,

$$\begin{aligned} \check{n}_N^f &\leq \check{n}_P^f \Leftrightarrow \\ P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) &\geq \log\left(\frac{\check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\check{n}_\vartheta\left(\lambda_{PP}\right) * \check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right) * \check{n}_\vartheta\left(\lambda_{NP}\right)}\right) \\ \check{n}_N^f &\leq \check{n}_B^f \Leftrightarrow \\ P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) &\geq \log\left(\frac{\check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{BN}\right)}\right) / \log\left(\frac{\check{n}_\vartheta\left(\lambda_{BP}\right) * \check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{BN}\right) * \check{n}_\vartheta\left(\lambda_{NP}\right)}\right) \end{aligned}$$

For the decision rules $(P_2) - (N_2)$, the three thresholds in these conditions are deduced as follows;

$$\alpha_2 = \log\left(\frac{\check{n}_\vartheta\left(\lambda_{BN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\check{n}_\vartheta\left(\lambda_{PP}\right) * \check{n}_\vartheta\left(\lambda_{BN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right) * \check{n}_\vartheta\left(\lambda_{BP}\right)}\right) \tag{34}$$

$$\beta_2 = \log\left(\frac{\check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{BN}\right)}\right) / \log\left(\frac{\check{n}_\vartheta\left(\lambda_{BP}\right) * \check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{BN}\right) * \check{n}_\vartheta\left(\lambda_{NP}\right)}\right) \tag{35}$$

$$\gamma_2 = \log\left(\frac{\check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\check{n}_\vartheta\left(\lambda_{PP}\right) * \check{n}_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right) * \check{n}_\vartheta\left(\lambda_{NP}\right)}\right) \tag{36}$$

Then, the decision rules $(P_2) - (N_2)$, can be re-expressed concisely as;

(P_2) . If $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \geq \alpha_2$ and $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \geq \gamma_2$, decide $\hbar \in POS\left(D\right)$;
(B_2) . If $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \leq \alpha_2$ and $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \geq \beta_2$, decide $\hbar \in BND\left(D\right)$;
(N_2) . If $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \leq \beta_2$ and $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) \leq \gamma_2$, decide $\hbar \in NEG\left(D\right)$;

From the neutral viewpoint, we finally determine the decision rule of the object $\hbar$ by comparing the conditional probability $P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right)$ and the thresholds $(\alpha_2, \beta_2, \gamma_2)$.

5.1.3. Method 3: A Negative Viewpoint

For decision rules $(P) - (N)$, the expected losses are $R\left(b_\cdot|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right) = (\rho_\cdot, \check{n}_\cdot, \nu_\cdot)\ (\cdot = P, B, N)$. With regard to the negative viewpoint, we straightly adopt the negative degree of FOFNs to analyze decision rules $(P) - (N)$. Under this situation, the negative degree of the expected losses have opposite directions with the expected losses. Following this scenario decision rules $(P) - (N)$ can be expressed as:

(P_3) . If $\nu_P^f \leq \nu_B^f$ and $\nu_P^f \leq \nu_N^f$, decide $\hbar \in POS\left(D\right)$;
(B_3) . If $\nu_B^f \leq \nu_P^f$ and $\check{n}_B^f \leq \nu_N^f$, decide $\hbar \in BND\left(D\right)$;
(N_3) . If $\nu_N^f \leq \nu_P^f$ and $\nu_N^f \leq \nu_B^f$, decide $\hbar \in NEG\left(D\right)$;

where, $\nu_\cdot = \nu_\vartheta\left(\lambda_{\cdot P}\right)^{P_r\left(D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right)} \nu_\vartheta\left(\lambda_{\cdot N}\right)^{P_r\left(\neg D|\widehat{N}_{\widetilde{E}(\hbar)}^{\beta}\right)}\ (\cdot = P, B, N)$. Under conditions of (16) and (19), we simplify the decision rules $(P_3) - (N_3)$. For the rule (P_3), the first condition is written as:

$$\begin{aligned}
\nu_P^f &\leq \nu_B^f \Leftrightarrow \nu_\vartheta\left(\lambda_{PP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{PN}\right) f.^{P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\geq \nu_\vartheta\left(\lambda_{BP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{BN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow \nu_\vartheta\left(\lambda_{PP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{PN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\geq \nu_\vartheta\left(\lambda_{BP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{BN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow \log\left(\nu_\vartheta\left(\lambda_{PP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{PN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\geq \log\left(\nu_\vartheta\left(\lambda_{BP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{BN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\Leftrightarrow \log\left(\nu_\vartheta\left(\lambda_{PP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) + \log\left(\nu_\vartheta\left(\lambda_{PN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\geq \log\left(\nu_\vartheta\left(\lambda_{BP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) + \log\left(v_\vartheta\left(\lambda_{BN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)}\right) \\
&\Leftrightarrow f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \log\left(\nu_\vartheta\left(\lambda_{PP}\right)\right) + f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \log\left(\nu_\vartheta\left(\lambda_{PN}\right)\right) \\
&\geq f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \log\left(\nu_\vartheta\left(\lambda_{BP}\right)\right) + f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \log\left(\nu_\vartheta\left(\lambda_{BN}\right)\right) \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{BP}\right)}\right)
\end{aligned}$$

Similarly, the second condition of rule (P_3) can be expressed as:

$$\begin{aligned}
\nu_P^f &\leq \nu_N^f \Leftrightarrow \nu_\vartheta\left(\lambda_{PP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{PN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\leq \nu_\vartheta\left(\lambda_{NP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \check{n}_\vartheta\left(\lambda_{NN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\check{n}_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right)
\end{aligned}$$

The first condition of rule (B_3) is the converse of the first condition of rule (P_3). It follows,

$$\nu_B^f \leq \nu_P^f \Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{BP}\right)}\right).$$

For the second condition of rule (B_3), we have;

$$\begin{aligned}
\nu_B^f &\leq \nu_N^f \Leftrightarrow \nu_\vartheta\left(\lambda_{BP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{BN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\leq \nu_\vartheta\left(\lambda_{NP}\right)^{f.P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \nu_\vartheta\left(\lambda_{NN}\right)^{f.P_r\left(\neg D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right)} \\
&\Leftrightarrow P_r\left(D|\widehat{N}^\beta_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{BP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right)
\end{aligned}$$

The first condition of rule (N_3) is the converse of the second condition of rule (P_3) and the second condition of rule (N_3) is the converse of the second condition of rule (B_3). It follows,

$$\nu_N^f \leq \nu_P^f \Leftrightarrow P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right)$$
$$\nu_N^f \leq \nu_B^f \Leftrightarrow P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{BP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right)$$

For the decision rules $(P_3) - (N_3)$, the three thresholds in these conditions are deduced as follows;

$$\alpha_3 = \log\left(\frac{\nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{BN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{BP}\right)}\right) \tag{37}$$

$$\beta_3 = \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{BP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{BN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right) \tag{38}$$

$$\gamma_3 = \log\left(\frac{\nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right)}\right) / \log\left(\frac{\nu_\vartheta\left(\lambda_{PP}\right) * \nu_\vartheta\left(\lambda_{NN}\right)}{\nu_\vartheta\left(\lambda_{PN}\right) * \nu_\vartheta\left(\lambda_{NP}\right)}\right) \tag{39}$$

Then, the decision rules $(P_3) - (N_3)$, can be re-expressed concisely as;

(P_3). If $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \alpha_3$ and $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \gamma_3$, decide $\hbar \in POS(D)$;
(B_3). If $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq \alpha_3$ and $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \geq \beta_3$, decide $\hbar \in BND(D)$;
(N_3). If $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq \beta_3$ and $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq \gamma_3$, decide $\hbar \in NEG(D)$;

From the negative viewpoint, we finally drive the decision rule of the object $\hbar$ by comparing the conditional probability $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ and the thresholds $(\alpha_3, \beta_3, \gamma_3)$.

5.1.4. Method 4–5: Based on Composite Viewpoint

With regards to Method 1, 2 and 3, it merely uses the positive neutral and negative degrees of FOFNs to generate decision rules with the positive viewpoint. From the Example 2, we find the inconsistency of Method 1, 2 and 3. For solving this problem, we required to synchronously consider the positive degree, neutral degree and the negative degree of FOFNs, which is known as a composite viewpoint. In order to compare the expected losses $R\left(b_{\cdot}|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) = (\rho_{\cdot}, \check{n}_{\cdot}, \nu_{\cdot})\ (\cdot = P, B, N)$, we introduce three different functions that compare the size of FOFNs. The first one is the score and the accuracy function, the second one is closeness index. These two methods are introduced as follows:

Method 4

In light of Definition (12), the score functions of the expected losses can be obtained as follows;

$$S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_P(\hbar)^f - \check{n}_P(\hbar)^f - \nu_P(\hbar)^f \tag{40}$$

$$S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_B(\hbar)^f - \check{n}_B(\hbar)^f - \nu_B(\hbar)^f \tag{41}$$

$$S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_N(\hbar)^f - \check{n}_N(\hbar)^f - \nu_N(\hbar)^f \tag{42}$$

where $\rho_{\cdot} = \left(1-\left(1-\rho_{\vartheta}\left(\lambda_{\cdot P}\right)^{f}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}\left(\lambda_{\cdot N}\right)^{f}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}$, $\check{n}_{\cdot} = \check{n}_{\vartheta}\left(\lambda_{\cdot P}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}\left(\lambda_{\cdot N}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}$ and $\nu_{\cdot} = \nu_{\vartheta}\left(\lambda_{\cdot P}\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}\left(\lambda_{\cdot N}\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}$ $(\cdot = P, B, N)$. Meanwhile, the accuracy functions of the expected losses can also be computed:

$$H\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_P\left(\hbar\right)^{f} + \check{n}_P\left(\hbar\right)^{f} + \nu_P\left(\hbar\right)^{f} \tag{43}$$

$$H\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_B\left(\hbar\right)^{f} + \check{n}_B\left(\hbar\right)^{f} + \nu_B\left(\hbar\right)^{f} \tag{44}$$

$$H\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \rho_N\left(\hbar\right)^{f} + \check{n}_N\left(\hbar\right)^{f} + \nu_N\left(\hbar\right)^{f} \tag{45}$$

For the rule (P), the first condition $R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ implies the following prerequisites:

$$\begin{aligned}
(C_1).S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_2).S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}$$

In the same way, the prerequisites for the second condition $R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right) \leq R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ of rule (P) are

$$\begin{aligned}
(C_3).S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_4).S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}$$

For the rule (B), we have

$$\begin{aligned}
(C_5).S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_6).S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}$$

$$\begin{aligned}
(C_7).S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_8).S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}$$

And for the rule (N), we have

$$\begin{aligned}
(C_9).S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_{10}).S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
(C_{11}).S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &< S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
(C_{12}).S\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &= S\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \wedge H\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)\\
&\leq H\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)
\end{aligned}
$$

Therefore, the decision rules $(P)-(N)$ can be re-expressed as $(P_4)-(N_4)$

(P_4). If $((C_1)\vee(C_2))\wedge((C_3)\vee(C_4))$, decide $\hbar \in POS(D)$;
(B_4). If $((C_5)\vee(C_6))\wedge((C_7)\vee(C_8))$, decide $\hbar \in POS(D)$;
(N_4). If $((C_9)\vee(C_{10}))\wedge((C_{11})\vee(C_{12}))$, decide $\hbar \in POS(D)$.

Method 5
In light of Definition (15). the closeness index of the expected losses can be determined as follows;

$$\Im\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \frac{1}{3}\left(1+\rho_P(\hbar)^f - \check{n}_P(\hbar)^f - \nu_P(\hbar)^f\right) \tag{46}$$

$$\Im\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \frac{1}{3}\left(1+\rho_B(\hbar)^f - \check{n}_B(\hbar)^f - \nu_B(\hbar)^f\right) \tag{47}$$

$$\Im\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) = \frac{1}{3}\left(1+\rho_N(\hbar)^f - \check{n}_N(\hbar)^f - \nu_N(\hbar)^f\right) \tag{48}$$

where $\rho_{\cdot} = \left(1-\left(1-\rho_{\vartheta}(\lambda_{\cdot P})^f\right)^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\left(1-\rho_{\vartheta}(\lambda_{\cdot N})^f\right)^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\right)^{\frac{1}{f}}$, $\check{n}_{\cdot} = \check{n}_{\vartheta}(\lambda_{\cdot P})^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\check{n}_{\vartheta}(\lambda_{\cdot N})^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}$ and $\nu_{\cdot} = \nu_{\vartheta}(\lambda_{\cdot P})^{P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}\nu_{\vartheta}(\lambda_{\cdot N})^{P_r\left(\neg D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)}$ $(\cdot = P, B, N)$.

Therefore, the decision rules $(P)-(N)$ can be expressed as $(P_5)-(N_5)$:
(P_5). If

$$
\begin{aligned}
\Im\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \text{ and}\\
\Im\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right), \text{ decide } \hbar \in POS(D);
\end{aligned}
$$

(B_5). If

$$
\begin{aligned}
\Im\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \text{ and}\\
\Im\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right), \text{ decide } \hbar \in BND(D);
\end{aligned}
$$

(N_5). If

$$
\begin{aligned}
\Im\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) \text{ and}\\
\Im\left(R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right) &\leq \Im\left(R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right), \text{ decide } \hbar \in NEG(D).
\end{aligned}
$$

6. Algorithm for the Multi-Attribute Decision Making with FOFCDTRSs

Input Decision-making table with FOF information and loss functions with FOFNs for risk or cost of actions in different states;

Step 1. Obtain FOF β-neighborhood $\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}$ and FO β-neighborhood $\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}$ from the given decision-making table with FOF information by using Definitions (1) and (17);

Step 2. Calculate the conditional probability $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ by the Formula (13).

Step 3. Give loss function with FOFNs for risk or cost of actions in different states, and then calculate the values of the thresholds $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ according to Formulas (31)–(39), respectively.

Step 4. Obtain the expected losses $R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ $(\cdot = P, B, N)$ by using the Formulas (28)–(30). According to the Formulas (40)–(45) and (46)–(48), we further acquire the values of the score and the accuracy function $H\left(R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)$ and the closeness index function $\Im\left(R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)\right)$.

Step 5. Based on the five methods in Section 5, the corresponding decision rules are used to calculate the positive domain POS (D), negative domain NEG (D) and boundary domain BND (D), respectively.

Step 6. Find and compare the optimal decision results.

6.1. An Illustrative Example

In this section , we will present the proposed MADM method based on FOFS models related to the evaluation and rank of heavy rainfall in the district of Lasbella district and adjoining areas of the Baluchistan, Pakistan.

A recent storm caused a spell of heavy rainfall in the Lasbella district, and adjoining areas of Baluchistan, Pakistan were hit with unprecedented flash floods in February 2019. In this flood a large number of roads which link the district of Lasbella with other parts of Baluchistan were destroyed. In this flood a large number of roads which link the district of Lasbella with other parts of Baluchistan were destroyed.

Such projects were carried out by a small number of well-established contractors, and the selection process was based solely on the tender price. In recent years, rising project complexity, technological capability, higher performance, security and financial requirements have demanded the use of multi-attribute decision-making methods. Pakistan's government has released a newspaper notice for this, and one construction company is responsible for choosing the best construction firm from a selection of six potential alternatives, $\hbar_1$ = Ahmed Construction, $\hbar_2$ = Matracon Pakistan Private (Pvt) Limited (Ltd), $\hbar_3$ = Eastern Highway Company, $\hbar_4$ = Banu Mukhtar Concrete Pvt. Ltd., $\hbar_5$ = Khyber Grace Pvt. Ltd., $\hbar_6$ = Experts Engineering services on the basis of the attributes, $\widetilde{E}_1$ = Technical capability, $\widetilde{E}_2$ = Higher performance, $\widetilde{E}_3$ = Safety, $\widetilde{E}_4$ = Financial requirements, $\widetilde{E}_5$ = Time saving, that is bid for these projects, and all criteria are of the type of benefit, so no need to normalized it. Then the Government's goal is to choose among them the best construction company for the task. Hence, as shown below, the following decision matrix was constructed given in Table 4:

Table 4. A tabular representation of FOFSs for $\widetilde{E}$.

	$\widetilde{E}_1$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{E}_4$	$\widetilde{E}_5$
$\hbar_1$	$(0.9, 0.1, 0.2)$	$(0.8, 0.2, 0.5)$	$(0.7, 0.3, 0.5)$	$(0.8, 0.2, 0.5)$	$(0.9, 0.1, 0.3)$
$\hbar_2$	$(0.8, 0.2, 0.4)$	$(0.3, 0.4, 0.5)$	$(0.7, 0.5, 0.3)$	$(0.6, 0.2, 0.1)$	$(0.5, 0.6, 0.2)$
$\hbar_3$	$(0.9, 0.3, 0.1)$	$(0.6, 0.3, 0.5)$	$(0.7, 0.2, 0.4)$	$(0.3, 0.4, 0.1)$	$(0.5, 0.4, 0.6)$
$\hbar_4$	$(0.8, 0.1, 0.5)$	$(0.6, 0.2, 0.7)$	$(0.5, 0.3, 0.4)$	$(0.7, 0.3, 0.2)$	$(0.8, 0.4, 0.3)$
$\hbar_5$	$(0.6, 0.5, 0.2)$	$(0.9, 0.4, 0.3)$	$(0.5, 0.3, 0.7)$	$(0.3, 0.2, 0.1)$	$(0.8, 0.5, 0.2)$
$\hbar_6$	$(0.8, 0.3, 0.5)$	$(0.6, 0.3, 0.1)$	$(0.9, 0.3, 0.2)$	$(0.6, 0.2, 0.3)$	$(0.5, 0.1, 0.4)$

Now, take the threshold $\beta = (0.6, 0.5, 0.4)$, then $\widetilde{E}$ is a FOF β-covering. Then, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_1)} = \widetilde{E}_1 \cap \widetilde{E}_5$, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_2)} = \widetilde{E}_1 \cap \widetilde{E}_3 \cap \widetilde{E}_4$, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_3)} = \widetilde{E}_1 \cap \widetilde{E}_3$, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_4)} = \widetilde{E}_4 \cap \widetilde{E}_5$, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_5)} = \widetilde{E}_1 \cap \widetilde{E}_2 \cap \widetilde{E}_5$, $\widetilde{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_6)} = \widetilde{E}_2 \cap \widetilde{E}_3 \cap \widetilde{E}_4$

According to the Definition (1), we get the FOF β-neighborhood as shown in Table 5.

Assume that the decision makers gives a evaluation threshold $\beta = (0.6, 0.5, 0.4)$. As a result, based on Table 5, and Equation (12), we have $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_1)} = (\hbar_1)$, $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_2)} = (\hbar_2, \hbar_3)$, $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_3)} = (\hbar_3)$, $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_4)} = (\hbar_4)$, $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_5)} = (\hbar_1, \hbar_2, \hbar_5)$, $\widehat{N}^{(0.6,0.5,0.4)}_{\widetilde{E}(\hbar_6)} = (\hbar_6)$.

Let the state set $D = (\hbar_1, \hbar_3, \hbar_5)$. By Equation (13), we have $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_1)}\right) = 1$, $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_2)}\right) = \frac{1}{2}$, $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_3)}\right) = 1$, $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_4)}\right) = 0$, $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_5)}\right) = \frac{2}{3}$, $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar_6)}\right) = 0$.

Table 5. A tabular representation of FOF β-neighborhood.

$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}$	$\hbar_1$	$\hbar_2$	$\hbar_3$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_1)}$	$(0.9, 0.1, 0.3)$	$(0.5, 0.2, 0.4)$	$(0.5, 0.3, 0.6)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_2)}$	$(0.7, 0.1, 0.5)$	$(0.6, 0.2, 0.4)$	$(0.3, 0.2, 0.4)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_3)}$	$(0.7, 0.1, 0.5)$	$(0.7, 0.2, 0.4)$	$(0.7, 0.2, 0.4)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_4)}$	$(0.8, 0.1, 0.5)$	$(0.5, 0.2, 0.2)$	$(0.3, 0.4, 0.6)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_5)}$	$(0.9, 0.1, 0.5)$	$(0.3, 0.2, 0.5)$	$(0.5, 0.3, 0.6)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_6)}$	$(0.7, 0.2, 0.5)$	$(0.3, 0.2, 0.5)$	$(0.3, 0.2, 0.5)$
	$\hbar_4$	$\hbar_5$	$\hbar_6$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_1)}$	$(0.8, 0.1, 0.5)$	$(0.6, 0.5, 0.2)$	$(0.5, 0.1, 0.5)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_2)}$	$(0.5, 0.1, 0.5)$	$(0.6, 0.5, 0.2)$	$(0.6, 0.2, 0.5)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_3)}$	$(0.5, 0.1, 0.5)$	$(0.3, 0.2, 0.7)$	$(0.8, 0.3, 0.5)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_4)}$	$(0.7, 0.3, 0.3)$	$(0.3, 0.2, 0.2)$	$(0.5, 0.1, 0.4)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_5)}$	$(0.6, 0.1, 0.7)$	$(0.6, 0.4, 0.3)$	$(0.5, 0.1, 0.5)$
$\widetilde{N}^{\beta}_{\widetilde{E}(\hbar_6)}$	$(0.5, 0.2, 0.7)$	$(0.3, 0.2, 0.7)$	$(0.6, 0.2, 0.3)$

Assume that the loss function for risk or cost of functions in different states D and $\neg D$ are in Table 6.

Table 6. The loss function 1 matrix in this Example.

	D	$\neg D$
b_P	$\vartheta(\lambda_{PP}) = (0, 0.1, 0.9)$	$\vartheta(\lambda_{PN}) = (0.85, 0.7, 0.1)$
b_B	$\vartheta(\lambda_{BP}) = (0.4, 0.4, 0.5)$	$\vartheta(\lambda_{BN}) = (0.7, 0.6, 0.3)$
b_N	$\vartheta(\lambda_{NP}) = (0.85, 0.8, 0.1)$	$\vartheta(\lambda_{NN}) = (0.02, 0.5, 0.75)$

Let $RP(\hbar) = R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$, $RB(\hbar) = R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ and $RN(\hbar) = R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$. Based on the Table 6 and Equations (28)–(30), we can get the expected losses $R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)(\cdot = P, B, N)$, which are shown in Table 7.

Table 7. Expected losses $R\left(b.|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)(\cdot = P, B, N)$.

	$RP(\hbar)$	$RB(\hbar)$	$RN(\hbar)$
$\hbar_1$	$(0, 0.1, 0.9)$	$(0.4, 0.4, 0.5)$	$(0.85, 0.8, 0.1)$
$\hbar_2$	$(0.745, 0.269, 0.299)$	$(0.529, 0.489, 0.387)$	$(0.746, 0.632, 0.274)$
$\hbar_3$	$(0, 0.1, 0.9)$	$(0.4, 0.4, 0.5)$	$(0.85, 0.8, 0.1)$
$\hbar_4$	$(0.85, 0.7, 0.1)$	$(0.7, 0.6, 0.3)$	$(0.02, 0.5, 0.75)$
$\hbar_5$	$(0.683, 0.199, 0.432)$	$(0.497, 0.458, 0.422)$	$(0.789, 0.684, 0.195)$
$\hbar_6$	$(0.85, 0.7, 0.1)$	$(0.7, 0.6, 0.3)$	$(0.02, 0.5, 0.75)$

In what follows, we shall adopt the above five decision methods to deal with this problem.

6.1.1. Decision-Making Based on Method 1

According to the Equations (31)–(33), we calculate the thresholds $\alpha_1, \beta_1, \gamma_1$, respectively. Concretely, $\alpha_1 = 0.889, \beta_1 = 0.321, \gamma_1 = 0.499$. Based on the Method 1, according to the decision rules $(P_1) - (N_1)$, we have $POS(D) = (\hbar_3)$, $BND(D) = (\hbar_1, \hbar_2, \hbar_5)$, $NEG(D) = (\hbar_4, \hbar_6)$.

6.1.2. Decision-Making Based on Method 2

According to the Equations (34)–(36), we calculate the thresholds $\alpha_2, \beta_2, \gamma_2$, respectively. Concretely, $\alpha_2 = 1, \beta_2 = 0.208, \gamma_2 = 0.139$. Based on the Method 2, according to the decision rules $(P_2) - (N_2)$, we have
$POS(D) = (\hbar_3)\,, BND(D) = (\hbar_1, \hbar_2, \hbar_5)\,, NEG(D) = (\hbar_4, \hbar_6)\,.$

6.1.3. Decision-Making Based on Method 3

According to the Equations (37)–(39), we calculate the thresholds $\alpha_3, \beta_3, \gamma_3$, respectively. Concretely, $\alpha_2 = 0.652, \beta_2 = 0.363, \gamma_2 = 0.478$. Based on the Method 3, according to the decision rules $(P_3) - (N_3)$, we have $POS(D) = (\hbar_3)\,, BND(D) = (\hbar_1, \hbar_{2,\hbar_5})\,, NEG(D) = (\hbar_4, \hbar_6)\,.$

6.1.4. Decision-Making Based on Method 4

Let $RP\,(\hbar) = R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right), RB\,(\hbar) = R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ and $RN\,(\hbar) = R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$. Based on the Method 4, according to the Equations (40)–(45), we calculate the score and accuracy function of expected losses, respectively. And the result are shown in Table 8.

Table 8. The score and accuracy functions of expected losses in this Example.

	$S\,(RP\,(\hbar))$	$S\,(RB\,(\hbar))$	$S\,(RN\,(\hbar))$	$H\,(RP\,(\hbar))$	$H\,(RB\,(\hbar))$	$H\,(RN\,(\hbar))$
$\hbar_1$	−0.656	−0.063	0.121	0.656	0.113	0.923
$\hbar_2$	0.295	−0.002	0.141	0.321	0.157	0.465
$\hbar_3$	−0.656	−0.063	0.121	0.656	0.113	0.923
$\hbar_4$	0.282	0.104	−0.064	0.762	0.378	0.064
$\hbar_5$	0.181	−0.015	0.151	0.253	0.136	0.606
$\hbar_6$	0.282	0.104	−0.064	0.762	0.378	0.064

So, according to the decision rules $(P_4) - (N_4), POS(D) = (\hbar_1, \hbar_3)\,, BND(D) = (\hbar_2, \hbar_5)\,, NEG(D) = (\hbar_4, \hbar_6)\,.$

6.1.5. Decision-Making Based on Method 5

Let $RP\,(\hbar) = R\left(b_P|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right), RB\,(\hbar) = R\left(b_B|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ and $RN\,(\hbar) = R\left(b_N|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$. Based on the Method 5, according to the Equations (46)–(48), we calculate the closeness index of the expected losses, respectively. And the result are shown as Table 9.

Table 9. The closeness index in this Example.

	$\Im\,(RP\,(\hbar))$	$\Im\,(RB\,(\hbar))$	$\Im\,(RN\,(\hbar))$
$\hbar_1$	0.115	0.312	0.373
$\hbar_2$	0.432	0.309	0.378
$\hbar_3$	0.115	0.312	0.373
$\hbar_4$	0.427	0.368	0.312
$\hbar_5$	0.394	0.328	0.389
$\hbar_6$	0.427	0.368	0.312

So, according to the decision rules $(P_5) - (N_5)$, we have $POS(D) = (\hbar_3)\,, BND(D) = (\hbar_1, \hbar_2, \hbar_5)\,, NEG(D) = (\hbar_4, \hbar_6)$

In the following, we orderly used five methods of Section 6 for deriving Three way-decsions.

(1) The contraction company selection with Method 1, 2, 3 (Positive, neutral and negative viwepoint): With the aid of the general method of Section 6, we first compute the FOF β-neighborhood $\widetilde{N}^{\beta}_{\widetilde{E}(\hbar)}$ and FO β-neighborhood $\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}$ from the given decision-making Table 4 by using the Definitions (1) and (17). We also, calculate the conditional probability $P_r\left(D|\widehat{N}^{\beta}_{\widetilde{E}(\hbar)}\right)$ by the Equation (13), and give loss function with FOFNs for risk or cost of actions in different states.

After that, find the values of the thresholds $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ according to Formulsa (31)–(39), respectively. On the basis of the decision rules $(P_1) - (N_1)$ to $(P_3) - (N_3)$, we can judge the corresponding decision rule for each company. At the moment, using the decision rules $(P_1) - (N_1)$ to $(P_3) - (N_3)$, we can predict that $(\hbar_3) \in POS(D), (\hbar_1, \hbar_2, \hbar_5) \in BND(D)$ and $(\hbar_4, \hbar_6) \in NEG(D)$.

(2) The contraction company selection with Method 4 (Score and Accuracy function): Based on the Method 4, according to the Equations (40)–(45), we calculate the score and accuracy function of expected losses, respectively. Then, using the decision rules $(P_4) - (N_4)$, we can predict that $(\hbar_1, \hbar_3) \in POS(D), (\hbar_2, \hbar_5) \in BND(D)$ and $(\hbar_4, \hbar_6) \in NEG(D)$.

(3) The contraction company selection with Method 5 (Closness index): We use the ranking method of the closness index function for the selection of construction company. Based on the Method 5, according to the Equations (46)–(48), we calculate the closeness index of the expected losses, respectively. Then, using the decision rules $(P_5) - (N_5)$, we can predict that $(\hbar_3) \in POS(D), (\hbar_1, \hbar_2) \in BND(D)$ and $(\hbar_4, \hbar_5, \hbar_6) \in NEG(D)$.

6.1.6. Sensitivity Analysis

When the $\beta = (0.6, 0.5, 0.4)$ and $f = 3$. The decision result of multi-attribute decision making will change with the change of loss function. Assume that there are three different loss functions as shown in Tables 6, 10 and 11. The decision results obtained by five methods under different loss functions are shown in Table 12 as follows.

Table 10. The loss function 2 matrix.

	D	$\neg D$
b_P	$\vartheta(\lambda_{PP}) = (0.3, 0.1, 0.7)$	$\vartheta(\lambda_{PN}) = (0.3, 0.2, 0.4)$
b_B	$\vartheta(\lambda_{BP}) = (0.4, 0.3, 0.3)$	$\vartheta(\lambda_{BN}) = (0.5, 0.4, 0.1)$
b_N	$\vartheta(\lambda_{NP}) = (0.1, 0.8, 0.1)$	$\vartheta(\lambda_{NN}) = (0.7, 0.3, 0.2)$

Table 11. The loss function 3 matrix.

	D	$\neg D$
b_P	$\vartheta(\lambda_{PP}) = (0.6, 0.2, 0.3)$	$\vartheta(\lambda_{PN}) = (0.3, 0.5, 0.1)$
b_B	$\vartheta(\lambda_{BP}) = (0.5, 0.1, 0.4)$	$\vartheta(\lambda_{BN}) = (0.4, 0.6, 0.3)$
b_N	$\vartheta(\lambda_{NP}) = (0.2, 0.7, 0.2)$	$\vartheta(\lambda_{NN}) = (0.2, 0.1, 0.7)$

According to the two different loss functions described in Tables 6, 10 and 11, we can use the five different decision methods in Section 5 to get different decision results, as shown in Table 12.

Table 12. Comparison of decision results of five methods (LF = loss function).

Loss Function	Method	$POS(D)$	$BND(D)$	$NEG(D)$
The LF-1 in Table 6	Method 1	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 2	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 3	$(\hbar_1, \hbar_3)$	$(\hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 4	$(\hbar_3)$	$(\hbar_1, \hbar_2,)$	$(\hbar_4, \hbar_5, \hbar_6)$
	Method 5	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_3)$	$(\hbar_4, \hbar_6)$
The LF-2 in Table 10	Method 1	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 2	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 3	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5, \hbar_6)$	$(\hbar_4)$
	Method 4	$(\hbar_1, \hbar_3)$	$(\hbar_2, \hbar_4)$	$(\hbar_5, \hbar_6)$
	Method 5	$(\hbar_3)$	$(\hbar_1, \hbar_2)$	$(\hbar_4, \hbar_5, \hbar_6)$
The LF- 3 in Table 11	Method 1	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 2	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$
	Method 3	$(\hbar_3)$	$(\hbar_1, \hbar_2)$	$(\hbar_4, \hbar_6)$
	Method 4	$(\hbar_3)$	$(\hbar_1, \hbar_2, \hbar_5)$	$(\hbar_4 \hbar_6)$
	Method 5	$(\hbar_3)$	$(\hbar_2, \hbar_5)$	$(\hbar_4, \hbar_6)$

It can be seen from Table 12, that on the basis of loss function $1, 2, 3$ the decision results of the five methods are the same, but only changes occur in the method 4 on the basis of loss function 3, thn the decision results of loss function 1 and loss function 2. Thus, Eastern Highway Company $\hbar_3$ is the best construction company for the selection of project.

6.1.7. Comparison and Analysis

To elaborate the validity and practicability of the created method in this essay, we conduct a collection of comparative analyzing with other previous decision methodologies including the method based upon covering-based Spherical fuzzy rough set Model hybrid with TOPSIS method proposed by Zeng et al. [77], the method based upon Spherical fuzzy Dombi aggregation operators proposed by Ashraf et al. [78], Spherical aggregation operators proposed by Ashraf and Abdullah [79] and the method based upon Spherical fuzzy Graphs proposed by Akram et al. [80]. We utilize these methods to cope with the Example in this paper, the score values and ranking of alternatives are displayed in Table 13. From it, we can attain the same sorting results of alternatives based on the previous methods and the designed method in this article, which can demonstrate the effectiveness of the propounded methods.

Table 13. Comparison Information.

Approaches	Score Value of Alternative						Ranking
	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$	$\hbar_6$	
Zeng et al. [77]	0.027	0.024	0.062	0.017	0.010	0.009	$\hbar_3 > \hbar_1 > \hbar_2 > \hbar_4 > \hbar_5 > \hbar_6$
Ashraf et al. [78]	0.603	0.520	0.823	0.391	0.476	0.314	$\hbar_3 > \hbar_1 > \hbar_2 > \hbar_5 > \hbar_4 > \hbar_6$
Ashraf & Abdullah [79]	0.293	0.463	0.537	0.235	0.114	0.079	$\hbar_3 > \hbar_2 > \hbar_1 > \hbar_4 > \hbar_5 > \hbar_6$
Akram et al. [80]	1.734	1.498	1.893	1.528	1.384	1.272	$\hbar_3 > \hbar_1 > \hbar_4 > \hbar_2 > \hbar_5 > \hbar_6$

It is noteworthy that the class of FOFSs extends the classes of PFSs and SFSs. Thus, it can express vague information more flexibly and accurately with increasing fraction. When $f = 1$, this model reduces to the PF model, and when $f = 2$, it becomes the SF model. Thus, a wider range of uncertain information can be expressed using the methods proposed in this paper, which are closer to real decision-making. This helps us to deal with MCDM problems and to sketch real scenarios more accurately. Hence our approach towards MCDM is more flexible and generalized, which provides a vast space of acceptable triplets given by decision-makers, according to the different attitudes, as compared to the PF model.

7. Conclusions

The FOFCDTRS model is an important tool in real life for handling uncertainties. In this paper, we combine the loss functions of DTRSs with CFOFSs in the fractional orthotriple fuzzy context. Therefore, a new approach is adopted to fractional orthotriple fuzzy sets with notions of covering rough set to presented the new method of FOFCDTRS through FOF β-neighborhoods. Then, we propose a new FOFCDTRS model and elaborate its respective properties. We set out five methods for resolving the predicted loss in the form of FOFNs and extract the related three-way decisions. At the same time we present an algorithm based on FOFCDTRSs for decision making with multiple attributes. Through example analysis, it is proved that the five methods proposed are correct and effective. Among them, Methods $1, 2$ and 3 have better stability.

In the next researches, we mainly focus on the following topics: (1) Extend FOFCDTRSs to the multi-period situation. (2) The application of FOFCDTRSs in big data processing and analysis. (3) Use the developed concept on new multi-attribute evaluation models to deal with fuzziness and uncertainty in multiple criteria decision- making topics such as planning choices, construction options, site selection, and decision-making problem in many other areas.

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