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On Metric-Type Spaces Based on Extended T-Conorms

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Abstract: Kirk and Shahzad introduced the class of strong b-metric spaces lying between the class of b-metric spaces and the class of metric spaces. As compared with b-metric spaces, strong b-metric spaces have the advantage that open balls are open in the induced topology and, hence, they have many properties that are similar to the properties of classic metric spaces. Having noticed the advantages of strong b-metric spaces Kirk and Shahzad complained about the absence of non-trivial examples of such spaces. It is the main aim of this paper to construct a series of strong b-metric spaces that fail to be metric. Realizing this programme, we found it reasonable to consider these metric-type spaces in the context when the ordinary sum operation is replaced by operation $\oplus$, where $\oplus$ is an extended t-conorm satisfying certain conditions.

Keywords: metric; b-metric; sb-metric; extended t-conorm; $\oplus$-metric; $\oplus$-b-metric; $\oplus$-sb-metric

1. Introduction

An important class of spaces was introduced by I.A. Bakhtin (under the name almost metric spaces) and rediscovered by S. Czerwik (under the name "b-metric spaces"). b-metric spaces generalize "classic" metric spaces by replacing in the definition of a metric the triangularity axiom $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$ with a more general axiom $d(x,z) \le k \cdot (d(x,y) + d(y,z))$ for all $x, y, z \in X$ where $k \ge 1$ is a fixed constant. The class of b-metric spaces includes such interesting and important for applications cases, as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$ defined by $d(x,y) = |x-y|^2$ or $d : C[a,b] \times C[a,b] \to \mathbb{R}^+$ defined by $d(f,g) = \int_a^b (f(x) - g(x))^2 dx$. A series of b-metrics can be obtained from an ordinary metric by the following construction. Let $k \ge 1$ be a fixed constant and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing mapping, such that $\varphi(0) = 0$ and $\varphi(a+b) \le k \cdot (\varphi(a) + \varphi(b))$ for all $a, b \in \mathbb{R}^+$. Further, let $\rho : X \times X \to \mathbb{R}_0^+$ be a metric on a set X, Then by setting $d_{\rho\varphi}(x,y) = (\varphi \circ \rho)(x,y) \quad x, y \in X$ we get a b-metric $d_{\rho\varphi}$ on the set X, which, in case $k > 1$ fails to be a metric.

Recently there were published several papers where the induced topology of b-metric spaces was applied. Unfortunately, in most papers the authors assumed that the open balls $B(a,r) = \{x \in X \mid d(a,x) < r\}$ are "really open", which is open in the induced topology. However, as it was noticed in [1] and soon rediscovered also in [2,3], it is generally not true. Some recent results concerning b-metric spaces can be found e.g., in [4–7].

To "remedy" this "shortage" of b-metric spaces, in [1], Kirk and Shahzad introduced the class of strong b-metric spaces (in this paper we call them sb-metric spaces). In the definition of an sb-metric space the third (triangularity) axiom is given by the inequality $d(x,y) \le d(x,z) + kd(z,y)$ for some $k \ge 1$. Obviously, the class of sb-metric spaces lies in between the class of metric and the class of b-metric spaces. As shown by Kirk and Shahzad [1], in sb-metric spaces open balls are really open in the induced topology. Thanks to this fact, sb-metric spaces have many useful properties common with ordinary metric spaces. Unfortunately, Kirk and Shahzad in [1] could not present any examples of strong b-metric spaces that fail to be metric. It was the first, original, aim of this paper to present a series of such examples, and this is done in Section 5.

However, having this primary aim in mind, we decide that it could be interesting, and probably useful, to develop our work on the basis of the operation $\oplus$, which is actually a kind of extended t-conorm. This allows for considering sb-metric spaces, and parallelly ordinary metric spaces and b-metric spaces in two ways: as ordinary metrics, if $\oplus$ in the triangularity axiom is an addition (that corresponds to the extension of Łukasiewicz t-conorm considered on the triangle $\{(x,y) \subset \mathbb{R}^2 \mid x,y \geq 0, x+y \leq 1\}$) and as ultra-metrics, in case when $\oplus$ is maximum (that corresponds to the maximum t-conorm).

The structure of the paper is a follows. In Section 2, we define the notion of an extended t-conorm. Some properties of extended t-conorms needed in the research are highlighted and examples are given. The Section 3 contains definitions of different metric-type operations and the related metric-type spaces. These spaces are reconsidered in the Section 4 from the categorical point of view. Section 5 contains a list of examples of non metric sb-metric spaces with different properties as well as examples of some metric-type spaces on the basis of extended t-conorms. Section 6 is devoted to the study of products and coproducts of metric-type spaces, specifically to products and coproducts of sb-metric spaces. In Section 7, we discuss some prospect for future work.

2. Extended T-Conorms

Definition 1. *Let $\mathbb{R}^+ = [0,\infty)$. A binary operation $\oplus : [0,\infty) \times [0,\infty) \to \mathbb{R}^+$ will be called extended t-conorm if, for all $\alpha, \beta, \gamma \in \mathbb{R}^+$, the following properties hold:*

($\oplus_1$) *$\oplus$ is commutative, that is $\alpha \oplus \beta = \beta \oplus \alpha$;*
($\oplus_2$) *$\oplus$ is associative, that is $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$;*
($\oplus_3$) *$\oplus$ is monotone, that is $\alpha \leq \beta \Longrightarrow \alpha \oplus \gamma \leq \beta \oplus \gamma$;*
($\oplus_4$) *0 is the neutral element for $\oplus$, that is $\alpha \oplus 0 = \alpha$.*

Remark 1. *Note that in case operation $\oplus$ is defined on $[0,1] \times [0,1]$ and it takes its values in $[0,1]$, then the definition of extended t-conorm reduces to the concept of a t-conorm [8]. Just for this observation, we refer to $\oplus$ as an extended t-conorm.*

We will also need the following special properties of operation $\oplus$:

Definition 2. *$\oplus$ is called distributive, if for all $\alpha, \beta, k \in \mathbb{R}^+$*

($\oplus_5$) *$k \cdot (\alpha \oplus \beta) = k \cdot \alpha \oplus k \cdot \beta$.*

$\oplus$ is called compressible, if ,

($\oplus_6$) $\alpha \leq \beta \oplus \gamma \Longrightarrow \frac{\alpha}{\alpha+1} \leq \frac{\beta}{\beta+1} \oplus \frac{\gamma}{\gamma+1}$.

$\oplus$ is called continuous at the bottom level, if this operation is continuous in all points of $\{0\} \times \mathbb{R}^+ \subset \mathbb{R}^+ \times \mathbb{R}^+$ (and, hence, by symmetry also on $\mathbb{R}^+ \times \{0\}$).

Below, we give some examples of extended t-conorms $\oplus$, which are mentioned in the paper.

Example 1. *Let $\alpha \oplus_L \beta = \alpha + \beta$. Thus, $\oplus_L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is an ordinary addition. It is obvious that $+$ satisfies all properties from Definition 1. One can easily see that this operation $\oplus_L$ is distributive, compressible, and is continuous on the whole space $\mathbb{R}^+ \times \mathbb{R}^+$. When restricted to the triangle $\{x+y \leq 1 \mid x,y \geq 0\} \subset [0,1] \times [0,1]$, operation $\oplus_L$ reduces to the Łukasiewicz t-conorm.*

Example 2. *Let $\alpha \oplus_M \beta = \alpha \vee \beta$,, where $\vee$ denotes the maximum. It is obvious that $\oplus_M$ satisfies all properties from Definition 1. Thus, $\oplus$ is the extension of the maximum t-conorm $\oplus_M$ from $[0,1] \times [0,1]$ to $\mathbb{R}^+ \times \mathbb{R}^+$. One can easily see that this operation $\oplus_M$ is distributive and it is continuous on the whole space $\mathbb{R}^+ \times \mathbb{R}^+$. The compressibility of $\vee$ follows from the next easily provable lemma:*

Lemma 1. *$b \leq c$ if and only if $\frac{b}{b+1} \leq \frac{c}{c+1}$.*

Example 3. *Let $\alpha \oplus_T \beta = \alpha \vee \beta \vee \alpha \cdot \beta$. It is obvious that $\oplus_T$ satisfies properties $\oplus_1$, $\oplus_3$ and $\oplus_4$ from Definition 1. We verify the associativity of $\oplus_T$, as follows.*

$$\begin{aligned}
\alpha \oplus_T (\beta \oplus_T \gamma) &= \alpha \oplus_T (\beta \vee \gamma \vee \beta \cdot \gamma) \\
&= \alpha \vee (\beta \vee \gamma \vee \beta \cdot \gamma) \vee \alpha \cdot (\beta \vee \gamma \vee \beta \cdot \gamma) \\
&= \alpha \vee \beta \vee \gamma \vee \beta \cdot \gamma \vee \alpha \cdot \beta \vee \alpha \cdot \gamma \vee \alpha \cdot \beta \cdot \gamma \\
&= \alpha \vee \beta \vee \alpha \cdot \beta \vee \gamma \vee \alpha \cdot \gamma \vee \beta \cdot \gamma \vee \alpha \cdot \beta \cdot \gamma \\
&= (\alpha \vee \beta \vee \alpha \cdot \beta) \vee \gamma \vee (\alpha \vee \beta \vee \alpha \cdot \beta) \cdot \gamma \\
&= (\alpha \vee \beta \vee \alpha \cdot \beta) \oplus_T \gamma \\
&= (\alpha \oplus_T \beta) \oplus_T \gamma.
\end{aligned}$$

The continuity of $\oplus_T$ on the whole $\mathbb{R}^+ \times \mathbb{R}^+$ is obvious. Unfortunately, $\oplus_T$ is not distributive:

$$\begin{aligned}
k \cdot (\alpha \oplus_T \beta) &= k \cdot (\alpha \vee \beta \vee \alpha \cdot \beta) = k \cdot \alpha \vee k \cdot \beta \vee k \cdot \alpha \cdot \beta \\
&\neq k \cdot \alpha \vee k \cdot \beta \vee k^2 \cdot \alpha \cdot \beta \\
&= k \cdot \alpha \oplus_T k \cdot \beta
\end{aligned}$$

and does not satisfy the property $\oplus_6$. For example $5 \leq 2 \oplus_T 3$ but $\frac{5}{5+1} \not\leq \frac{2}{2+1} \oplus_T \frac{3}{3+1}$.

Example 4. *We define operation $\oplus_P : [0,1] \times [0,1] \to [0,1]$ by setting $a \oplus b = a + b - a \cdot b$. Obviously it is just the product t-conorm on the closed interval, and hence the properties of Definition 1 are valid.*

We do not know whether it is possible to extend the product t-conorm to the whole square $\mathbb{R}^+ \times \mathbb{R}^+$ preserving properties $\oplus_1 - \oplus_4$.

Example 5. *The h-shifted arithmetic sum, where $h > 0$ is a fixed constant and it is defined as*

$$\alpha \oplus \beta = \beta \oplus \alpha = \begin{cases} \alpha & , \beta = 0 \\ \beta & , \alpha = 0 \\ \alpha + \beta + h & , \textit{otherwise} \end{cases}$$

and satisfies all properties from Definition 1. However, it is not distributive, since

$$k \cdot (\alpha \oplus \beta) = k \cdot (\alpha + \beta + h) = k \cdot \alpha + k \cdot \beta + k \cdot h \neq k \cdot \alpha + k \cdot \beta + h = (k \cdot \alpha \oplus k \cdot \beta)$$

and is not continuous at any point on $\{0\} \times \mathbb{R}^+ \cup \mathbb{R}^+ \times \{0\}$.

3. Metric-Type Structures Based on Extended T-Conorms

To make our exposition coherent, in the following definition, we specify the terminology concerning metric and its generalizations that can be found in the literature (see e.g., [9–11]):

Definition 3. *A mapping $d : X \times X \to \mathbb{R}^+$ is called semi pseudometric if it satisfies the following two properties*

(m_1) $d(x,x) = 0 \ \forall\, x \in X$;
(m_2) $d(x,y) = d(y,x) \ \forall\, x, y \in X$;

A semi-metric is a semi pseudometric satisfying the following strengthen version of the first axiom:

$(m_{1'}) d(x,y) = 0 \Longleftrightarrow x = y$.

Definition 4. *Given an extended t-conorm* $\oplus : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$*, a semi (pseudo)metric* $d : X \times X \to \mathbb{R}^+$ *is called*

(m_3) *$\oplus$-based (pseudo)metric, or just $\oplus$-(pseudo)metric, if*
$d(x,y) \le d(x,z) \oplus d(z,y) \forall x,y,z \in X;$

(mb_3) *$\oplus$-based b-(pseudo)metric, or just $\oplus$-b-(pseudo)metric if there exists* $k \ge 1$ *such that*
$d(x,y) \le k \cdot (d(x,z) \oplus d(z,y)) \ \forall \ x,y,z \in X;$

(msb_3) *$\oplus$-based sb-(pseudo)metric or just $\oplus$-sb-(pseudo)metric, if there exists* $k \ge 1$*, such that*
$d(x,y) \le d(x,z) \oplus k \cdot d(z,y) \ \forall \ x,y,z \in X.$

In particular, in case constant k is fixed in conditions (mb_3) *and* (msb_3)*, we may speak, respectively, of $\oplus$-bk-(pseudo)metric and $\oplus$-sbk-(pseudo)metric.*

Remark 2. *In case* $\oplus = +$*, that is an arithmetic addition, $\oplus$-(pseudo)metric, $\oplus$-b-(pseudo)metric and $\oplus$-sb-(pseudo)metric reduce to the definitions of an ordinary (pseudo)-metric, b-(pseudo)metric [12–14] and sb-(pseudo)-metric [1] (Actually Kirk and Shahazad [1] call this kind of a mapping by strong b-metric. However, we consider this term to be inappropriate, since it may lead to a misunderstanding when extending this notion to the case of fuzzy metric-type structures since it comes into collision with the concepts of a strong fuzzy metric and a strong b-fuzzy metric widely used in the literature). In this case, we usually omit sign $\oplus$.*

In case $\oplus = \vee$*, which is maximum, $\oplus$-(pseudo)metric reduces to the definition of an ultra-(pseudo)-metric, see, e.g., [15]. Accordingly, $\oplus$ b-(pseudo)metric and $\oplus$-(pseudo)metric in this case could be called ultra-b-(pseudo)metric and ultra-sb-(pseudo)metric, respectively.*

By applying other extended t-conorm based operators, we obtain new versions of metric-type structures some of which will be considered in this paper.

Having these general definitions for different types of metric-type structures, in the sequel we will mainly be interested in $\oplus$- sb-(pseudo)metrics, since just sb-(pseudo)metrics are the main purpose of this paper. However, for comparison, we will sometimes comment the versions of the obtained results for $\oplus$-sb-(pseudo)metrics in case of the $\oplus$-(pseudo)metrics from one side and in the case of b-(pseudo)metrics from the other. In particular, we will be interested in what properties of an extended t-conorm $\oplus$ are important for the validity of different properties of metric-type structures.

In order to make exposition more homogeneous, we will restrict to the case of $\oplus$-sb-metrics, which is with assumption of axiom $(m_{1'})$. A reader can easily reformulate and verify the validity of our results for the case of $\oplus$- sb-pseudometrics, which is without the assumption of axiom $(m_{1'})$.

4. Categories of $\oplus$-Metric-Type Spaces

To view the $\oplus$-metric-type spaces considered in the previous section in the framework of a category, we have to define their morphisms, which is "continuous' mappings. To do this, we first define an open ball in a $\oplus$-metric type space.

Definition 5. *Given a set X, an extended t-conorm $\oplus$ and an $\oplus$-based metric-type structure* $d : X \times X \to \mathbb{R}^+$*. We define an open ball* $B(a,r)$ *with center* $a \in X$ *and radius* $r > 0$ *in the space* (X,d) *as follows:*

$$B(a,r) = \{x | x \in X, d(a,x) < r\}.$$

We call an open ball $B(a,r)$ *really open, if, for every* $x \in B(a,r)$*, there exists* $\varepsilon > 0$*, such that* $B(x,\varepsilon) \subseteq B(a,r)$*.*

Note that saying that a ball $B(a,r)$ is really open just means that it is open in the topology obtained from the family $\{B(x,r) \mid x \in X, r > 0\}$ as a base.

Theorem 1. *Let $\oplus$ be a continuous at the bottom extended t-conorm. Subsequently, open balls in $\oplus$-metric spaces and in $\oplus$-sb-metric spaces are really open.*

We prove the theorem in the case of $\oplus$-sb-metric spaces. Patterned after this proof, a reader can easily prove it for $\oplus$-metric spaces. Certainly, this fact for $\oplus$-metric spaces in cases of ordinary metrics ($\oplus = +$) and ultrametrics ($\oplus = \vee$) is well known.

Proof. Consider the open ball $B(x,r)$ where $x \in X$ and $r > 0$. Let $y \in B(x,r)$. Then $d(x,y) = \alpha < r$ and there exists $\beta > 0$, such that $\alpha \oplus \beta < r$ since $\oplus$ is continuous on $\{0\} \times \mathbb{R}^+ \cup \mathbb{R}^+ \times \{0\}$. (Indeed, consider $l_\alpha : \mathbb{R}^+ \to \mathbb{R}^+, l_\alpha(x) = \alpha \oplus x$. Notice that $l_\alpha(0) = \alpha < r$ and by assumption l_α is continuous at 0. Therefore, there exists $\beta > 0$, such that $l_\alpha(\beta) = \alpha \oplus \beta < r$). Now, we shall show that $B\left(y, \frac{\beta}{k}\right) \subseteq B(x,r)$. Let $z \in B\left(y, \frac{\beta}{k}\right)$. Subsequently, we have

$$d(x,z) \leq d(x,y) \oplus k \cdot d(y,z) < \alpha \oplus k \cdot \frac{\beta}{k} = \alpha \oplus \beta < r$$

and this means that $B\left(y, \frac{\beta}{k}\right) \subseteq B(x,r)$. □

Remark 3. *The statement of the previous theorem actually means that open balls are open in the topology generated by the base $\{B(a,r) \mid a \in X, r > 0\}$. This might not be true for $\oplus$-b-metric spaces, even in case $\oplus = +$, see e.g., [2,3].*

Corollary 1. *In case $\oplus$ is continuous at the bottom, $\oplus$-metric and $\oplus$-sb-metric spaces are first countable.*

Theorem 2. *Let $\oplus$ be a continuous at the bottom extended t-conorm. Subsequently, closed balls defined as $B[x,r] = \{y | y \in X, d(x,y) \leq r\}$ in $\oplus$-metric spaces and in $\oplus$-sb-metric spaces are closed.*

Proof. We prove the theorem in the case of $\oplus$-sb-metric spaces. Let d be a $\oplus$-*sbk* metric on X, $y \in X - B[x,r]$ where $x, y \in X, r > 0$. We shall show that $B(y, \frac{r_1}{k}) \subset X - B[x,r]$, where $r_1 = d(x,y) - r > 0$. For any $z \in B(y, \frac{r_1}{k})$, we have

$$r + r_1 = d(x,y) \leq d(x,z) + k \cdot d(z,y) < d(x,z) + k \cdot \frac{r_1}{k}$$

and

$$d(x,z) > r$$

means that $z \in X - B[x,r]$. Therefore, $X - B[x,r]$ is open and $B[x,r]$ is closed in X. □

Definition 6. *Two $\oplus$-sb-metrics ($\oplus$-b-metrics) are called equivalent if they induce the same topology.*

Remark 4. *Because the topologies induced by $\oplus$-sb-metrics are first countable, topologies $\mathcal{T}_d$ and $\mathcal{T}_{d'}$ coincide if and only if a sequence $(x_n)_{n \in \mathbb{N}}$ converges to a point x_0 in topology $\mathcal{T}_d$ if and only if $(x_n)_{n \in \mathbb{N}}$ converges to x_0 in topology $\mathcal{T}_{d'}$*

Definition 7. *Let $\oplus$ be a continuous at the bottom extended t-conorm and let (X, d_X) and (Y, d_Y) be $\oplus$-metric or $\oplus$-sb-metric spaces. A mapping $f : (X, d_X) \to (Y, d_Y)$ is called continuous if it is continuous as a mapping $f : (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ where to $\mathcal{T}_{d_X}$ and $\mathcal{T}_{d_Y}$ are the topologies induced by d_X and d_Y respectively.*

Because, in these cases, open balls are really open and hence form bases for the corresponding first countable topologies, we easily get the following result:

Theorem 3. *The following are equivalent for a mapping $f : (X, d_X) \to (Y, d_Y)$ where d_X and d_Y are $\oplus$-metrics or $\oplus$-sb-metrics and $\oplus$ is a continuous extended t-conorm:*

1. *$f : (X, d_X) \to (Y, d_Y)$ is continuous;*
2. *for every $a \in X$, every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(a), f(x)) < \varepsilon$ whenever $d_X(a, x) < \delta$;*
3. *if a sequence $x_1, x_2, \dots x_n \dots$ converges to a point x_0 in the space (X, d_X), then the sequence $f(x_1), f(x_2), \dots f(x_n) \dots$ converges to the point $f(x_0)$ in the space (Y, d_Y).*

Theorem 4. *If the extended t-conorm $\oplus$ is continuous, then an $\oplus$-sb-metric of the space (X, d) is continuous as the mapping $d : X \times X \to \mathbb{R}^+$.*

Proof. Because the topology induced by an $\oplus$-sb-metric is first countable, it is sufficient to prove that, if $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ are sequences in X and $\lim_n d(x_n, x_0) = 0$, $\lim_n d(y_n, y_0) = 0$, then $\lim_n d(x_n, y_n) = d(x_0, y_0)$. We do it as follows:

$$\begin{aligned} d(x_0, y_0) &\leq k \cdot d(x_0, x_n) \oplus d(x_n, y_0) \\ &\leq k \cdot d(x_0, x_n) \oplus k \cdot d(y_n, y_0) \oplus d(x_n, y_n) \\ &\leq k \cdot d(x_0, x_n) \oplus k \cdot d(y_n, y_0) \oplus k \cdot d(x_n, x_0) \oplus d(x_0, y_n) \\ &\leq k \cdot d(x_0, x_n) \oplus k \cdot d(y_n, y_0) \oplus k \cdot d(x_n, x_0) \oplus k \cdot d(y_n, y_0) \oplus d(x_0, y_0) \end{aligned}$$

Taking limits on the both sides of the inequality, we conclude first the existence of $\lim_n d(x_n, y_n)$ and then that $\lim_n d(x_n, y_n) = d(x_0, y_0)$. □

Now we are ready to define some categories for $\oplus$-metric type spaces under the assumption that $\oplus$ is a continuous extended t-conorm.

1. The objects of the category $\oplus$-**Metr** are pairs (X, d) where X is a set and d is an $\oplus$-metric on it. The morphisms of the category $\oplus$-**Metr** are continuous mappings $f : (X, d_X) \to (Y, d_Y)$.
2. The objects of the category $\oplus$-**Mtrz** are pairs $(X, \mathcal{T}_d)$ where X is a set and $\mathcal{T}_d$ is a topology induced by some $\oplus$-metric d. The morphisms of the category $\oplus$-**Mtrz** are continuous mappings $f : (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$.
3. The objects of the category $\oplus$-**SbMetr** of $\oplus$-sb-metric spaces are pairs (X, d) where X is a set and d is an $\oplus$-sb-metric on it. The morphisms of the category $\oplus$-**SbMetr** are continuous mappings $f : (X, d_X) \to (Y, d_Y)$.

   By $\oplus$-**SbkMetr**, we denote the full subcategory of the category $\oplus$-**SbMetr**, whose objects are $\oplus$-sbk-metric spaces.
4. The objects of the category $\oplus$-**SbMtrz** are pairs $(X, \mathcal{T}_d)$, where X is a set and $\mathcal{T}_d$ is a topology that is induced by some $\oplus$-sb-metric d. The morphisms of the category $\oplus$-**SbMtrz** are continuous mappings $f : (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$. Let $\oplus$-**SbkMtrz** be the full subcategory of $\oplus$-**SbMtrz** whose objects are obtained by $\oplus$-sbk-metrics.
5. On the basis of $\oplus$-b-metrics we introduce two categories .

   (a) (see [3]) Let $\mathcal{S}_d$ be the family of all unions of open balls, that is

   $$\mathcal{S}_d = \{U \subseteq X : \exists B_d(a_i, \varepsilon_i), i \in I \text{ such that } U = \bigcup_{i \in I} B_d(a_i, \varepsilon_i)\}.$$

   The family $\mathcal{S}_d$ is obviously a supratopology (see e.g., [16,17]), that is closed under taking arbitrary unions. $\mathcal{S}_d$ need not be a topology: the intersection of even two elements $U_1, U_2 \in \mathcal{S}_d$ need not be in $\mathcal{S}_d$ since an open ball need not be open in $\mathcal{S}_d$. Let $\oplus$-**b-MetrS** be a category whose objects are pairs (X, d) where X is a set and d is an $\oplus$-b-metric on it and whose morphisms are continuous mappings $f : (X, \mathcal{S}_{d_X}) \to (Y, \mathcal{S}_{d_Y})$

   (b) (see [3]) We call a set $U \subseteq X$ $\mathcal{T}_d$-open if for every $x \in X$ there exists $\varepsilon > 0$, such that $B(a, \varepsilon) \subseteq U$. One can easily notice that $U_1, U_2 \in \mathcal{T}_d \implies U_1 \cap U_2 \in \mathcal{T}_d$ and the union of any family of $\mathcal{T}_d$-open sets is $\mathcal{T}_d$-open. Thus $\mathcal{T}_d$ is indeed a topology on X.

Obviously, each $U \in \mathcal{T}_d$ can be expressed as a union of some open balls and for this reason it belongs to $\mathcal{S}_d$. Thus, $\mathcal{T}_d \subseteq \mathcal{S}_d$. On the other hand not every open ball $B(a,\varepsilon)$ needs to be $\mathcal{T}_d$-open and, hence, generally $\mathcal{S}_d \neq \mathcal{T}_d$. Let **$\oplus$-b-MetrT** be the category whose objects are pairs (X,d) where X is a set and d is an $\oplus$-b-metric on it and whose morphisms are continuous mappings $f : (X,\mathcal{T}_{d_X}) \to (Y,\mathcal{T}_{d_Y})$.

5. Examples

5.1. Examples of Sb-Metric Spaces

In this subsection, we assume that $\oplus = +$. Henceforth the resulting spaces are "classical" sb-metric spaces [1].

Example 6. *Let $X = \{x_n : n \in \{0,1,2,3,\cdots\}\} \cup \{0\}$ where $x_n = \dfrac{1}{2^n}, n \in \{0,1,2,3,\cdots\}$ and $d : X \times X \to \mathbb{R}$ be a distance function defined as follows:*

$$d(x,y) = \begin{cases} \frac{1}{2}\left|x-y\right| & \text{, } x \text{ and } y \text{ are consecutive terms} \\ \left|x-y\right| & \text{, otherwise} \end{cases}$$

Then d is an sb4-metric.

Proof. Notice that $d\left(\dfrac{1}{2^n}, \dfrac{1}{2^{n-1}}\right) = \dfrac{1}{2}\left(\dfrac{1}{2^{n-1}} - \dfrac{1}{2^n}\right) = \dfrac{1}{2^{n+1}}$.
For any $x,y,z \in X$ such that none of the pairs $\{x,y\}, \{x,z\}, \{z,y\}$ is consecutive it holds

$$\begin{aligned} d(x,y) &= \left|x-y\right| \leq |x-z| + \left|z-y\right| \\ &\leq |x-z| + 4 \cdot \left|z-y\right| \\ &= d(x,z) + 4 \cdot d(z,y). \end{aligned}$$

Accordingly, we only need to investigate those x, y and z for which at least one of the pairs $\{x,y\}, \{x,z\}, \{z,y\}$ is consecutive.
Let $x,y \in X$ be given. Without the loss of generality, we assume that $x < y$:
If $z = x$ or $z = y$, then the equality $d(x,y) = d(x,z) + d(z,y)$ is obvious.
If $x < y < z$, then it means that x and z are not consecutive and we have

$$d(x,y) = d(y,x) \leq \left|x-y\right| \leq |x-z| = d(x,z) = d(z,x)$$

and

$$\begin{aligned} d(x,y) &\leq d(x,z) + 4 \cdot d(z,y) \\ d(y,x) &\leq d(y,z) + 4 \cdot d(z,x). \end{aligned}$$

If $z < x < y$, then it means that z and y are not consecutive and we have

$$d(x,y) = d(y,x) \leq \left|x-y\right| \leq \left|y-z\right| = d(y,z) = d(z,y)$$

and

$$\begin{aligned} d(x,y) &\leq d(x,z) + 4 \cdot d(z,y) \\ d(y,x) &\leq d(y,z) + 4 \cdot d(z,x). \end{aligned}$$

If $x < z < y$, then it means that x and y are not consecutive and $d(x,y) = \left|x-y\right|$. Here, we have three cases.
Case 1: x and z are consecutive, but y and z are not consecutive.

For this case, $x \neq 0$ since x and z are consecutive. Let $x = \frac{1}{2^n}, z = \frac{1}{2^{n-1}} < y$. Here, notice that $y > \frac{1}{2^{n-2}}$. Subsequently,

$$\begin{aligned} d(x,z) + 4 \cdot d(z,y) &= \frac{1}{2^{n+1}} + 4 \cdot \left(y - \frac{1}{2^{n-1}}\right) \\ &= 4 \cdot y - \frac{15}{2^{n+1}} > y - \frac{2}{2^{n+1}} = y - \frac{1}{2^n} = d(x,y), \end{aligned}$$

$$\begin{aligned} d(y,z) + 4 \cdot d(z,x) &= \left(y - \frac{1}{2^{n-1}}\right) + 4 \cdot \left(\frac{1}{2^{n+1}}\right) \\ &= y > y - \frac{1}{2^n} = d(y,x). \end{aligned}$$

Case 2: x and z are not consecutive, but y and z are consecutive.
Let $x < z = \frac{1}{2^n}$ and $y = \frac{1}{2^{n-1}}$. Here, notice that $x < \frac{1}{2^{n+1}}$. Afterwards,

$$\begin{aligned} d(x,z) + 4 \cdot d(z,y) &= \left(\frac{1}{2^n} - x\right) + 4 \cdot \left(\frac{1}{2^{n+1}}\right) \\ &= \frac{6}{2^{n+1}} - x > \frac{4}{2^{n+1}} - x = \frac{1}{2^{n-1}} - x = d(x,y), \end{aligned}$$

$$\begin{aligned} d(y,z) + 4 \cdot d(z,x) &= \frac{1}{2^{n+1}} + 4 \cdot \left(\frac{1}{2^n} - x\right) \\ &= \frac{9}{2^{n+1}} - 4 \cdot x > \frac{4}{2^{n+1}} - x = \frac{1}{2^{n-1}} - x = d(y,x). \end{aligned}$$

Case 3: x and z are consecutive and y and z are consecutive.
For this case again $x \neq 0$ since x and z are consecutive. Let $x = \frac{1}{2^n}, z = \frac{1}{2^{n-1}}$ and $y = \frac{1}{2^{n-2}}$. Subsequently,

$$\begin{aligned} d(x,z) + 4 \cdot d(z,y) &= \frac{1}{2^{n+1}} + 4 \cdot \left(\frac{1}{2^n}\right) \\ &= \frac{9/2}{2^n} \geq \frac{3}{2^n} = \frac{1}{2^{n-2}} - \frac{1}{2^n} = d(x,y), \end{aligned}$$

$$d(y,z) + 4 \cdot d(z,x) = \frac{1}{2^n} + 4 \cdot \left(\frac{1}{2^{n+1}}\right) = \frac{3}{2^n} = \frac{1}{2^{n-2}} - \frac{1}{2^n} = d(y,x).$$

□

Remark 5. *In above example, d is not a metric. To see this, consider $x = \frac{1}{2^2}, z = \frac{1}{2^1}$ and $y = 1$. Afterwards,*

$$d\left(\frac{1}{2^2}, \frac{1}{2^1}\right) + d\left(\frac{1}{2^1}, 1\right) = \frac{1}{2^3} + \frac{1}{2^2} = \frac{3}{8} \ngeq \frac{3}{4} = 1 - \frac{1}{2^2} = d\left(\frac{1}{2^2}, 1\right).$$

On the other hand, the induced topology $\mathcal{T}_d$ is the same as the topology induced on X by the ordinary topology on the real line. More precisely $\mathcal{T}_d = \mathcal{T}_{d'}$, where d' is the ordinary metric. Hence, $\mathcal{T}_d$ is not a discrete space. However, any element X except 0 is an isolated point.

Example 7. *Let $X_a = \{a\} \times [0,1], X_b = \{b\} \times [0,1], X_c = \{c\} \times [0,1]$ and $X = X_a \cup X_b \cup X_c$. We denote $x = \{i\} \times \overline{x} \in X$ where $\overline{x} \in [0,1]$ and $i \in \{a,b,c\}$. Define $d : X \times X \to [0,5]$, as follows:*

$$d(x,y) = d(y,x) = \begin{cases} |\overline{x} - \overline{y}| & , x, y \in X_i \\ 1 & , x \in X_a, y \in X_b \\ 2 & , x \in X_a, y \in X_c \\ 5 & , x \in X_b, y \in X_c \end{cases}.$$

Subsequently, d is an sb3-metric.

Proof. For any $x, y, z \in X$, we will show that

$$d(x,y) \leq d(x,z) + 3 \cdot d(z,y).$$

We need to consider five cases:
Case 1: if $x, y, z \in X_i$, then it is obvious since $d(x,y)$ is usual metric on each i-level.
Case 2: if $x, y \in X_i$ and $z \in X_j$, where $i \neq j$, then $d(x,y) \leq 1, d(x,z) \geq 1, d(y,z) \geq 1$ and

$$\begin{aligned} d(x,y) &\leq 1 \leq d(x,z) + 3 \cdot d(z,y), \\ d(y,x) &\leq 1 \leq d(y,z) + 3 \cdot d(z,x). \end{aligned}$$

Case 3: if $x, z \in X_i$ and $y \in X_j$ where $i \neq j$, then $d(x,y) = d(z,y)$ and

$$\begin{aligned} d(x,y) &= d(z,y) \leq d(x,z) + 3 \cdot d(z,y), \\ d(y,x) &= d(y,z) \leq d(y,z) + 3 \cdot d(z,x). \end{aligned}$$

Case 4: if $x \in X_i$ and $y, z \in X_j$ where $i \neq j$, then $d(x,y) = d(x,z)$ and

$$\begin{aligned} d(x,y) &= d(x,z) \leq d(x,z) + 3 \cdot d(z,y), \\ d(y,x) &= d(z,x) \leq d(y,z) + 3 \cdot d(z,x). \end{aligned}$$

Case 5: if $x \in X_i, y \in X_j$ and $z \in X_k$, where $i \neq j \neq k$, then we patterned the solution of Example 2.1 in [18]. Let fix $i = a, j = b$ and $k = c$ (note that other combinations would give the same results). Subsequently, we need to check the following:

$$\begin{aligned} d(x,y) + 3 \cdot d(y,z) &= 1 + 3 \cdot 5 \geq 2 = d(x,z), \\ d(z,y) + 3 \cdot d(y,x) &= 5 + 3 \cdot 1 \geq 2 = d(z,x), \\ d(x,z) + 3 \cdot d(z,y) &= 2 + 3 \cdot 5 \geq 1 = d(x,y), \\ d(y,z) + 3 \cdot d(z,x) &= 5 + 3 \cdot 2 \geq 1 = d(y,x), \\ d(y,x) + 3 \cdot d(x,z) &= 1 + 3 \cdot 2 \geq 5 = d(y,z), \\ d(z,x) + 3 \cdot d(x,y) &= 2 + 3 \cdot 1 = 5 = d(z,y). \end{aligned}$$

□

Remark 6. *In the above example, notice that every i-level is an ordinary metric and, between the levels, it behaves as sb3-metric. Moreover, the induced topology does not contain any isolated point.*

Example 8. *Let X be the unit disc and S^1 be the unit circle in $\mathbb{R}^2$ with the center in the origin $0 = (0_1, 0_2)$ of the plane and let the distance function $d : X \times X \to \mathbb{R}$ be defined, as follows: for $x = (x_1, x_2)$ and $y = (y_1, y_2)$*

$$d(x,y) = d(y,x) = \begin{cases} 3 & , x, y \in S^1 \\ d'(x,y) & , otherwise \end{cases}$$

where d' is the post office metric, which is $d'(x,y) = \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$. Then d is an sb2-metric which fails to be a metric.

Proof. Because other cases are obvious, the only case that we need investigate is the case when $x, y \in S^1$ and $z \notin S^1$.

$$\begin{aligned} d(x,z) + 2\cdot d(z,y) &= 1 + \sqrt{z_1^2+z_2^2} + 2\cdot\left(1+\sqrt{z_1^2+z_2^2}\right) \\ &= 3 + 3\cdot\sqrt{z_1^2+z_2^2} = 3 + 3\cdot d(z,0) \\ &\geq 3 = D(x,y). \end{aligned}$$

□

Remark 7. *This example can be generalized for any other value of $k \geq 1$ if we define $d(x,y) = k+1$ for $x, y \in S^1$.*

5.2. Examples of $\oplus$-Metric Type Spaces

Example 9. *Consider the discrete space $X = \{a,b,c\}$ where the distance function d defined, as follows:*
$d(a,a) = d(b,b) = d(c,c) = 0,$
$d(a,b) = d(b,a) = 2, d(b,c) = d(c,b) = 4, d(a,c) = d(c,a) = 8$
Notice that d is neither a metric nor an ultra metric.

(1) *If $\oplus = +$,then d is an $\oplus$-sb2-metric (simply sb2-metric).*
(2) *If $\oplus = \vee$ then d is an $\oplus$-sb4-metric (might be called ultra sb4-metric).*
(3) *If $\oplus = \oplus_T$ (see Example 2.7) then d is an $\oplus$-metric. Note that $\oplus_T$ is not distributive. However, in this case, distributivity is not a necessary condition.*
(4) *Let $\oplus$ be the h-shifted arithmetic sum for an arbitrary h. If $h \geq 2$, then d is an $\oplus$-metric. If $h < 2$, then d is an $\oplus$-sb- $\frac{4-h}{2}$-metric.*

Example 10. *Let $X = [0,1]$, $k > 0$ and $\oplus = +$. Define $d : X \times X \to \mathbb{R}$ as follows:*

$$d(x,y) = d(y,x) = \begin{cases} k & , x = 0, y = 1 \\ |x-y| & , otherwise. \end{cases}$$

(a) If $k = 1$, then d is an $\oplus$-metric.
(b) If $k < 1$, then d is an $\oplus$-b$\frac{1}{k}$-metric.
(c) If $k > 1$, then d is an $\oplus$-bk-metric.

Proof. The case (a) is obvious.
(b) Let $k \in (0,1)$. Then the only cases we need to investigate are whether the equalities

$$d(x,0) \leq \frac{1}{k}\cdot(d(0,1) \oplus d(1,x))$$

and

$$d(1,x) \leq \frac{1}{k}\cdot(d(1,0) \oplus d(0,x)).$$

hold. We get them, as follows:

$$\frac{1}{k}\cdot(d(0,1) \oplus d(1,x)) = \frac{1}{k}\cdot(k+1-x) = 1 + \frac{1-x}{k} \geq x = d(0,x)$$

and

$$\frac{1}{k}\cdot\left(d\left(1,0\right)\oplus d\left(0,x\right)\right)=\frac{1}{k}\cdot\left(k+x\right)=1+\frac{x}{k}\geq 1-x=d\left(1,x\right).$$

Therefore, d is an $\oplus$-$b\frac{1}{k}$-metric space. On the other hand,

$$d\left(1,0\right)\oplus d\left(0,x\right)=k+x\ngeq 1-x=d\left(1,x\right)$$

and

$$d\left(1,0\right)\oplus k\cdot d\left(0,x\right)=k+x\ngeq 1-x=d\left(1,x\right).$$

Hence, d is neither an $\oplus$-metric and nor an $\oplus$-*sb*-metric.
(c) Let $k>1$. Subsequently, the only case we need to investigate is

$$d\left(0,1\right)\leq k\cdot\left(d\left(0,x\right)\oplus d\left(x,1\right)\right).$$

Here, again, we have

$$d\left(0,x\right)\oplus d\left(x,1\right)=x+\left(1-x\right)=1\ngeq k=d\left(0,1\right)$$

and

$$d\left(0,x\right)\oplus k\cdot d\left(x,1\right)=x+k\cdot\left(1-x\right)\ngeq k=d\left(0,1\right).$$

Hence, d is neither an $\oplus$-metric and nor an $\oplus$-*sb*-metric. However,

$$k\cdot\left(d\left(0,x\right)\oplus d\left(x,1\right)\right)=k\cdot\left(x+\left(1-x\right)\right)=k=d\left(0,1\right)$$

and, hence, d is an $\oplus$-*bk*-metric. □

Example 11. *Now, we consider the above example where $\oplus$ is h-shifted arithmetic sum.*
(a) If $k=1$, then d is an $\oplus$-metric.
(b) If $k<1$ and $k+h\geq 1$, then d is an $\oplus$-metric.
(c) If $k<1$ and $k+h<1$, then d is an $\oplus$-$b\frac{1}{k}$-metric.
(d) If $k>1$ and $1+h\geq k$, then d is an $\oplus$-metric.
(e) If $k>1$ and $1+h<k$, then d is an $\oplus$-$b\frac{k}{1+h}$ metric.

Proof. The case (a) is obvious, since $\oplus\geq +$.
(b) Let $k<1$ and $k+h\geq 1$, then we have

$$d\left(0,1\right)\oplus d\left(1,x\right)=k\oplus\left(1-x\right)=k+1-x+h\geq 2-x\geq x=d\left(0,x\right)$$

and

$$d\left(1,0\right)\oplus d\left(0,x\right)=k\oplus x=k+x+h\geq 1+x\geq 1-x=d\left(1,x\right).$$

Therefore, d is an $\oplus$-metric.
(c) Let $k<1$ and $k+h<1$, then we have

$$\begin{aligned}\frac{1}{k}\cdot\left(d\left(0,1\right)\oplus d\left(1,x\right)\right) &= \frac{1}{k}\cdot\left(k\oplus\left(1-x\right)\right)=\frac{1}{k}\cdot\left(k+1-x+h\right)\\ &= 1+\frac{1-x+h}{k}\geq x=d\left(0,x\right)\end{aligned}$$

and

$$\begin{aligned} \frac{1}{k}\cdot(d(1,0)\oplus d(0,x)) &= \frac{1}{k}\cdot(k\oplus x)=\frac{1}{k}\cdot(k+x+h) \\ &= 1+\frac{x+h}{k}\geq 1-x=d(1,x). \end{aligned}$$

Hence, d is an $\oplus$-$b\frac{1}{k}$-metric.
(d) Let $k>1$ and $1+h\geq k$, then we have

$$d(0,x)\oplus d(x,1)=(x\oplus(1-x))=(x+1-x+h)\geq k=d(0,1)$$

means that d is an $\oplus$-metric.
(e) Let $k>1$ and $1+h<k$, then we have

$$\frac{k}{1+h}\cdot(d(0,x)\oplus d(x,1))=\frac{k}{1+h}\cdot(x\oplus(1-x))=\frac{k}{1+h}\cdot(x+1-x+h)=k=d(0,1).$$

Consequently, d is an $\oplus$-$b\frac{k}{1+h}$-metric. □

Remark 8. *Similarly, it can be shown that d is not an $\oplus$-$sb\frac{1}{k}$-metric and d is not an $\oplus$-$sb\frac{k}{1+h}$-metric in the cases (c) and (e) in the above example, respectively.*

Here, we want to emphasize that continuity of $\oplus$ at the bottom level is essential for the openness of the open balls for $\oplus$-sb metric spaces as well as for $\oplus$-metric spaces.

Example 12. *Let $X=[0,1]$. Define $d: X\times X\to\mathbb{R}$, as follows:*

$$d(x,y)=d(y,x)=\begin{cases} \frac{1}{2} & ,x=0,y=1 \\ |x-y| & ,\text{otherwise.} \end{cases}$$

If we choose $\oplus$ as the h-shifted arithmetic sum for $h=1$, then by Example 11, d is an $\oplus$-metric. However, the open balls $B(0,r)=[0,r)\cup\{1\}$ where $r\in\left(\frac{1}{2},1\right)$ are not open in the induced topology.

6. Products of $\oplus$-Sb-Metric Spaces

Here, we distinguish the cases of finite and infinite (countable) number of spaces.

6.1. Products of Finite Families of $\oplus$-Sb-Metric Spaces

Let $\{(X_i,d_i): i=1,\ldots,n\}$ be a family of $\oplus$-sb-metrics spaces. Further, assume that (X_i,d_i) is an $\oplus$-sbk$_i$-metric space. Because the family is finite, we may take $k=\max\{k_1,\ldots,k_n\}$. Subsequently, all d_i are $\oplus$-sbk-metrics. We define $X=\prod_{i=1}^n X_i$ and $d: X\times X\to[0,\infty)$ by $d(x,y)=\bigoplus_{i=1}^n d(x_i,y_i)$ where $x,y\in X$ and x_i,y_i are i-th coordinates of x and y, respectively.

Theorem 5. *(X,d) is the product of the family $\{(X_i,d_i): i=1,\cdot,n\}$ in the category $\oplus$-**SbMetr**, where $\oplus$ is a distributive continuous on the bottom extended t-conorm. Besides, the topology $\mathcal{T}_d$ that is induced by the $\oplus$-sb-metric d on X coincides with the product of the topologies $\mathcal{T}_{d_i}$ induced by $\oplus$-sb-metrics d_i.*

Proof. First, we have to show that d is an $\oplus$-sb-metric. Properties $(m_{1'})$ and (m_2) of d follow obviously from the corresponding properties of all $d_i, i=1,\ldots,n$, respectively. To show the validity of axiom (msb_3), we fix $x,y,z\in X$, and applying the axiom (msb_3) for every d_i and disributivity of $\oplus$, we are reasoning, as follows:

$$\begin{aligned} d(x,y) &= \bigoplus_{i=1}^{n} d_i(x_i,y_i) \leq \bigoplus_{i=1}^{n} (d_i(x_i,z_i) \oplus k \cdot d_i(z_i,y_i)) \\ &= \bigoplus_{i=1}^{n} d(x_i,z_i) \oplus k \cdot \bigoplus_{i=1}^{n} d(z_i,y_i) = d(x,z) \oplus k \cdot d(z,y). \end{aligned}$$

Further, let (Z,ρ) be an $\oplus$-sb-metric space and let for every $i = 1,\ldots,n$ a mapping $\varphi_i : Z \to X_i$ be continuous. Subsequently, from the construction, it is clear that, by setting $\varphi(z) = (\varphi_i(z) \in X_i)_{i=1,\ldots,n}$, we obtain a continuous mapping $\varphi : Z \to X$, such that $p_i \circ \varphi = \varphi_i : Z \to X_i$ where $p_i : X \to X_i$ is the projection. Thus, (X,d) is the product of the family $\{(X_i,d_i) : i = 1,\cdot,n\}$ in the category $\oplus$-**SbMetr**. To show the second statement of the theorem, notice first that all of the projections $p_i : (X,d) \to (X_i,d_i)$ are continuous in $\oplus$-**SbMetr**. Indeed, let $\varepsilon > 0$ and a point $a \in \prod_i X_i$ be given. Because $\oplus$ is monotone and 0 is the neutral element of $\oplus$, we have $a,b \leq a \oplus b$ for any $a,b \in [0,1]$. Therefore, $d_i(a_i,x_i) < \varepsilon$ for any $i = 1,\ldots n$ whenever $d(a,x) < \varepsilon$. Hence, the topology induced by the $\oplus$-sb-metric d is stronger or equal than the topology of the product of topologies $\mathcal{T}_{d_i}$ induced by $\oplus$-sb-metrics d_i.

To prove the converse inequality, we show that every set $U \subseteq X$ that is open in $\mathcal{T}_d$ is also open in the product topology $\prod_i \mathcal{T}_{d_i}$. Let $a \in U$, we fix $\varepsilon > 0$ such that $B_d(a,\varepsilon) \subseteq U$. Subsequently, for each coordinate i, take the ball $B_{d_i}(a_i,\frac{\varepsilon}{n})$. Applying distributivity and monotonicity of the t-conorm $\oplus$, we conclude that $d_i(a_i,x_i) \leq \frac{\varepsilon}{n}$ for all $i - 1,\ldots,n$ implies $d(a,x) \leq \varepsilon$. However, this means that

$$\bigcap\nolimits_{i=1}^{n} p_i^{-1} B_{d_i}\left(a_i,\frac{\varepsilon}{n}\right) \subseteq B_d(a,\varepsilon)$$

and hence U is open in the product topology. □

In case all d_i in the previous theorem are $\oplus$-sbk-metrics with a fixed k, we get the following corollary:

Corollary 2. *(X,d) defined in the previous theorem is the product of the family $\{(X_i,d_i) : i = 1,\ldots,n\}$ in the category $\oplus$-**SbkMetr**.*

Remark 9. *Patterned after the above proof it can also be proved in the context of $\oplus$-metric for any distributive continuous on the bottom extended t-conorm. Certainly, in case $\oplus = +$ this result can be found in almost any book on topology or functional analysis. Concerning the category of $\oplus$-b-metric spaces, we have the following:*

*Let $\{(X_i,d_i) : i = 1,\ldots,n\}$ be a family of $\oplus$-b-metrics. We define $X = \prod_{i=1}^{n} X_i$ and $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \bigoplus_{i=1}^{n} d_i(x_i,y_i)$ where $x,y \in X$ and x_i,y_i are i-th coordinates of x and y, respectively. Subsequently, patterned after the proof of Theorem 5, one can show that (X,d) is the product of the family $\{(X_i,d_i) : i = 1,\cdot,n\}$ in the category $\oplus$-**bMetr**. However, for the reasons discussed above, we cannot confirm the topological part of the previous theorem.*

6.2. Products of Infinite Families of $\oplus$-Sb-Metrics

In case of infinite products, we have to restrict our results to the cases when $\oplus = +$ and $\oplus = \vee$. Patterned after the standard proof of the theorem about the product of a countable family of ordinary metrics, see e.g., [19], we have to replace a given $\oplus$-sb-metric by an equivalent bounded $\oplus$-sb-metric, which is a metric that induces the same topology on the underlying set X. In order to do this, we ask the additional assumption that $\oplus$ is compressible, which is

$(\oplus_6)$ $a \leq b \oplus c \implies \frac{a}{1+a} \leq \frac{b}{1+b} \oplus \frac{c}{1+c}$

Lemma 2. *Operations $\oplus = " + "$ and $\oplus = " \vee "$ are compressible.*

Proof. Straightforward. □

We consider the cases $\oplus = +$ and $\oplus = \vee$ separately. First, we consider the case $\oplus = +$, which is $\oplus$-sb-metrics are "ordinary" sb-metrics.

Theorem 6. *Let $\rho : X \times X \to \mathbb{R}^+$ be an sbk-metric. Subsequently, by setting $d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$ a bounded sbk-metric is defined. Besides, sbk-metrics d and ρ are equivalent.*

Proof. Obviously,

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$$

satisfies properties (m_1') and (m_2). To show that axiom (msb_3) holds for d, we refer to the compressibility of $\oplus = +$ and the validity of this axiom for ρ and have

$$\begin{aligned} d(x,y) &= \frac{\rho(x,y)}{1+\rho(x,y)} \leq \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{k \cdot \rho(z,y)}{1+k \cdot \rho(z,y)} \\ &\leq \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{k \cdot \rho(z,y)}{1+\rho(z,y)} = d(x,z) + k \cdot d(z,y). \end{aligned}$$

The equivalence of the $\oplus$-sbk-metrics d and ρ is obvious, since they have the same convergent sequences. □

Let $\{(X_i, \rho_i) : i \in \mathbb{N}\}$ be a countable family of sbk-metric spaces. Let $d_i(x,y) = \frac{\rho_i(x,y)}{1+\rho_i(x,y)}$. We define $X = \prod_{i \in \mathbb{N}} X_i$ and $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} d_i(x_i, y_i)$$

where $x, y \in X$ and x_i, y_i are i^{th} coordinates of x and y, respectively. Further, let $p_i : X \to X_i$ be the corresponding projection.

Theorem 7. *(X,d) is the product of the family $\{(X_i, d_i) : i \in \mathbb{N}\}$ in* **SbkMetr***. The topology that is induced by $\oplus$-sbk-metric d coincides with the topology of the product of the topologies $\mathcal{T}_{d_i}$.*

Proof. We are reasoning patterned after the proof of Theorem 4.2.2 in [19]. Since $d_i(x_i, y_i) \leq 1$ for every $i \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} \frac{1}{2^i} d_i(x_i, y_i) \leq 1$ and hence the definition of the sum is correct. To show that d is thus defined an sbk-metric, note that the validity of axioms $(m1')$ and $(m2)$ is obvious. We verify the validity of axiom (mbs_3), as follows. Because all d_i are sbk-metrics (with the same k) for the points $x, y, z \in X$, we have:

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + k \cdot d_i(z_i, y_i) \; \forall i \in \mathbb{N}.$$

Multiplying these inequalities by $\frac{1}{2^i}$ and summing them up, we obtain

$$\begin{aligned} d(x,y) &= \sum \frac{1}{2^i} d_i(x_i, y_i) \leq \sum \left(\frac{1}{2^i} (d_i(x_i, z_i) + k \cdot d_i(z_i, y_i) \right) \\ &= \sum \frac{1}{2^i} d_i(x_i, z_i) + \sum \frac{1}{2^i} k \cdot d_i(z_i, y_i) = d(x,z) + k \cdot d(z,y). \end{aligned}$$

Further, let (Z, δ) be a sbk-metric space and let for every $i \in \mathbb{N}$ a mapping $\varphi_i : (Z, \delta) \to (X_i, d_i)$ be continuous. Subsequently, from the construction, it is clear that, by setting $\varphi(z) = (\varphi_i(z) \in X_i)_{i \in \mathbb{N}}$, we obtain a continuous mapping $\varphi : Z \to X$, such that $p_i \circ \varphi = \varphi_i : Z \to X_i$ where $p_i : X \to X_i$ is the projection. Thus, (X,d) is indeed the product of the spaces (X_i, d_i) in the category **SbkMetr**.

The proof that topology $\mathcal{T}_d$ induced by sbk-metric d coincides with the product of the topologies $\mathcal{T}_{d_i}$ induced by sbk-metrics d_i can be done repeating verbatim the proof of Theorem 4.2.2 in [19]. □

Corollary 3. *(X, d) is the product of the family $\{(X_i, \rho_i) : i \in \mathbb{N}\}$ in* **SbkMtrz***.*

Let now $\{(X_i, \rho_i) : i \in \mathbb{N}\}$ be a countable family of $\vee$-sbk-metric spaces and let $d_i(x, y) = \frac{\rho_i(x,y)}{1+\rho_i(x,y)}$. We define $X = \prod_{i\in\mathbb{N}} X_i$ and $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \bigvee_{i\in\mathbb{N}} d_i(x_i, y_i)$$

where $x, y \in X$ and x_i, y_i are i^{th} coordinates of x and y, respectively.

Theorem 8. *(X, d) is the product of the family $\{(X_i, d_i) : i \in \mathbb{N}\}$ in $\vee$-***SbkMetr***. The topology $\mathcal{T}_d$ induced by $\vee$-sbk-metric d coincides with the product of the topologies $\mathcal{T}_{d_i}$ induced by $\vee$-sbk-metrics d_i. Hence (X, d) is also the product of $\{(X_i, d_i) : i \in \mathbb{N}\}$ in $\vee$-***SbkMtrz***.*

Proof. Because all $d_i(x_i, y_i) \le 1$, definition of d is correct and $d(x, y) \le 1$. The properties (m_1') (reflexivity) and (m_2) (symmetry) of d are obvious. We prove the property (mbs_3) for d, as follows.

Let $(x, y, z) \in X$, then

$$\begin{aligned} d(x, y) &= \bigvee\nolimits_{i=1}^{\infty} (d_i(x_i, y_i)) \le \bigvee\nolimits_{i=1}^{\infty} (d_i(x_i, z_i) \vee k \cdot d_i(z_i, y_i)) \\ &= \bigvee\nolimits_{i=1}^{\infty} (d_i(x_i, z_i) \vee \bigvee\nolimits_{i=1}^{\infty} k \cdot d_i(z_i, y_i)) = d(x, z) \vee k \cdot d(z, y). \end{aligned}$$

Further, let (Z, δ) be an $\vee$-sbk-metric space and let for every $i \in \mathbb{N}$ a mapping $\varphi_i : Z \to X_i$ be continuous. Subsequently, from the construction it is clear that by setting $\varphi(z) = (\varphi_i(z) \in X_i)_{i\in\mathbb{N}}$ we obtain a continuous mapping $\varphi : Z \to X$ such that $p_i \circ \varphi = \varphi_i : Z \to X_i$ where $p_i : X \to X_i$ are the projections. Hence, the defined construction is indeed a product in the category of bounded $\vee$-sbk-metric spaces.

To see that the topology $\mathcal{T}_d$ induced by d on the product $\prod_{i=1}^{\infty}(X_i, d_i)$ coincides with the product of topologies $\mathcal{T}_{d_i}$, just notice that, as a base for the topology $\mathcal{T}_d$, we may take the family of all balls

$$\left\{B(a, \varepsilon) = \bigwedge\nolimits_{i=1}^{n} p_i^{-1} B_i(a_i, \varepsilon) \mid a \in X, \varepsilon > 0, n \in \mathbb{N}\right\}.$$

□

Corollary 4. *(X, d) is the product of the family $\{(X_i, \rho_i) : i \in \mathbb{N}\}$ in $\vee$-***SbkMtrz***.*

Remark 10. *The analogous statement of Theorems 5 and 7 in the category of metric spaces is a classic result that can be found in almost every textbook in functional analysis and topology. Patterned after the proof of Theorem 5, analogous result for $\oplus$-metric spaces can also be proved in the case of any distributive continuous at the bottom extended t-conorm.*

*Unfortunately, for the reasons discussed above we have no any results about products in categories of $\oplus$-b-metric spaces $\oplus$-***b-MetrS** *and $\oplus$-***b-MetrT***.*

6.3. Co-Products (Direct Sums) of Families of $\oplus$-Sbk-Metrics

Speaking about co-products (or direct sums) of $\oplus$-sbk-metrics, we do not distinguish between finite and infinite cases, because, in both cases, we need to assume that the $\oplus$-sbk-metrics are bounded. The only difference between the finite and infinite cases is that in the infinite case we have to assume that the parameter k in all $\oplus$-sbk-metrics is the same, while, in the finite case, we may take k as the largest of all k_i. Accordingly, in the proof we assume that k is the same for all d_i.

Let $\{(X_i, \rho_i) : i \in I\}$ be a family of $\oplus$-sbk-metric spaces and let $d_i(x, y) = \frac{\rho_i(x,y)}{1+\rho_i(x,y)}$. We define $X = \coprod_{i \in I} X_i$ (that is X is the disjoint union of the sets X_i and $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d_i(x, y)$ if $x, y \in X_i$ and $d(x, y) = 1$ otherwise.

Theorem 9. *$d : X \times X \to \mathbb{R}$ is an $\oplus$-sbk-metric and (X, d) is the coproduct of the family $\{(X_i, d_i) : i \in I\}$ in the category $\oplus$-**SbkMetr**, where $\oplus$ is a compressible continuous on the bottom extended t-conorm. The topology that is induced by d coincides with the topology of the coproduct (direct sum) of the spaces (X_i, d_i).*

Proof. We first show that d is an $\oplus$-sbk-metric. The validity of properties (m_1') and (m_2) for d is obvious. To verify (msb_3), let $x, y, z \in X$ and consider the three cases.

(1) If there exists $i \in I$ such that $x, y, z \in X_i$, then the conclusion follows, since d_i is an $\oplus$-sbk-metric.

(2) If $x, y \in X_i$, but $z \notin X_i$, then $d(x, y) < 1$, but $d(x, z) \oplus d(z, y) \geq 1$. For example, in case $\oplus = +$ we have $d(x, z) + d(z, y) = 2, d(x, z) \vee d(z, y) = 1$.

(3) If $x \in X_i, y \in X_j, i \neq j$, then $d(x, y) = 1$ while $d(x, z) \oplus d(z, y) > 1$.

From the construction it is clear that the inclusion mappings $q_i : (X_i, d_i) \to (X, d)$ are continuous and that the topology that is induced by d coincides with the coproduct (direct sum) of the topologies induced by d_i.

Finally, let (Z, δ) be an $\oplus$-sbk-metric space and let for every $i \in \mathbb{N}$ a mapping $\varphi_i : X_i \to Z$ be continuous. Subsequently, from the construction it is clear that by setting $\varphi(x) = \varphi_i(x)$ if $x \in X_i$, we obtain a continuous mapping $\varphi : X \to Z$, such that $\varphi \circ q_i = \varphi_i$. $\square$

Corollary 5. *(X, d) is the coproduct of the family $\{(X_i, \rho_i : i \in I\}$ in the category $\oplus$-**SbkMtrz**, where $\oplus$ is a compressible continuous on the bottom extended t-conorm.*

7. Conclusions

In the paper, we have used the concept of an extended t-conorm $\oplus$ in order to define $\oplus$-metric, $\oplus$-b-metric and $\oplus$-sb-metric spaces. In case when $\oplus$ is the Łukasiewicz t-norm extended from the triangle $x, y \geq 0, x + y \leq 1$, they become "classic" metrics, b-metrics, and sb-metrics, respectively. The structure of these spaces was discussed and the property of continuity for mappings of such spaces was studied. $\oplus$-metric, $\oplus$-b-metric, and $\oplus$-sb-metric spaces, and their continuous mappings allow us to speak about several categories. We study some properties of these categories specifically the existence of products and coproducts in these categories. Throughout the paper our main interest is in $\oplus$-sb-metric spaces, in particular in case of sb-metric spaces. Answering a question post in [1], we constructed a series of sb-metric spaces that fail to be metric.

Concerning the plans for the future work, we see the following:

1. To study completeness and Baire property for $\oplus$-sb-metric spaces.
2. To investigate topological properties of sb-metric spaces. It is clear that they are Hausdorff. However, are they regular, completely regular, normal? What additional properties can be proved for separable sb-metric spaces?
3. Characterize compact subsets in sb-metric spaces.
4. We plan to study further categorical properties of sb-metric spaces and sb-metrizable spaces
5. To study the relations between the categories of metric-type spaces.
6. Consider the perspective for studying $\oplus$-metric-type structures for a general extended t-conorm. By now we have the full bodied theory only in case $\oplus$ is distributive and continuous. Unfortunately, we only have two examples of extended t-norms with such properties: $\otimes = +$ and $\otimes = \vee$. Are there other distributive continuous extended t-norms (except of obvious modification of the previous two)?

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