mathematics

Review

History, Developments and Open Problems on Approximation Properties

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Abstract: In this paper, we give a comprehensive review of the classical approximation property. Then, we present some important results on modern variants, such as the weak bounded approximation property, the strong approximation property and p-approximation property. Most recent progress on E-approximation property and open problems are discussed at the end.

Keywords: Banach space; unconditional basis; approximation property

1. Classical Approximation Property

The paper was intended to give a short yet comprehensive review for the classical approximation property and variants of it. The first section is about classical approximation property, and some of the statements of the theorems are chosen from the beautiful review given by Casazza [1]. The second section focuses on the weak bounded approximation property and the strong approximation property introduced by Lima and Oja [2,3]. Besides the results of Lima and Oja, we also selected some of the key theorems from our earlier paper [4] which solved one of Oja's conjectures. In the last section, we begin with the recent results on the p-approximation property and then discuss the most recent progress on the brand new E-approximation property.

The earliest recorded appearance of the approximation property was in Banach's book [5]. In 1955, Grothendieck [6] first studied approximation property systematically. The approximation property is closely related to another fundamental property of Banach spaces, namely the basis property. We start from here.

Definition 1. *A sequence of vectors (x_i) in a Banach space X is a Schauder basis for X if every vector x in X has a unique representation*

$$x = \sum a_i x_i,$$

where the summation converges in norm.

Many classical Banach spaces have bases, such as ℓ_p $(1 \le p < \infty)$, c_0 or L_p $(1 \le p < \infty)$. So a natural question is

Question 1. *Does every separable Banach space have a basis?*

This question was answered negatively by P. Enflo [7] in 1973. He constructed a Banach space that fails a weaker property—the approximation property.

Definition 2. *Let X be a Banach space. We say that X has the approximation property (AP) if for every $\epsilon > 0$ and every compact set $K \subset X$, there exists a finite rank operator T on X so that $\|Tx - x\| < \epsilon$ for every $x \in K$, where $\mathcal{F}(X)$ is the space of finite rank operators on X.*

In the same year, Pelczynski and Figiel [8] built Banach spaces without the approximation property by simplifying Enflo's construction. The simplification avoided using the deep theorem of Grothendieck that reflexive spaces with the approximation property have the bounded approximation property. Later, Szarek [9] constructed a Banach space X with a finite dimensional decomposition which does not have a basis.

Question 2. *Given a separable Banach space, can you embed it into a superspace with a basis?*

Theorem 1 (Banach)**.** *Every separable Banach space embeds isometrically into $C[0,1]$.*

Although every separable Banach space embeds into a superspace with a basis, there are separable Banach spaces which never embed complementably in a space with a basis. There is a simple criterion due to Pelczynski and Wojtaszczyk [10] which was improved later by Pelczynski [11] and Johnson, Rosenthal and Zippin [12] for checking when a Banach space is complemented in a Banach space with a basis.

Theorem 2. *A Banach space X is isomorphic to a complemented subspace of a Banach space with a basis if and only if X has the bounded approximation property.*

Definition 3. *A Banach space X is said to have the bounded approximation property (BAP) if there exists a $\lambda \geq 1$ so that for every compact set K in X and every $\epsilon > 0$, there is a $T \in \mathcal{F}(X)$ so that $\|Tx - x\| < \epsilon$ and $\|T\| \leq \lambda$ for every $x \in K$, where $\mathcal{F}(X)$ is the space of finite rank operators on X.*

It was shown in [12] that if the dual of X has a basis, then X itself has a basis. Actually, in this situation X^* has a boundedly complete basis and X has a shrinking basis, but the converse fails. To see this, we need a result of Lindenstrauss [13].

Theorem 3 (Lindenstrauss)**.** *If X is a separable Banach space, there is a separable Banach space Z with Z^{**} having a boundedly complete basis and Z^{**}/Z is isomorphic to X. Moreover, Z^{***} is isomorphic to $Z^* \oplus X^*$.*

Corollary 1. *There exists a Banach space X with a basis such that X^* fails the approximation property and is separable.*

Proof. We take a separable reflexive Banach space X failing the approximation property and choose Z as in the theorem above for X^*. Then Z^{***} is isomorphic to $Z^* \oplus X^{**} = Z^* \oplus X$ and fails the approximation property. □

Grothendieck [6] systematically studied the variants of the approximation property. One important tool was the topology of uniform convergence on compact sets. Let τ be the topology of uniform convergence on compact sets on the space of bounded linear operators $L(X,Y)$. It is generated by the seminorms $\|T\|_K = \sup\{\|Tx\| : x \in K\}$, where K ranges over compact subsets of X. We write $(L(X,Y),\tau)$ for this space. Grothendieck characterized the dual space of $(L(X,Y),\tau)$.

Theorem 4 (Grothendieck)**.** *The continuous linear functionals on $(L(X,Y),\tau)$ consist of all functionals Φ of the form*

$$\Phi(T) = \sum_{i=1}^{\infty} y_i^*(Tx_i), x_i \in X, y_i^* \in X^*, \sum_{i=1}^{\infty} \|x_i\|\|y_i^*\| < \infty.$$

Grothendieck also gave a set of equivalent conditions for the approximation property.

Theorem 5 (Grothendieck)**.** *For a Banach space X, the following are equivalent:*

(1) X has the approximation property;
(2) There is a net $(T_\lambda)_{\lambda\in\Lambda}$ of finite rank operators on X converging to the identity operator on X with respect to τ;
(3) $\mathcal{F}(Y,X)$ is dense in $(L(Y,X),\tau), \forall Y$;
(4) $\mathcal{F}(X,Y)$ is dense in $(L(X,Y),\tau), \forall Y$;
(5) For every sequence of $(x_i)\in X$ and $(x_i^*)\in X^*$ such that $\sum\|x_i\|\|x_i^*\|<\infty$ and $\sum x_i^*(x)x_i=0$, for all $x\in X$, we have $\sum x_i^*(x_i)=0$;
(6) For every Banach space Y, every compact operator $T\in L(Y,X)$ and $\epsilon>0$, there is a $T_1\in\mathcal{F}(Y,X)$ such that $\|T-T_1\|<\epsilon$.

An immediate consequence of the theorem above is that if X does not have the approximation property, then we can find a separable subspace $Y\subset X$ so that every space Z sitting between X and Z fails the approximation property. Another consequence is the following result.

Theorem 6. *X^* has the approximation property implies that X has the approximation property. Therefore when X is reflexive space, X^* has the approximation property if and only if X has the approximation property.*

If we change the order of X and Y in (6), we get another theorem of Grothendieck.

Theorem 7. *Let X be a Banach space. Then, X^* has the approximation property if and only for every Banach space Y, every $\epsilon>0$ and every compact operator $T\in L(X,Y)$, there is a finite rank operator $T_1: X\to Y$ such that $\|T-T_1\|<\epsilon$.*

After Enflo constructed the first Banach space failing the approximation property, a number of important examples followed up. Figiel [14] and Davie [15,16] showed that there are subspaces of $\ell_p, p>2$ which fail the approximation property and Szankowski [17] then found subspaces of $\ell_p, 1\le p<2$ failing the approximation property. All the proofs of these results used a general criterion of Enflo, which characterizes Banach spaces failing the approximation. Actually this criterion characterizes Banach spaces failing a weaker property—the compact approximation property.

Definition 4. *Let X be a Banach space. We say that X has the compact approximation property (CAP) if for every $\epsilon>0$ and for every compact set $K\subset X$, there exists a compact operator T on X so that $\|Tx-x\|<\epsilon$ for every $x\in K$, where $\mathcal{K}(X)$ is the space of compact operators on X.*

Szankowski produced a fantastic set of examples of Banach spaces without the approximation property. The first example was the construction of a Banach lattice [18] failing the approximation property.

Theorem 8 (Szankowski)**.** *Let $1\le r<p<\infty$. Then, there exists a sublattice of $\ell_p(L_r[0,1])$ failing the compact approximation property.*

Szankowski's next construction [17] was of subspaces of ℓ_p $(p\neq 2)$ failing the approximation property. The importance of his construction is the explicit representation of the subspace in ℓ_p.

Theorem 9 (Szankowski)**.** *Let $1\le p\neq 2$. Then, there exists a subspace of ℓ_p failing the compact approximation property.*

Szankowski [19], using another spectacular construction, solved the following important problem.

Theorem 10. *For any infinite dimensional Hilbert space H, $B(H)$ fails the approximation property.*

This space was very natural and is different from the previous constructed examples. Later, Godefroy and Saphar [20] showed that $B(H)/K(H)$ also fails the approximation property.

Theorem 11. *The following conditions for a Banach space X are equivalent:*

(1) *X has the bounded approximation property;*
(2) *There is a net of finite rank operators (T_α) on X which tends strongly to the identity on X and satisfy $\sup_\alpha\{\|T_\alpha\|\} < \infty$;*
(3) *There exists a $\lambda \geq 1$ so that for every finite dimensional subspace E of X, there is a finite rank operator T on X such that $Tx = x$ and $\|T\| \leq \lambda, \forall x \in E$.*

For separable Banach spaces, we have the following corollary.

Corollary 2. *Let X be a separable Banach space. Then, X has the λ-bounded approximation property if and only if there exists a sequence of operators $(T_n) \subset \mathcal{F}(X)$ converging to the identity in the strong operator topology so that $\limsup_n \|T_n\| \leq \lambda$ and $T_m T_n = T_n$ for all $n < m$.*

Definition 5. *A Banach space X is said to have the λ-duality bounded approximation property if for every $\epsilon > 0$ and every pair of finite dimensional subspaces F of X^* and E of X, there exists an operator $T \in \mathcal{F}(X)$ with $\|Tx - x\| \leq \epsilon\|x\|$, for all $x \in E$, $\|T\| \leq \lambda$, and $\|T^*x^* - x^*\| \leq \epsilon\|x^*\|$ for all $x^* \in F$.*

In [21], Johnson proved the following result.

Theorem 12 (Johnson)**.** *If X^* has the λ-bounded approximation property, then X has the λ-duality bounded approximation property.*

The next theorem was given by Grothendieck [6].

Theorem 13 (Grothendieck)**.** *A separable dual space with the approximation property has the metric approximation property.*

Using the theorem above, a result of Johnson [22] and a result of Lindenstrauss [23], we have the following corollary.

Corollary 3. *Every reflexive Banach space with the approximation property has the metric approximation property.*

In [22], Johnson obtained a characterization of the bounded approximation property for non-separable spaces.

Theorem 14 (Johnson)**.** *A non-separable Banach space X has the λ-bounded approximation property if and only if every separable subspace of X is contained in a separable subspace with the λ-bounded approximation property.*

The following is a result of Lindenstauss [23].

Theorem 15 (Lindenstrauss)**.** *Every reflexive Banach space has the separable complementation property with norm one projections.*

Question 3. *If X is a (non-separable) dual space with the approximation property, does X have the metric approximation property?*

Figiel and Johnson [24] proved the following:

Theorem 16 (Figiel, Johnson)**.** *Let $1 \leq \lambda < \infty$. If X is a Banach space which has the λ-bounded approximation property in every equivalent norm, then X^* has $2\lambda(1+4\lambda)$-bounded approximation property.*

In the case of the metric approximation property, the theorem above was strengthened by Johnson [22].

Theorem 17 (Johnson)**.** *If X has the metric approximation property in every equivalent norm, then X^* has the metric approximation property.*

Corollary 4. *Let X be a Banach space with a separable dual. Then the following are equivalent:*

(1) *X^* has the approximation property;*
(2) *X^* has the metric approximation property;*
(3) *X has the metric approximation property in all equivalent norms.*

With a little more efforts, we get another consequence of the theorems above.

Corollary 5.

(1) *There exists a Banach space with the bounded approximation property but fails the metric approximation property.*
(2) *There exists a separable Banach space with the approximation property but fails the bounded approximation property.*

Proof. (1) Let X be a separable Banach space with a basis whose dual is separable and fails the approximation property. Then, for every $n \in \mathbb{N}$, there is an equivalent norm $|\cdot|_n$ on X so that $(X, |\cdot|_n)$ fails the n-bounded approximation property. Then, $(X, |\cdot|_n)$ has the bounded approximation property and fails the metric approximation property.

(2) Let $Y = (\oplus_n (X, |\cdot|_n))_{\ell_2}$. Then, Y has the approximation property and fails the bounded approximation property. □

Here is another important open problem.

Question 4. *If a Banach space X has the bounded approximation property, does X have the metric approximation property in an equivalent norm?*

2. Weak BAP and Strong AP

A long standing open problem in the study of approximation property is whether the BAP and the AP are equivalent for dual spaces.

Question 5. *Are the BAP and AP equivalent for dual spaces?*

In order to attack the problem, Oja and Lima introduced the strong AP and weak BAP which sit between AP and BAP. In 2008, Oja [3] introduced the strong approximation property.

Definition 6. *A Banach space X is said to have the strong approximation property (strong AP) if for every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$, there exists a bounded net (U_α) in $\mathcal{F}(X, Y)$ such that $\|U_\alpha x - Rx\| \longrightarrow 0$ for every $x \in X$.*

Lima and Oja [2] then introduced a weaker form of the BAP.

Definition 7. *For $\lambda \geq 1$, a Banach space X is said to have the weak λ-bounded approximation property (weak λ-BAP) if for every Banach space Y and every weakly compact operator $R : X \to Y$, $id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda\|R\|\}}^{\tau_c}$.*

Theorem 18 (Oja). *If X^* has the AP, then X has the strong AP. If X has the strong AP, then X has the AP.*

Theorem 19 (Lima, Oja). *If X has the λ-BAP, then X has the weak λ-BAP. If X has the weak BAP, then X has the strong AP.*

The formal implications between these approximation properties are:

$$\lambda\text{-BAP} \Longrightarrow \text{weak } \lambda\text{-BAP} \Longrightarrow \text{strong AP} \Longrightarrow \text{AP}.$$

Theorem 20 (Lima and Oja [2]). *The weak metric approximation property and the approximation property are equivalent for every dual space.*

Theorem 21 (Oja [25]). *The λ-bounded approximation property and the weak λ-bounded approximation property are equivalent for every Banach space with separable dual.*

Lima and Oja [2] believed that the weak BAP and the BAP are not equivalent and Oja [3] conjectured that the strong AP and the weak BAP are not equivalent.

Oja's Conjecture. *There exists a Banach space which has the strong AP but fails the weak BAP.*

However, the authors [4] gave a negative answer to Oja's conjecture.

Theorem 22 (Kim and Zheng). *The weak BAP and the strong AP are equivalent for every Banach space.*

One of the main tools to prove the above theorem is the following:

Theorem 23 (Kim and Zheng). *Suppose that X^{**} or Y^* has the Radon–Nikodym property. If X has the strong AP, then, for every $R \in \mathcal{L}(X, Y)$, there exists a $\lambda_R > 0$ such that $id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

The Radon–Nikodym property in the assumption is needed since we used a representation of the dual space of compact operators by Feder and Saphar [26].

Theorem 24 (Feder and Saphar). *Suppose that X^{**} or Y^* has the Radon–Nikodym property. If $f \in (\mathcal{K}(X, Y), \|\cdot\|)^*$, then for every $\varepsilon > 0$, there exist sequences (y_n^*) in Y^* and (x_n^{**}) in X^{**} with $\sum_n \|y_n^*\|\|x_n^{**}\| < \|f\| + \varepsilon$ such that*

$$f(U) = \sum_n x_n^{**}(U^* y_n^*) \ \text{ for } U \in \mathcal{K}(X, Y).$$

Lemma 1 (Kim and Zheng). *Let X be a Banach space. Then, the following statements are equivalent:*

- *(a) X has the strong AP;*
- *(b) For every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$, there exists a $\lambda_R > 0$ such that $R \in \overline{\{RS : S \in \mathcal{F}(X), \|RS\| \leq \lambda_R\}}^{\tau_c}$;*
- *(c) For every Banach space Y and every $R \in \mathcal{K}(X, Y)$, there exists a $\lambda_R > 0$ such that $id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

Lemma 2. *Let X be a Banach space and $\lambda > 0$. Then, X has the weak λ-BAP if and only if for every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$,*

$$R \in \overline{\{RS : S \in \mathcal{F}(X), \|RS\| \leq \lambda\|R\|\}}^{\tau_c}.$$

Proof of the Main Theorem. Suppose that X fails the weak BAP. Then, by the above lemma, for every $m \in \mathbb{N}$, there exist a separable reflexive Banach space Y_m and $R_m \in \mathcal{K}(X, Y_m)$ such that

$$R_m \notin \overline{\{R_m S : S \in \mathcal{F}(X), \|R_m S\| \leq m^2 \|R_m\|\}}^{\tau_c}.$$

We may assume $\|R_m\| = 1$ for all m. Now, we define the map $R : X \to (\sum_m \oplus Y_m)_{c_0}$ by

$$Rx = \left(\frac{R_m}{m}(x)\right)_{m=1}^{\infty}.$$

Then, the map is well defined and linear, and it is regular to check that $R \in \mathcal{K}(X, (\sum_m \oplus Y_m)_{c_0})$. By the assumption and our previous lemma, there exists a $\lambda_R > 0$ such that

$$id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}.$$

Hence for every m, we have

$$R_m \in \overline{\{R_m S : S \in \mathcal{F}(X), \|RS\| \leq \lambda_R\}}^{\tau_c} \subset \overline{\{R_m S : S \in \mathcal{F}(X), \|R_m S\| \leq m\lambda_R\}}^{\tau_c},$$

which is a contradiction.

□

Recall that the formal implications between these approximation properties are:

$$\text{BAP} \Longrightarrow \text{weak-BAP} \Longrightarrow \text{strong AP} \Longrightarrow \text{AP}.$$

Figiel and Johnson [24] first showed that there is a Banach space with the AP but fails the BAP. Then Figiel, Johnson and Pelczynski [27] proved that c_0 and ℓ_1 have subspaces with the AP but failing the BAP. In 2015, this result was improved by Chen, Kim and Zheng [28].

Theorem 25. *Each of c_0 and ℓ_1 has a subspace which has the AP but fails the weak BAP.*

The proof of the above theorem uses the following result whose proof involves certain properties of the weak BAP and some of the techniques in [27].

Theorem 26 (Chen, Kim and Zheng)**.** *If X has the AP but X^* fails the AP, then there exists a subspace Y of $(\sum_n X)_{c_0}$ (or $(\sum_n X)_{\ell_p}$ $(1 \leq p < \infty)$) such that Y has the AP but it fails to have the weak BAP.*

Figiel, Johnson and Pelczy ński [27] defined the bounded approximation property of pairs.

Definition 8. *Let Y be a subspace of a Banach space X and $\lambda \geq 1$. (X, Y) is said to have the λ-BAP if for every finite-dimensional subspace F of X and every $\varepsilon > 0$ there exists an $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda + \varepsilon$ such that $Sx = x$ for all $x \in F$ and $S(Y) \subset Y$.*

The pair (X, X) has the λ-BAP if and only if X has the λ-BAP [12]. This notion was first extended to the bounded compact approximation property for pairs by Chen and Zheng [29]. Then Chen, Kim and Zheng [28] defined a weaker notion of the bounded approximation property of pairs.

Definition 9. The pair (X, Y) is said to have the weak λ-BAP if for every Banach space Z and $R \in \mathcal{W}(X, Z)$, for every finite-dimensional subspace F of X and for every $\varepsilon > 0$, there exists an $S \in \mathcal{F}(X)$ with $\|RS\| \leq (\lambda + \varepsilon)\|R\|$ such that $Sx = x$ for all $x \in F$ and $S(Y) \subset Y$.

Theorem 27. *Let X be a Banach space. Then the following statements are equivalent.*

(a) $id_{X^*} \in \overline{\mathcal{F}^*(X)}^{\tau_c}$;
(b) *For every finite-codimensional subspace Y of X, the pair (X, Y) has the weak λ-BAP;*
(c) *There exists a $\lambda \geq 1$ satisfying that for every finite-codimensional subspace W of X, there exists a finite-codimensional subspace Y of X with $Y \subset W$ such that the pair (X, Y) has the weak λ-BAP.*

Lemma 3. *Let X be a Banach space. Let Y be a finite-codimensional subspace of X. If Y has the weak λ-BAP, then the pair (X, Y) has the weak 3λ-BAP.*

Corollary 6. *Let X be a Banach space. If $id_{X^*} \notin \overline{\mathcal{F}^*(X)}^{\tau_c}$, then for any $\lambda \geq 1$, there exists a finite-codimensional subspace Y_λ of X such that Y_λ does not have the weak λ-BAP.*

3. Banach Approximation Property

Compactness is a topological property which also has geometric characterizations in Banach spaces. One of the most famous such characterizations is probably the Grothendieck's criterion [6], which can be stated as the following:

A subset K of a Banach space X is relatively compact if and only if for every $\varepsilon > 0$, there exists $(x_n)_n \in c_0(X)$ with $\|(x_n)_n\|_\infty \leq \sup_{x \in K} \|x\| + \varepsilon$ such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_1} \right\},$$

where $c_0(X)$ is the space of sequences in X which converge to 0 in norm. We denote by B_Z the closed unit ball of a Banach space Z.

Motivated by this, Sinha and Karn [30] introduced the notion of p-compact sets. Let $1 \leq p < \infty$ and let $1/p + 1/p^* = 1$. A subset K of X is said to be *p-compact* if there exists $(x_n)_n \in \ell_p(X)$, which is the Banach space with the norm $\|\cdot\|_p$ of all X-valued absolutely p-summable sequences, such that

$$K \subset p\text{-}co(x_n)_n := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}} \right\}.$$

Every p-compact set is relatively compact.

An operator $T : Y \to X$ is called p-compact if there exists $(x_n)_n \in \ell_p(X)$ such that $T(B_Y) \subset p\text{-}co(x_n)_n$. The set of p-compact operators from Y into X is denoted by $\mathcal{K}_p(Y, X)$ and the $\mathcal{K}_p$-norm is given by

$$\|T\|_{\mathcal{K}_p} := \inf \{ \|(x_n)_n\|_p : (x_n)_n \in \ell_p(X), T(B_Y) \subset p\text{-}co(x_n)_n \}.$$

Delgado, Piñeiro and Serrano [31,32] showed that $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal.

The concept of p-compact sets naturally leads to that of the p-AP.

Definition 10. *For $1 \leq p \leq \infty$, a Banach space X is said to have the p-AP if for every p-compact subset K of X and every $\epsilon > 0$, there exists a finite rank operator T on X such that $\sup_{x \in K} \|Tx - x\| \leq \epsilon$.*

In fact, the ∞-AP means the AP. We can see easily that if X has the q-AP, then X has the p-AP for $1 \leq p < q \leq \infty$. An interesting result given by Sinha and Karn [30] is that every Banach space has the 2-AP and hence has the p-AP for every $1 \leq p \leq 2$. In 2009, Delgado, Oja, Pineiro and Serrano [33] proved that if X^{**} has the p-AP, then X has the p-AP.

One year later, Choi and Kim [34] got the following nice result which is parallel to the result for AP.

Theorem 28 (Choi and Kim)**.** *Let X be a Banach space and $2 < p < \infty$. If X^* has the p-AP, then so does X.*

One of the crucial tools in their proof is a representation theorem of the dual of $(\mathcal{L}(X,Y),\tau_p)$ which is an analogue of Grothendieck's representation theorem.

Theorem 29 (Choi and Kim)**.** *Let $1 < p < \infty$. Then $(\mathcal{L}(X,Y),\tau_p)^*$ consists of all linear functionals f of the form*

$$f(T) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n^j y_j^*(Tx_n),$$

where $(x_n) \in \ell_p(X), z_j = (\lambda_n^j)_{n=1}^{\infty} \in \ell_{p^*}$ for each $j \in \mathbb{N}$ and $(y_j^*) \in Y^*$ with $\sum_{j=1}^{\infty} \|z_j\|_{p^*}\|y_j^*\| < \infty$.

In the same paper, Choi and Kim showed that the converse of the theorem is not true in general. They first used the Davie space [15] to prove the following:

Theorem 30 (Choi and Kim)**.** *Let $2 < p < \infty$. There exists a subspace ℓ_q which fails the p-AP whenever $q > 2p/(p-2)$.*

As a consequence of Theorems 30 and 3, Choi and Kim gave a new result, as we can see in the next theorem.

Theorem 31 (Choi and Kim)**.** *For every $2 < p < \infty$, there exists a separable Banach space X_p such that X_p^{**} has a basis but X_p^{***} fails the p-AP.*

Most recently, Kim and Zheng [35] introduced a more general type of approximation property called the E-approximation property and its variants. In the same paper, characterizations of these properties were given and representation theorems for the dual of $\mathcal{L}(X,Y)$ under the topology of uniform convergence on E-compact sets and E^u-compact sets will also be presented.

Let E be a Banach space with a normalized basis $(e_n)_n$ and $(e_n^*)_n$ denote the sequence of coordinate functionals. Let X be a Banach space and define

$$E^w(X) := \Big\{(x_n)_n \text{ in } X : \sum_{n=1}^{\infty} x^*(x_n)e_n \text{ converges in } E \text{ for each } x^* \in X^*\Big\},$$

$$E^u(X) := \Big\{(x_n)_n \text{ in } X : \lim_{l\to\infty}\sup_{x^*\in B_{X^*}} \Big\|\sum_{n\geq l} x^*(x_n)e_n\Big\|_E = 0\Big\},$$

$$E(X) := \Big\{(x_n)_n \text{ in } X : \sum_{n=1}^{\infty} \|x_n\|e_n \text{ converges in } E\Big\}.$$

Clearly, $E^u(X) \subset E^w(X)$ and if $(e_n)_n$ is 1-unconditional, then $E(X) \subset E^u(X)$. A subset K of X is said to be E^u*-compact* if there exists $(x_n)_n \in E^u(X)$ such that

$$K \subset E\text{-}co(x_n)_n := \Big\{\sum_{n=1}^{\infty} e^*(e_n)x_n : e^* \in B_{E^*}\Big\}.$$

When $(e_n)_n$ is 1-unconditional, K is said to be E*-compact* if there exists $(x_n)_n \in E(X)$ such that $K \subset E\text{-}co(x_n)_n$.

Definition 11. *A Banach space X is said to have the E^u-approximation property (E^u-AP) (respectively, E-AP whenever $(e_n)_n$ is 1-unconditional) if for E^u-compact subset (respectively, E-compact subset) K of X and every $\epsilon > 0$, there exists an $S \in \mathcal{F}(X,X)$ such that*

$$\sup_{x\in K} \|x - Sx\| \leq \epsilon.$$

It follows from the definition that the p-AP is simply the ℓ_p-AP and

$$\text{AP} \Leftrightarrow c_0\text{-AP} \Leftrightarrow c_0^u\text{-AP}.$$

The following fact can be verified easily from the definition.

Lemma 4. *Every E^u-compact set is a relatively compact set. If the basis (e_n) for E is 1-unconditional, then every E-compact set is E^u-compact.*

We denote by τ_{E^u} (respectively, τ_E whenever $(e_n)_n$ is 1-unconditional) the locally convex topology on the space $\mathcal{L}(X,Y)$ of all operators from a Banach space X to a Banach space Y defined by the seminorms $\sup_{x\in K}\|Tx\|$, where the supremums are taken over all E^u-compact (respectively, E-compact) subsets of X. Then, a Banach space X has the E^u-AP (respectively, E-AP) is equivalent to

$$id_X \in \overline{\mathcal{F}(X,X)}^{\tau_{E^u}} \text{ (respectively, } id_X \in \overline{\mathcal{F}(X,X)}^{\tau_E}).$$

When $(e_n)_n$ is 1-unconditional, since every E-compact set is an E^u-compact set, τ_{E^u} is stronger than τ_E.

An operator $R : Y \to X$ is said to be E^u*-compact* (respectively, E*-compact* whenever $(e_n)_n$ is 1-unconditional) if $R(B_Y)$ is an E^u-compact (respectively, E-compact) subset of X. We denote by $\mathcal{K}_{E^u}(Y,X)$ (respectively, $\mathcal{K}_E(Y,X)$) the collection of all E^u-compact (respectively, E-compact) operators from Y to X. Also, we let

$$\|R\|_{\mathcal{K}_{E^u}} := \inf\{\|(x_n)_n\|_{E^w(X)} : R(B_Y) \subset E\text{-}co(x_n)_n\}$$

for $R \in \mathcal{K}_{E^u}(Y,X)$. The following theorems characterize E^u-AP and E-AP using E^u-compact and E-compact operators [35].

Theorem 32 (Kim, Zheng)**.** *The following statements are equivalent for a Banach space X.*

(a) *X has the E^u-AP;*
(b) *For every Banach space Y and every $R \in \mathcal{K}_{E^u}(Y,X)$,*

$$R \in \overline{\{SR : S \in \mathcal{F}(X,X)\}}^{\|\cdot\|_{\mathcal{K}_{E^u}}}$$

;
(c) *For every quotient space Y of E^* and every injective $R \in \mathcal{K}_{E^u}(Y,X)$,*

$$R \in \overline{\mathcal{F}(Y,X)}^{\tau_{E^u}}.$$

Theorem 33 (Kim, Zheng)**.** *Suppose that $(e_n)_n$ is 1-unconditional. The following statements are equivalent for a Banach space X.*

(a) *X has the E-AP;*
(b) *For every Banach space Y and every $R \in \mathcal{K}_E(Y,X)$,*

$$R \in \overline{\{SR : S \in \mathcal{F}(X,X)\}}^{\|\cdot\|_{\mathcal{K}_{E^u}}}$$

;
(c) *For every quotient space Y of E^* and every injective $R \in \mathcal{K}_E(Y,X)$,*

$$R \in \overline{\mathcal{F}(Y,X)}^{\tau_{E^u}}.$$

Let X and Y be Banach spaces. We denote by $X \otimes Y$ the algebraic tensor product of X and Y. The normed space $X \otimes Y$ equipped with a norm α is denoted by $X \otimes_\alpha Y$ and its completion is denoted by $X \hat{\otimes}_\alpha Y$. One of the most classical norms on $X \otimes Y$ is the *injective tensor norm* which is denoted by ε. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup \left\{ \left| \sum_{n=1}^{m} x^*(x_n) y^*(y_n) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^{m} x_n \otimes y_n$ is any representation of u. In [35], the following identification was attained.

Proposition 1 (Kim, Zheng)**.** *For any Banach space X,*

$$E^u(X) = E \hat{\otimes}_\varepsilon X$$

holds isometrically.

Recall that the *projective tensor norm* π is defined by

$$\pi(u; X, Y) := \inf \left\{ \sum_{n=1}^{m} \|x_n\| \|y_n\| : u = \sum_{n=1}^{m} x_n \otimes y_n, m \in \mathbb{N} \right\},$$

for all $u \in X \otimes Y$. $X \hat{\otimes}_\pi Y$ is the completion of $X \otimes Y$ under the projective tensor norm. If $u \in X \hat{\otimes}_\pi Y$, then there exist sequences $(x_n)_n$ in X and $(y_n)_n$ in Y with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

converges in $X \hat{\otimes}_\pi Y$. It is well known that

$$(X \hat{\otimes}_\varepsilon Y)^* = X^* \hat{\otimes}_\pi Y^*$$

whenever X^* is separable and has the AP. The next theorem is a characterization of the dual of $\mathcal{L}(X, Y)$ under the topology τ_{E^u}.

Theorem 34 (Kim, Zheng)**.** *Suppose that $(e_n)_n$ is shrinking. Let X and Y be Banach spaces. Then we have*

$$(\mathcal{L}(X, Y), \tau_{E^u})^*$$
$$= \left\{ f(\cdot) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n^j y_j^*(\cdot\, x_n) : (x_n)_n \in E^u(X), \sum_{j=1}^{\infty} \left\| \sum_{n=1}^{\infty} \lambda_n^j e_n^* \right\|_{E^*} \|y_j^*\|_{Y^*} < \infty \right\}.$$

A similar result holds for the dual of $\mathcal{L}(X, Y)$ under the topology τ_E.

Theorem 35 (Kim, Zheng)**.** *Suppose that $(e_n)_n$ is* 1*-unconditional and shrinking. Let X and Y be Banach spaces. Then we have*

$$(\mathcal{L}(X, Y), \tau_E)^*$$
$$= \left\{ f(\cdot) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \lambda_n^j y_j^*(\cdot\, x_n) : (x_n)_n \in E(X), \sum_{j=1}^{\infty} \left\| \sum_{n=1}^{\infty} \lambda_n^j e_n^* \right\|_{E^*} \|y_j^*\|_{Y^*} < \infty \right\}.$$

Using the representations above, Kim and Zheng [35] obtained another characterization of the E^u-AP when E has a shrinking basis.

Theorem 36 (Kim, Zheng)**.** *Suppose that $(e_n)_n$ is shrinking. The following statements are equivalent for a Banach space X.*

(a) *X has the E^u-AP;*
(b) *For every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$,*

$$R \in \overline{\{RS : S \in \mathcal{F}(X, X)\}}^{\tau_{E^u}}$$

;
(c) *For every Banach space Y and every $R \in \mathcal{K}(X, Y)$,*

$$R \in \overline{\{RS : S \in \mathcal{F}(X, X)\}}^{\tau_{E^u}}$$

;
(d) *For every separable reflexive Banach space Y,*

$$\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau_{E^u}}.$$

A similar result was attained for E^u-AP in [35].

Theorem 37 (Kim, Zheng)**.** *Suppose that $(e_n)_n$ is 1-unconditional and shrinking. The following statements are equivalent for a Banach space X.*

(a) *X has the E-AP;*
(b) *For every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$,*

$$R \in \overline{\{RS : S \in \mathcal{F}(X, X)\}}^{\tau_E}$$

;
(c) *For every Banach space Y and every $R \in \mathcal{K}(X, Y)$,*

$$R \in \overline{\{RS : S \in \mathcal{F}(X, X)\}}^{\tau_E}$$

;
(d) *For every separable reflexive Banach space Y,*

$$\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau_E}.$$

An interesting application of the above theorem is the following result [35].

Theorem 38 (Kim, Zheng)**.** *Let $1 < p, q < \infty$ and let X be a Banach space. If X^* has the $(\sum \ell_q)_p$-AP, then X has the p-AP.*

Remark 1. *In [34], one of the main theorems proved by Choi and Kim is that if X^* has the p-AP, then X has the p-AP. In Theorem 38, if $p = q$, then we recover the theorem of Choi and Kim.*

Theorem 39 (Kim, Zheng)**.** *Let $1 < p \leq q < \infty$ and let X be a Banach space. If X^* has the $(\sum \ell_p)_q$-AP, then X has the p-AP.*

Theorem 39 does not tell us what happens when $q < p$. So the problem below is left unsolved.

Problem 1. *Let $1 < q < p < \infty$ and X be a Banach space. If X^* has the $(\sum \ell_p)_q$-AP, does X have the p-AP?*

Theorems 38 and 39 do not include the cases when $p = 1, p = \infty, q = 1$ or $q = \infty$. So the following problems are still open.

Problem 2. *Let $1 < q < \infty$ and X be a Banach space. If X^* has the $(\sum \ell_q)_1$-AP, does X have the q-AP?*

Problem 3. *Let $1 < q < \infty$ and X be a Banach space. If X^* has the $(\sum \ell_q)_{c_0}$-AP, does X have the q-AP?*

Problem 4. *Let $1 < p < \infty$ and X be a Banach space. If X^* has the $(\sum \ell_1)_p$-AP, does X have the p-AP?*

Problem 5. *Let $1 < p < \infty$ and X be a Banach space. If X^* has the $(\sum c_0)_p$-AP, does X have the p-AP?*

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