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Coefficient Related Studies for New Classes of Bi-Univalent Functions

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Abstract: Using the recently introduced Sălăgean integro-differential operator, three new classes of bi-univalent functions are introduced in this paper. In the study of bi-univalent functions, estimates on the first two Taylor–Maclaurin coefficients are usually given. We go further in the present paper and bounds of the first three coefficients $|a_2|$, $|a_3|$ and $|a_4|$ of the functions in the newly defined classes are given. Obtaining Fekete–Szegő inequalities for different classes of functions is a topic of interest at this time as it will be shown later by citing recent papers. So, continuing the study on the coefficients of those classes, the well-known Fekete–Szegő functional is obtained for each of the three classes.

Keywords: bi-univalent functions; Sălăgean integral and differential operator; coefficient bounds; Fekete–Szegő problem

MSC: 30C45; 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$. Let $\mathcal{S} \subset \mathcal{A}$ denote the class of all functions in $\mathcal{A}$ which are univalent in U .

The Koebe One-Quarter Theorem [1] ensures that the image of the unit disk under every $f \in \mathcal{S}$ function contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z, \quad (z \in U),$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if $U \subset f(U)$ and if both f and f^{-1} are univalent in U . Let Σ denote the class of bi-univalent functions in U given by (1).

Studying the class of bi-univalent functions begun some time ago, around the year 1970 as it can be seen from papers [2–4]. The topic resurfaced as interesting in the last decade, many papers being published since 2011, for example, [5,6]. Interesting results related to coefficient estimates for certain special classes of univalent functions appeared like the ones published in [7–13].

The operators have been used ever since the beginning of the study of complex functions. Many known results have been proved easier by using them and new results could be obtained especially related to starlikeness and convexity of certain functions. Introducing new classes of analytic functions is the most common outcome of the study that involves operators.

The study of bi-univalent functions using operators is also an approach that is in trend nowadays as it can be seen in the very recent results from papers [14,15] and a particular interest is shown to obtaining the Fekete–Szegő functional for the special classes that are being introduced as it can be seen in the very recent paper [16].

The study on coefficients of the functions in certain special classes is a topic that has its origin at the very beginning of the study of univalent functions. A main result in the theory of univalent functions is Gronwall's Area Theorem stated in 1914 and used for obtaining bounds on the coefficients of the class of meromorphic functions. An analogous problem for the class $\mathcal{S}$ was solved by Bieberbach and its famous conjecture stated in 1916, only proven in 1984, has stimulated the development of different methods in the geometric theory of functions of a complex variable. Just as in the case of the classes studied by Gronwall and Bieberbach, in the study of bi-univalent functions, estimates on the first two Taylor–Maclaurin coefficients are usually given. We extend the study and manage to give estimates on the fourth coefficient too, concerning the functions in the classes introduced in the present paper.

Another aspect of the novelty of the results contained in the present paper is given by the operator used in defining the three new classes for which coefficient estimates are obtained. The operator was previously defined in the paper [17] as a new type of operator introduced by mixing the two forms of the well-known Sălăgean operator, its differential and integral forms.

Definition 1. [18] For $f \in \mathcal{A}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$, the Sălăgean differential operator $\mathcal{D}^n$ is defined by

$$\begin{aligned}\mathcal{D}^n &: \mathcal{A} \rightarrow \mathcal{A}, \\ \mathcal{D}^0 f(z) &= f(z), \\ \mathcal{D}^{n+1} f(z) &= z (\mathcal{D}^n f(z))', z \in U.\end{aligned}$$

Remark 1. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

Definition 2. [18] For $f \in \mathcal{A}$, $n \in \mathbb{N}_0$, the Sălăgean integral operator I^n is defined by

$$\begin{aligned}I^0 f(z) &= f(z), \\ I^1 f(z) &= If(z) = \int_0^z f(t) t^{-1} dt, \dots, \\ I^{n+1} f(z) &= I(I^n f(z)), z \in U.\end{aligned}$$

The I^1 is the Alexander operator used for the first time in [19], the I^n operator is called the generalized Alexander operator.

Remark 2. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \quad (3)$$

$z \in U$, ($n \in \mathbb{N}_0$) and $z (I^{n+1} f(z))' = I^n f(z)$.

Remark 3. We have $\mathcal{D}^n I^n f(z) = I^n \mathcal{D}^n f(z) = f(z)$, $f \in \mathcal{A}$, $z \in U$.

Definition 3. [17] Let $\xi \geq 0$, $n \in \mathbb{N}_0$. Denote by $\mathcal{D}I^n$ the operator given by

$$\begin{aligned} \mathcal{D}I^n : \mathcal{A} &\rightarrow \mathcal{A}, \\ \mathcal{D}I^n f(z) &= \left(1 - \tilde{\xi}\right) \mathcal{D}^n f(z) + \tilde{\xi} I^n f(z), z \in U. \end{aligned}$$

Remark 4. [17] If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathcal{D}I^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1 - \tilde{\xi}) + \tilde{\xi} \frac{1}{k^n} \right] a_k z^k = z + \sum_{k=2}^{\infty} \Gamma_k a_k z^k, z \in U, \quad (4)$$

where $\Gamma_k = k^n (1 - \tilde{\xi}) + \tilde{\xi} \frac{1}{k^n}$, $k \geq 2$.

This generalized operator is the linear combination of the Sălăgean differential and Sălăgean integral operator.

In 1933, Fekete and Szegő [20] proved that

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} 4\mu - 3, & \mu \geq 1, \\ 1 + 2 \exp \left[\frac{-2\mu}{1-\mu} \right], & 0 \leq \mu < 1, \\ 3 - 4\mu, & \mu < 0, \mu \in \mathbb{R}. \end{cases}$$

holds for any normalized univalent function and the result is sharp. The problem of maximizing the absolute value of the functional $|a_3 - \mu a_2^2|$ is called the Fekete–Szegő problem. Many authors obtained Fekete–Szegő inequalities for different classes of functions: [21–23].

In order to prove the original results from the main results part of the paper, the following lemmas are used:

We denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the open unit disk U , for example,

$$\mathcal{P} = \{f \in \mathcal{A} \mid f(0) = 1, \Re f(z) > 0, z \in U\}.$$

Lemma 1. [24] If $h \in \mathcal{P}$ then $|c_k| \leq 2$, $\forall k$, where $h(z) = 1 + c_1 z + c_2 z^2 + \dots$ for $z \in U$.

Lemma 2. [1] Let $p \in \mathcal{P}$ be of the form $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2} \text{ and } |c_k| \leq 2, \forall k \in \mathbb{N}.$$

Lemma 3. [25] If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, $z \in U$ is a function with positive real part in U and μ is a complex number, then

$$\left| c_2 - \mu c_1^2 \right| \leq 2 \max \{1; |2\mu - 1|\}.$$

The result is sharp for the function given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}, \quad z \in U.$$

2. Main Results

Using the operator shown in Definition 3, we introduce three new classes as follows:

Definition 4. For $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ a function $f(z)$ given by (1) is said to be in the class $\mathcal{P}_{\Sigma}^{\alpha}(\lambda)$ if the following conditions are satisfied:

$$\left| \arg \left(\frac{z (\mathcal{D}I^n f(z))' + \lambda z^2 (\mathcal{D}I^n f(z))''}{(1-\lambda) \mathcal{D}I^n f(z) + \lambda z (\mathcal{D}I^n f(z))'} \right) \right| < \frac{\alpha\pi}{2} \quad (5)$$

and

$$\left| \arg \left(\frac{w (\mathcal{D}I^n g(w))' + \lambda w^2 (\mathcal{D}I^n g(w))''}{(1-\lambda) \mathcal{D}I^n g(w) + \lambda w (\mathcal{D}I^n g(w))'} \right) \right| < \frac{\alpha\pi}{2}, \quad (6)$$

where $z, w \in U$ and the function g is given by (2).

Example 1. If $\lambda = n = 0$ we have the well-known class of strongly bi-starlike functions of order α :

$$\left| \arg \frac{z(f(z))'}{f(z)} \right| < \frac{\alpha\pi}{2}, \quad \left| \arg \frac{w(g(w))'}{g(w)} \right| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1.$$

Example 2. If $\lambda = 1$ and $n = 0$ we have the class of strongly bi-convex functions of order α :

$$\left| \arg \left(1 + \frac{z(f(z))''}{(f(z))'} \right) \right| < \frac{\alpha\pi}{2}, \quad \left| \arg \left(1 + \frac{w(g(w))''}{(g(w))'} \right) \right| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1.$$

Definition 5. For $0 \leq \beta < 1$, $0 \leq \lambda \leq 1$ a function $f(z)$ given by (1) is said to be in the class $\mathcal{Q}_{\Sigma}^{\beta}(\lambda)$ if the following conditions are satisfied:

$$\Re \left(\frac{z (\mathcal{D}I^n f(z))' + \lambda z^2 (\mathcal{D}I^n f(z))''}{(1-\lambda) \mathcal{D}I^n f(z) + \lambda z (\mathcal{D}I^n f(z))'} \right) > \beta \quad (7)$$

and

$$\Re \left(\frac{w (\mathcal{D}I^n g(w))' + \lambda w^2 (\mathcal{D}I^n g(w))''}{(1-\lambda) \mathcal{D}I^n g(w) + \lambda w (\mathcal{D}I^n g(w))'} \right) > \beta, \quad (8)$$

where $z, w \in U$ and the function g is given by (2).

Example 3. If $\lambda = n = 0$ we have the well-known class of bi-starlike functions of order β :

$$\Re \left(\frac{z(f(z))'}{f(z)} \right) > \beta, \quad \Re \left(\frac{w(g(w))'}{g(w)} \right) > \beta, \quad 0 \leq \beta < 1.$$

Example 4. If $\lambda = 1$ and $n = 0$ we have the class of bi-convex functions of order β :

$$\Re \left(1 + \frac{z(f(z))''}{(f(z))'} \right) > \beta, \quad \Re \left(1 + \frac{w(g(w))''}{(g(w))'} \right) > \beta, \quad 0 \leq \beta < 1.$$

Definition 6. Let $h, l : U \rightarrow \mathbb{C}$ be analytic functions and

$$\min \{ \Re (h(z)), \Re (l(z)) \} > 0, \quad (z \in U) \quad h(0) = l(0) = 1.$$

A function $f(z)$ given by (1) is said to be in the class $\mathcal{P}_{\Sigma}^{h,l}$ if the following conditions are satisfied:

$$\frac{z (\mathcal{D}I^n f(z))' + \lambda z^2 (\mathcal{D}I^n f(z))''}{(1-\lambda) \mathcal{D}I^n f(z) + \lambda z (\mathcal{D}I^n f(z))'} \in h(U) \quad (9)$$

and

$$\frac{w (\mathcal{D}I^n g(w))' + \lambda w^2 (\mathcal{D}I^n g(w))''}{(1-\lambda) \mathcal{D}I^n g(w) + \lambda w (\mathcal{D}I^n g(w))'} \in l(U), \quad (10)$$

where $z, w \in U$ and the function g is given by (2).

Remark 5. If we let $h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ and $l(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}$, $0 < \alpha \leq 1$ then the class $\mathcal{P}_{\Sigma}^{h,l}$ reduces to the class denoted by $\mathcal{P}_{\Sigma}^{\alpha}(\lambda)$.

Remark 6. If we let $h(z) = \frac{1+(1-2\beta)z}{1-z}$ and $l(z) = \frac{1-(1-2\beta)z}{1+z}$, $0 \leq \beta < 1$ then the class $\mathcal{P}_{\Sigma}^{h,l}$ reduces to the class denoted by $\mathcal{Q}_{\Sigma}^{\beta}(\lambda)$.

Remark 7. The classes introduced in this paper are defined in the classical way. All subclasses of bi-univalent functions are defined, the connection with the classes of bi-starlike and bi-convex functions being illustrated in the examples above. Being defined using relations related to arguments and real part of the functions contained, a geometric interpretation could be given for the classes. For the class in Definition 4, the geometrical image is in the first trigonometric dial, the section between two lines that converge at the origin having its maximum image the entire dial. The class in Definition 5 has its image in the half right plane. The first two classes defined are connected through the relation obtained for $\alpha = 1$ and $\beta = 0$, $\mathcal{P}_{\Sigma}^1(\lambda) = \mathcal{Q}_{\Sigma}^0(\lambda)$. The results for the class of functions $\mathcal{P}_{\Sigma}^{h,l}$ would generalize and improve the results for the classes of functions from Definitions 4 and 5. For special uses of parameters, new conditions for bi-starlikeness and bi-convexity could be established. Future interpretations are left to the imagination of interested researchers.

3. Coefficient Estimates

First, we give the coefficient estimates for the class $\mathcal{P}_{\Sigma}^{\alpha}(\lambda)$ given in Definition 4.

Theorem 1. Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and let $f(z)$ given by (1) be in the class $\mathcal{P}_{\Sigma}^{\alpha}(\lambda)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{|4\alpha\Gamma_3(1+2\lambda) + \Gamma_2^2(1+\lambda)^2(1-3\alpha)|}}, \quad (11)$$

$$|a_3| \leq \frac{\alpha}{\Gamma_3(1+2\lambda)} + \frac{4\alpha^2}{\Gamma_2^2(1+\lambda)^2} \quad (12)$$

and

$$|a_4| \leq \frac{2\alpha(2\alpha^2+1)}{9\Gamma_4(1+3\lambda)} - \frac{10\alpha(2\alpha-1)}{3[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda)-5\Gamma_4(1+3\lambda)]} + \frac{8\alpha^3\Gamma_2(1+\lambda)[3(1+2\lambda)\Gamma_3-(1+\lambda)^2\Gamma_2^2]}{3\Gamma_4(1+3\lambda)[4\alpha\Gamma_3(1+2\lambda)+\Gamma_2^2(1+\lambda)^2(1-3\alpha)]\sqrt{|4\alpha\Gamma_3(1+2\lambda)+\Gamma_2^2(1+\lambda)^2(1-3\alpha)|}}, \quad (13)$$

where Γ_k , $k \geq 2$ are defined in (4).

Proof. It follows from (5) and (6) that

$$\frac{z(\mathcal{D}I^n f(z))' + \lambda z^2(\mathcal{D}I^n f(z))''}{(1-\lambda)\mathcal{D}I^n f(z) + \lambda z(\mathcal{D}I^n f(z))'} = [p(z)]^\alpha \quad (14)$$

and

$$\frac{w(\mathcal{D}I^n g(w))' + \lambda w^2(\mathcal{D}I^n g(w))''}{(1-\lambda)\mathcal{D}I^n g(w) + \lambda w(\mathcal{D}I^n g(w))'} = [q(z)]^\alpha, \quad (15)$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \quad (16)$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \dots \quad (17)$$

Equating the coefficients in (14) and (15), we get

$$(1+\lambda)\Gamma_2 a_2 = \alpha p_1 \quad (18)$$

$$2(1+2\lambda)\Gamma_3 a_3 - \Gamma_2^2 a_2^2 (1+\lambda)^2 = \frac{1}{2} [\alpha(\alpha-1)p_1^2 + 2\alpha p_2] \quad (19)$$

$$\begin{aligned} 3\Gamma_4 a_4 (1+3\lambda) - 3\Gamma_2 \Gamma_3 a_2 a_3 (1+\lambda)(1+2\lambda) + \Gamma_2^3 a_2^3 (1+\lambda)^3 = \\ = \frac{1}{6} p_1^3 (\alpha-2)(\alpha-1)\alpha + p_1 p_2 (\alpha-1)\alpha + p_3 \alpha \end{aligned} \quad (20)$$

$$-(1+\lambda)\Gamma_2 a_2 = \alpha q_1 \quad (21)$$

$$2(1+2\lambda)\Gamma_3 (2a_2^2 - a_3) - \Gamma_2^2 a_2^2 (1+\lambda)^2 = \frac{1}{2} [\alpha(\alpha-1)q_1^2 + 2\alpha q_2] \quad (22)$$

$$\begin{aligned} -3\Gamma_4 (5a_2^3 - 5a_2 a_3 + a_4) (1+3\lambda) + 3\Gamma_2 \Gamma_3 a_2 (2a_2^2 - a_3) (1+\lambda)(1+2\lambda) - \\ - \Gamma_2^3 a_2^3 (1+\lambda)^3 = \frac{1}{6} q_1^3 (\alpha-2)(\alpha-1)\alpha + q_1 q_2 (\alpha-1)\alpha + q_3 \alpha. \end{aligned} \quad (23)$$

From (18) and (21), we get

$$p_1 = -q_1 \quad (24)$$

and

$$2\Gamma_2^2 a_2^2 (1+\lambda)^2 = \alpha^2 (p_1^2 + q_1^2). \quad (25)$$

From (19), (22) and (25), we obtain

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{4\alpha\Gamma_3 (1+2\lambda) + \Gamma_2^2 (1+\lambda)^2 (1-3\alpha)}.$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we get (11).

To find the bound on $|a_3|$, first we subtract (22) from (19):

$$4a_3\Gamma_3 (1+2\lambda) - 4\Gamma_3 (1+2\lambda) a_2^2 = \alpha (p_2 - q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 - q_1^2). \quad (26)$$

From (24), (25) and (26) follows that

$$a_3 = \frac{\alpha (p_2 - q_2)}{4\Gamma_3 (1+2\lambda)} + \frac{\alpha^2 (p_1^2 + q_1^2)}{2\Gamma_2^2 (1+\lambda)^2}, \quad (27)$$

and applying Lemma 1 we get (12).

To find the bound on $|a_4|$, first we subtract (23) from (20) and using (24) we get

$$6(1+3\lambda)\Gamma_4 a_4 + 15\Gamma_4(1+3\lambda)a_2(a_2^2 - a_3) - 6\Gamma_2\Gamma_3 a_2^3(1+\lambda)(1+2\lambda) + 2\Gamma_2^3 a_2^3(1+\lambda)^3 = \frac{1}{3}p_1^3(\alpha-2)(\alpha-1)\alpha + p_1\alpha(\alpha-1)(p_2+q_2) + \alpha(p_3-q_3). \quad (28)$$

Now we add (20) and (23) and using (24) we get

$$\begin{aligned} & -15\Gamma_4(1+3\lambda)a_2(a_2^2 - a_3) + 6\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda)a_2(a_2^2 - a_3) = \\ & = p_1\alpha(\alpha-1)(p_2-q_2) + \alpha(p_3+q_3), \end{aligned}$$

or equivalently

$$a_2(a_2^2 - a_3) = \frac{p_1\alpha(\alpha-1)(p_2-q_2) + \alpha(p_3+q_3)}{3[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda)]}. \quad (29)$$

Substituting (29) in (28) and applying Lemma 1 we get (13). $\square$

Now we calculate the Fekete–Szegő functional for the the class $\mathcal{P}_\Sigma^\alpha(\lambda)$.

Theorem 2. Let f of the form (1) be in the class $\mathcal{P}_\Sigma^\alpha(\lambda)$. Then

$$|a_3 - \tilde{\xi}a_2^2| \leq \begin{cases} \frac{\alpha}{\Gamma_3(1+2\lambda)}; & \left| \frac{\alpha(1-\tilde{\xi})}{4\alpha\Gamma_3(1+2\lambda) + \Gamma_2^2(1+\lambda)^2(1-3\alpha)} \right| \leq \frac{1}{4\Gamma_3(1+2\lambda)}, \\ \left| \frac{4\alpha^2(1-\tilde{\xi})}{4\alpha\Gamma_3(1+2\lambda) + \Gamma_2^2(1+\lambda)^2(1-3\alpha)} \right|; & \left| \frac{\alpha(1-\tilde{\xi})}{4\alpha\Gamma_3(1+2\lambda) + \Gamma_2^2(1+\lambda)^2(1-3\alpha)} \right| \geq \frac{1}{4\Gamma_3(1+2\lambda)}. \end{cases}$$

Proof. From Theorem 1 we use the value of a_2^2 and a_3 to calculate $a_3 - \tilde{\xi}a_2^2$.

$$a_3 - \tilde{\xi}a_2^2 = \alpha \left[p_2 \left(h(\tilde{\xi}) + \frac{1}{4\Gamma_3(1+2\lambda)} \right) + q_2 \left(h(\tilde{\xi}) - \frac{1}{4\Gamma_3(1+2\lambda)} \right) \right],$$

where $h(\tilde{\xi}) = (1-\tilde{\xi}) \frac{\alpha}{4\alpha\Gamma_3(1+2\lambda) + \Gamma_2^2(1+\lambda)^2(1-3\alpha)}$.

Then

$$|a_3 - \tilde{\xi}a_2^2| \leq \begin{cases} \frac{\alpha}{\Gamma_3(1+2\lambda)}; & |h(\tilde{\xi})| \leq \frac{1}{4\Gamma_3(1+2\lambda)}, \\ 4\alpha|h(\tilde{\xi})|; & |h(\tilde{\xi})| \geq \frac{1}{4\Gamma_3(1+2\lambda)}. \end{cases}$$

$\square$

Theorem 3. Let $0 \leq \beta < 1$, $0 \leq \lambda \leq 1$ and let $f(z)$ given by (1) be in the class $\mathcal{Q}_\Sigma^\beta(\lambda)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2(1+2\lambda)\Gamma_3 - \Gamma_2^2(1+\lambda)^2}}, \quad (30)$$

$$|a_3| \leq \frac{1-\beta}{\Gamma_3(1+2\lambda)} + \frac{4(1-\beta)^2}{\Gamma_2^2(1+\lambda)^2} \quad (31)$$

and

$$|a_4| \leq \frac{2(1-\beta)}{3\Gamma_4(1+3\lambda)} - \frac{10(1-\beta)}{3[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda)-5\Gamma_4(1+3\lambda)]} + \frac{4\Gamma_2(1-\beta)(1+\lambda)[3\Gamma_3(1+2\lambda)-\Gamma_2^2(1+\lambda)^2]}{3\Gamma_4(1+3\lambda)[2\Gamma_3(1+2\lambda)-\Gamma_2^2(1+\lambda)^2]} \sqrt{\frac{1-\beta}{|4(1+2\lambda)\Gamma_3-2\Gamma_2^2(1+\lambda)^2|}}, \quad (32)$$

where $\Gamma_k, k \geq 2$ are defined in (4).

Proof. It follows from (5) and (6) that

$$\frac{z(\mathcal{D}I^n f(z))' + \lambda z^2(\mathcal{D}I^n f(z))''}{(1-\lambda)\mathcal{D}I^n f(z) + \lambda z(\mathcal{D}I^n f(z))'} = \beta + (1-\beta)p(z) \quad (33)$$

and

$$\frac{w(\mathcal{D}I^n g(w))' + \lambda w^2(\mathcal{D}I^n g(w))''}{(1-\lambda)\mathcal{D}I^n g(w) + \lambda w(\mathcal{D}I^n g(w))'} = \beta + (1-\beta)q(w), \quad (34)$$

where $p(z)$ and $q(w)$ have the forms (16) and (17).

Equating the coefficients in (33) and (34), we get

$$(1+\lambda)\Gamma_2 a_2 = (1-\beta)p_1 \quad (35)$$

$$2(1+2\lambda)\Gamma_3 a_3 - \Gamma_2^2 a_2^2 (1+\lambda)^2 = (1-\beta)p_2 \quad (36)$$

$$3\Gamma_4 a_4 (1+3\lambda) - 3\Gamma_2\Gamma_3 a_2 a_3 (1+\lambda)(1+2\lambda) + \Gamma_2^3 a_2^3 (1+\lambda)^3 = (1-\beta)p_3 \quad (37)$$

$$-(1+\lambda)\Gamma_2 a_2 = (1-\beta)q_1 \quad (38)$$

$$2(1+2\lambda)\Gamma_3 (2a_2^2 - a_3) - \Gamma_2^2 a_2^2 (1+\lambda)^2 = (1-\beta)q_2 \quad (39)$$

$$-3\Gamma_4 (5a_2^3 - 5a_2 a_3 + a_4) (1+3\lambda) + 3\Gamma_2\Gamma_3 a_2 (2a_2^2 - a_3) (1+\lambda)(1+2\lambda) - \Gamma_2^3 a_2^3 (1+\lambda)^3 = (1-\beta)q_3. \quad (40)$$

From (35) and (38), we get

$$p_1 = -q_1 \quad (41)$$

and

$$2\Gamma_2^2 a_2^2 (1+\lambda)^2 = (1-\beta)^2 (p_1^2 + q_1^2). \quad (42)$$

From (36) and (39), we obtain

$$a_2^2 = \frac{(1-\beta)(p_2 + q_2)}{4\Gamma_3(1+2\lambda) - 2\Gamma_2^2(1+\lambda)^2}.$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we get (30).

To find the bound on $|a_3|$, first we subtract (39) from (36):

$$4a_3\Gamma_3(1+2\lambda) - 4\Gamma_3(1+2\lambda)a_2^2 = (1-\beta)(p_2 - q_2). \quad (43)$$

From (42) and (43) follows that

$$a_3 = \frac{(1-\beta)(p_2 - q_2)}{4\Gamma_3(1+2\lambda)} + \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2\Gamma_2^2(1+\lambda)^2}, \quad (44)$$

and applying Lemma 1 we get (31).

To find the bound on $|a_4|$, first we subtract (40) from (37) and using (41) we get

$$\begin{aligned} 6(1+3\lambda)\Gamma_4 a_4 + 15\Gamma_4(1+3\lambda)a_2(a_2^2 - a_3) - 6\Gamma_2\Gamma_3 a_2^3(1+\lambda)(1+2\lambda) + \\ + 2\Gamma_2^3 a_2^3(1+\lambda)^3 = (1-\beta)(p_3 + q_3). \end{aligned} \quad (45)$$

Now we add (37) and (40) and using (41) we get

$$\begin{aligned} -15\Gamma_4(1+3\lambda)a_2(a_2^2 - a_3) + 6\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda)a_2(a_2^2 - a_3) = \\ = (1-\beta)(p_3 - q_3), \end{aligned}$$

or equivalently

$$a_2(a_2^2 - a_3) = \frac{(1-\beta)(p_3 + q_3)}{3[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda)]}. \quad (46)$$

Substituting (46) in (45) and applying Lemma 1 we get (32). $\square$

Theorem 4. Let f of the form (1) be in the class $\mathcal{Q}_{\Sigma}^{\beta}(\lambda)$. Then

$$|a_3 - \tilde{\xi}a_2^2| \leq \begin{cases} \frac{1-\beta}{\Gamma_3(1+2\lambda)}; & \left| \frac{1-\tilde{\xi}}{4\alpha\Gamma_3(1+2\lambda)-2\Gamma_2^2(1+\lambda)^2} \right| \leq \frac{1}{4\Gamma_3(1+2\lambda)}, \\ \left| \frac{4(1-\beta)(1-\tilde{\xi})}{4\alpha\Gamma_3(1+2\lambda)-2\Gamma_2^2(1+\lambda)^2} \right|; & \left| \frac{1-\tilde{\xi}}{4\alpha\Gamma_3(1+2\lambda)-2\Gamma_2^2(1+\lambda)^2} \right| \geq \frac{1}{4\Gamma_3(1+2\lambda)}. \end{cases}$$

Proof. From Theorem 3 we use the value of a_2^2 and a_3 to calculate $a_3 - \tilde{\xi}a_2^2$.

$$a_3 - \tilde{\xi}a_2^2 = (1-\beta) \left[p_2 \left(h(\tilde{\xi}) + \frac{1}{4\Gamma_3(1+2\lambda)} \right) + q_2 \left(h(\tilde{\xi}) - \frac{1}{4\Gamma_3(1+2\lambda)} \right) \right],$$

where $h(\tilde{\xi}) = (1-\tilde{\xi}) \frac{1}{4\Gamma_3(1+2\lambda) - 2\Gamma_2^2(1+\lambda)^2}$.

Then

$$|a_3 - \tilde{\xi}a_2^2| \leq \begin{cases} \frac{1-\beta}{\Gamma_3(1+2\lambda)}; & \left| h(\tilde{\xi}) \right| \leq \frac{1}{4\Gamma_3(1+2\lambda)} \\ 4(1-\beta) \left| h(\tilde{\xi}) \right|; & \left| h(\tilde{\xi}) \right| \geq \frac{1}{4\Gamma_3(1+2\lambda)}. \end{cases}$$

$\square$

Theorem 5. Let $0 \leq \lambda \leq 1$ and let $f(z)$ given by (1) be in the class $\mathcal{P}_{\Sigma}^{h,l}$. Then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |l'(0)|^2}{2\Gamma_2^2(1+\lambda)^2}}, \sqrt{\frac{|h''(0)| + |l''(0)|}{4|2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|}} \right\} \quad (47)$$

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |l'(0)|^2}{2\Gamma_2^2(1+\lambda)^2} + \frac{|h''(0)| + |l''(0)|}{8\Gamma_3(1+2\lambda)}, \frac{|h''(0)| |4\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2| + |l''(0)| \Gamma_2^2(1+\lambda)^2}{8\Gamma_3(1+2\lambda) |2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|} \right\} \quad (48)$$

and

$$|a_4| \leq \min \left\{ \frac{|h'''(0)| + |l'''(0)|}{36} \left| \frac{1}{\Gamma_4(1+3\lambda)} - \frac{5}{2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda)} \right| + \right. \\ \left. + \frac{|h'(0)|^2 + |l'(0)|^2}{\Gamma_2^2(1+\lambda)^2} \sqrt{\frac{|h'(0)|^2 + |l'(0)|^2}{2}} \frac{|3\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|}{6\Gamma_4(1+3\lambda)} \right\}, \quad (49) \\ \frac{|h'''(0)| + |l'''(0)|}{36} \left| \frac{1}{\Gamma_4(1+3\lambda)} - \frac{5}{2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda)} \right| + \\ \left. \frac{|h''(0)| + |l''(0)|}{|2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|} \sqrt{\frac{|h''(0)| + |l''(0)|}{|2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|}} \frac{\Gamma_2(1+\lambda)|3\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2|}{24\Gamma_4(1+3\lambda)} \right\},$$

where $\Gamma_k, k \geq 2$ are defined in (4).

Proof. For a start, we write the equivalent forms of the argument inequalities in (9) and (10).

$$\frac{z(\mathcal{D}I^n f(z))' + \lambda z^2(\mathcal{D}I^n f(z))''}{(1-\lambda)\mathcal{D}I^n f(z) + \lambda z(\mathcal{D}I^n f(z))'} = h(z) \quad (50)$$

and

$$\frac{w(\mathcal{D}I^n g(w))' + \lambda w^2(\mathcal{D}I^n g(w))''}{(1-\lambda)\mathcal{D}I^n g(w) + \lambda w(\mathcal{D}I^n g(w))'} = l(w), \quad (51)$$

where $h(z)$ and $l(w)$ satisfy the conditions of Definition 6 and have the following Taylor–Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots, \quad (52)$$

$$l(w) = 1 + l_1 w + l_2 w^2 + \dots. \quad (53)$$

Substituting from (52) and (53) into (50) and (51), respectively, and equating the coefficients, we get

$$(1+\lambda)\Gamma_2 a_2 = h_1 \quad (54)$$

$$2(1+2\lambda)\Gamma_3 a_3 - \Gamma_2^2 a_2^2 (1+\lambda)^2 = h_2 \quad (55)$$

$$3\Gamma_4 a_4 (1+3\lambda) - 3\Gamma_2\Gamma_3 a_2 a_3 (1+\lambda)(1+2\lambda) + \Gamma_2^3 a_2^3 (1+\lambda)^3 = h_3 \quad (56)$$

$$-(1+\lambda)\Gamma_2 a_2 = l_1 \quad (57)$$

$$2(1+2\lambda)\Gamma_3 (2a_2^2 - a_3) - \Gamma_2^2 a_2^2 (1+\lambda)^2 = l_2 \quad (58)$$

$$-3\Gamma_4 (5a_2^3 - 5a_2 a_3 + a_4) (1+3\lambda) + 3\Gamma_2\Gamma_3 a_2 (2a_2^2 - a_3) (1+\lambda)(1+2\lambda) - \\ - \Gamma_2^3 a_2^3 (1+\lambda)^3 = l_3. \quad (59)$$

From (54) and (57), we get

$$h_1 = -l_1 \quad (60)$$

and

$$2\Gamma_2^2 a_2^2 (1+\lambda)^2 = h_1^2 + l_1^2. \quad (61)$$

Adding (55) and (58), we obtain

$$4\Gamma_3 a_2^2 (1+2\lambda) - 2\Gamma_2^2 a_2^2 (1+\lambda)^2 = h_2 + l_2. \quad (62)$$

Therefore, from (61) and (62), we get

$$a_2^2 = \frac{h_1^2 + l_1^2}{2\Gamma_2^2(1+\lambda)^2} \quad (63)$$

and

$$a_2^2 = \frac{h_2 + l_2}{2 \left[2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right]}. \quad (64)$$

We find from (63) and (64) that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |l'(0)|^2}{2\Gamma_2^2(1+\lambda)^2}$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |l''(0)|}{4 \left[2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right]}.$$

So we get the desired estimate on the coefficient $|a_2|$ as asserted in (47).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (58) from (55), we get

$$4\Gamma_3(1+2\lambda)a_3 - 4\Gamma_3(1+2\lambda)a_2^2 = h_2 - l_2. \quad (65)$$

Substituting the value of a_2^2 from (63) into (65), it follows that

$$a_3 = \frac{h_2 - l_2}{4\Gamma_3(1+2\lambda)} + \frac{h_1^2 + l_1^2}{2\Gamma_2^2(1+\lambda)^2}.$$

So

$$|a_3| \leq \frac{|h'(0)|^2 + |l'(0)|^2}{2\Gamma_2^2(1+\lambda)^2} + \frac{|h''(0)| + |l''(0)|}{8\Gamma_3(1+2\lambda)}.$$

On the other hand, upon substituting the value of a_2^2 from (64) into (65), it follows that

$$a_3 = \frac{h_2 \left[4\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right] + l_2\Gamma_2^2(1+\lambda)^2}{4\Gamma_3(1+2\lambda) \left[2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right]}.$$

Consequently, we have

$$|a_3| \leq \frac{|h''(0)| \left| 4\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right| + |l''(0)|\Gamma_2^2(1+\lambda)^2}{8\Gamma_3(1+2\lambda) \left| 2\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right|}.$$

To find the bound on $|a_4|$, first we add (56) and (59) and using (60) we get

$$a_2(a_2^2 - a_3) = \frac{h_3 + l_3}{3 \left[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda) \right]}. \quad (66)$$

Now we subtract (59) from (56) and using (60) the result is

$$6(1+3\lambda)\Gamma_4a_4 + 15\Gamma_4(1+3\lambda)a_2(a_2^2 - a_3) - 6\Gamma_2\Gamma_3a_2^3(1+\lambda)(1+2\lambda) + 2\Gamma_2^3a_2^3(1+\lambda)^3 = h_3 - l_3,$$

if we substitute (66) we have

$$a_4 = \frac{h_3 - l_3}{6\Gamma_4(1+3\lambda)} - \frac{5(h_3 + l_3)}{6[2\Gamma_2\Gamma_3(1+\lambda)(1+2\lambda) - 5\Gamma_4(1+3\lambda)]} + \\ + 2\Gamma_2(1+\lambda) \left[3\Gamma_3(1+2\lambda) - \Gamma_2^2(1+\lambda)^2 \right] \frac{1}{6\Gamma_4(1+3\lambda)} \cdot a_2^3. \quad (67)$$

Finally, if we use (63) then (64) in (67) the result is (49). $\square$

4. Conclusions

The original results of this paper are about coefficient estimates given the three original classes defined here. The classes are defined in the paper using an interesting new type of integro-differential operator, Sălăgean integro-differential operator. Since the only study done on them was related to coefficient estimates, they could be of particular interest for further studies related to different other aspects.

As it can be seen in Examples 1–4, for certain use of the parameters of the class given in Definition 4, strongly bi-starlikeness and strongly bi-convexity is proven. Similar studies related to starlikeness, convexity, and close-to-convexity of all the classes defined in the paper using values for the parameters can be conducted. With these studies, more could be found out about an intuitive or high level interpretation of the three function classes defined.

With the introduction of Definitions 1 and 2, it is worth investigating the possibility of applying the Lie algebra method in the work [26] to the complex plane. In the present paper, estimates for coefficient $|a_4|$ are given going further than estimates for coefficients $|a_2|$ and $|a_3|$ which are usually obtained in the study of bi-univalent functions.

It remains an open problem to obtain estimates on bound of $|a_n|$, $(n \in \mathbb{R} - \{1, 2, 3, 4\})$ for the classes that have been introduced here. Particular uses of coefficient estimates could lead to potentially interesting new results. The results from this paper could also inspire further research related to integro-differential operators used for introducing new classes of bi-univalent functions.

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