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Connections between Weighted Generalized Cumulative Residual Entropy and Variance

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Abstract: A shift-dependent information measure is favorable to handle in some specific applied contexts such as mathematical neurobiology and survival analysis. For this reason, the weighted differential entropy has been introduced in the literature. In accordance with this measure, we propose the weighted generalized cumulative residual entropy as well. Despite existing apparent similarities between these measures, however, there are quite substantial and subtle differences between them because of their different metrics. In this paper, particularly, we show that the proposed measure is equivalent to the generalized cumulative residual entropy of the cumulative weighted random variable. Thus, we first provide expressions for the variance and the new measure in terms of the weighted mean residual life function and then elaborate on some characteristics of such measures, including equivalent expressions, stochastic comparisons, bounds, and connection with the excess wealth transform. Finally, we also illustrate some applications of interest in system reliability with reference to shock models and random minima.

Keywords: weighted generalized cumulative residual entropy; non-homogeneous Poisson process; excess wealth transform; shock model; variance

MSC: 60E05; 62B10; 62N05; 94A17

1. Introduction and Preliminaries

One of the most important issues in several fields such as biology, survival analysis, reliability engineering, econometrics, statistics, and demography is the investigation of distribution functions on the ground on partial information. Few examples of relevant activities involve model selection, estimation, tests of hypotheses, inequality/poverty evaluation, and portfolio analysis. Stochastic dominance and other order relations are useful to attain partial rankings of distributions, whose advantage is the avoidance of strong cardinalization sometimes induced by other criteria applied to rank distributions. The notions of uncertainty and information are relative and involve comparison of distributions in terms of a probabilistic point of view in which the comparisons are always explicit. A pioneer measure of uncertainty in Information Theory is the concept of entropy which was introduced and studied by Shannon [1] for discrete random variables. For an absolutely continuous non-negative random variable X, the (differential) entropy is $H(X) = -\mathbb{E}[\log f(X)] = -\int_0^\infty f(x)\log f(x)\mathrm{d}x$, where f is the probability density function (PDF) and "log" is the natural logarithm, with $0\log 0 = 0$. Namely, bearing in mind possible applications to stochastic modeling in biology, X represents the random lifetime of a living organisms. More generally, it describes the operating time of a suitable system.

The differential entropy allows for having equal importance (or weights) to every event of the form $\{X = x\}$. However, in certain situations, they provide different qualitative features. This remark motivated Di Crescenzo and Longobardi [2] to define the *weighted entropy* of X as

$$H^w(X) = -\mathbb{E}[X \log f(X)] = -\int_0^\infty x f(x) \log f(x)\, \mathrm{d}x. \tag{1}$$

The term x in (1) can be viewed as a weight function, so that $H^w(X)$ provides a "length-biased" shift-dependent information measure. Clearly, it assigns greater importance to large outcomes of X. Furthermore, in analogy with (1), Misagh et al. [3] studied another weighted measure called *weighted cumulative residual entropy* (WCRE). For a non-negative random variable X with survival function $\overline{F}(x) = \mathbb{P}(X > x)$, the WCRE is

$$\mathcal{E}^w(X) = -\int_0^\infty x\overline{F}(x) \log \overline{F}(x)\, \mathrm{d}x = \int_0^\infty x\overline{F}(x)\Lambda(x)\, \mathrm{d}x, \tag{2}$$

where

$$\Lambda(x) = -\log \overline{F}(x) = \int_0^x \lambda(t)\, \mathrm{d}t, \quad x \geq 0, \qquad \text{and} \qquad \lambda(t) = \frac{f(t)}{\overline{F}(t)}, \quad t > 0,$$

denote, respectively, the cumulative hazard function and the hazard rate function of X.

We recall that several results on weighted entropies including stochastic ordering, aging classes properties, effects of linear transformations, bounds, and their relationships with some well-known concepts are investigated and discussed in Mirali and Baratpour [4] and Mirali et al. [5]. Moreover, numerous results for the WCRE are derived by Suhov and Yasaei Sekeh [6] for general weights.

Recently, the WCRE was extended in Tahmasebi [7] by the weighted generalized cumulative residual entropy (WGCRE) expressed as

$$\mathcal{E}_n^\phi(X) = \int_0^\infty \phi(x) \frac{[\Lambda(x)]^n}{n!} \overline{F}(x)\, \mathrm{d}x, \tag{3}$$

for all $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$, and for any non-negative weight function $\phi(x)$. In particular, by taking $\phi(x) \equiv 1$ in (3), we immediately derive the generalized cumulative residual entropy (GCRE), see Psarrakos and Navarro [8]. Various characteristics of the generalized measure (3) and its connection with some other notions have been studied in Tahmasebi [7]. See Kayal [9] for some characterizations of weighted GCRE for the linear weight function $\phi(x) = x$. Despite investigations on such measure, the analysis of its exact meaning and interpretation can still be improved. In this paper, we show that the WGCRE is the GCRE of the transformed random variable $\psi(X)$, namely the cumulative weighted random variable, where $\psi(x) = \int_0^x \phi(t)\, \mathrm{d}t$. Moreover, we show that they are suitable alternatives for the standard deviation of transformed random variables.

Therefore, this is the organization of the paper: In Section 2, we first introduce the cumulative weight function and the weighted mean residual life that plays a crucial role in developing our results, and then provide examples and investigate general properties, and sufficient conditions for the monotonicity of the latter function. In particular, we show that the variance of a transformed random variable is equal to the mean of the square of the weighted mean residual life of the same random variable. We also provide some inequalities for the mentioned notion that involve the variance and the cumulative residual entropy.

In Section 3, we elaborate the concept of WGCRE, with special attention to its intimate connection with the non-homogeneous Poisson process (NHPP). Indeed, it can be expressed as the mean of the weighted mean residual life of the epoch times of a NHPP. Then, we obtain several results related to the WGCRE, with special emphasis on its monotonicity, bounds, a recurrence relation, and two probabilistic meanings.

Section 4 is devoted to ordering results that involve the weighted mean residual life function. Specifically, we first define the weighted mean residual life stochastic order and then apply this notion to several stochastic comparisons for the variance and the WGCRE of transformed random variables having suitable aging distributions related to the well-known stochastic orders.

In Section 5, we focus on some connections with the excess wealth transform, by showing that the weighted mean residual life function is strictly related to the transformed (or weighted) excess wealth function. This allows us to reconsider some results investigated in the other sections under the light of the excess wealth transform.

Section 6 deals with applications in survival analysis and reliability. We consider a system subject to shocks governed by a non-homogeneous Poisson process, and for two such systems we show some comparison results related to the variance and WGCRE in terms of their random lifetimes, under suitable ordering conditions for the random numbers of shocks survived by the systems. We finally provide similar results for random minima, by bearing in mind applications to the random lifetimes of series systems composed by a random number of identical components.

Finally, some concluding remarks are provided in Section 7.

The main results of this paper involve typical notions such as the hazard rate, the mean residual life function, and various stochastic orders.

We use the terms increasing and decreasing in non-strict sense, and use $g'(x)$ for the derivative of $g(x)$. For simplicity, we write $g^n(x)$ instead of $[g(x)]^n$ for any given function g, we denote by $\sigma^2(X)$ or $Var(X)$ the variance of X. The expectations are assumed to exist whenever used. Furthermore, for the random lifetimes X and Y, their survival functions will be $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x) = 1 - G(x)$, with PDFs $f(x)$ and $g(x)$, respectively.

Throughout the paper, we denote by $\mathcal{S}(\mathbb{R}_+)$ the set of nonnegative absolutely continuous random variables having support $\mathbb{R}_+ = (0, \infty)$, and by $\mathcal{S}(\mathbb{N})$ the set of integer-valued random variables taking values in $\mathbb{N} = \{1, 2, \dots\}$.

2. Variance of a Transformed Random Variable

In what follows, we assume that $\phi(t)$ is a non-negative and differentiable function in $[0, \infty)$. The *cumulative weight function* is defined as

$$\psi(x) = \int_0^x \phi(s)\, \mathrm{d}s, \qquad x \geq 0. \tag{4}$$

It has a crucial role in developing our results. Specifically, given the random lifetime X, hereafter we analyze various properties of the transformed random variable $\psi(X)$, where the latter may be viewed as an increasing time-change of X. By virtue of (4), we get an expression for the variance of the cumulative weighted random variable $\psi(X)$. It is clear that (4) is an increasing function of $x \geq 0$ with $\psi(0) = 0$ since $\psi'(x) = \phi(x) \geq 0$; in addition, if $\phi(x)$ is increasing (decreasing) in $x > 0$, then clearly $\psi(x)$ is convex (concave). If $X \in \mathcal{S}(\mathbb{R}_+)$ represents the lifetime of a system or a component, then assuming that the system has survived up to age t, the residual lifetime is defined by $X_t = [X - t \,|\, X > t]$. Indeed, the distribution X_t is the same of $X - t$ conditional on $X > t$. For all $t \geq 0$, the *mean residual life* (MRL) function of X is defined as

$$m(t) = \mathbb{E}[X - t | X > t] = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)\, \mathrm{d}x. \tag{5}$$

This topic is attracting interest especially in various contexts; see, e.g., Eryilmaz [10] and Goliforushani et al. [11] and references therein. As pointed out by Hall and Wellner [12] (see also Toomaj and Di Crescenzo [13]), the variance of X is represented by means of the MRL function as follows:

$$\sigma^2(X) = \mathbb{E}[m^2(X)]. \tag{6}$$

With references to (6), henceforward, we develop the case under consideration for the transformed random variable $\psi(X)$. To this purpose, we introduce the *weighted mean residual life* (WMRL) function by

$$\begin{aligned} m_\psi(t) := m_{\psi(X)}(t) &= \mathbb{E}[\psi(X) - \psi(t) | X > t] \\ &= \tfrac{1}{\overline{F}(t)} \int_t^\infty \phi(x) \overline{F}(x)\, \mathrm{d}x, \end{aligned} \tag{7}$$

for all $t \geq 0$. Hence, $m_\psi(t)$ represents the mean residual life of the system in the transformed time scale. In addition, we assume that

$$\mathbb{E}[\psi(X)] = m_\psi(0) = \int_0^\infty \phi(x) \overline{F}(x)\, \mathrm{d}x < \infty, \tag{8}$$

to ensure the finiteness of $m_\psi(t)$. In particular, when $\psi(t) = t$, and hence $\phi(t) = 1$, then (7) coincides with the MRL function (5).

Remark 1. *The effect of transformations based on the function (4) is relevant in several contexts. For instance, in reliability theory, $\psi(X)$ may represent the lifetime of a system under time-dependent scale transformation that sometimes is called the accelerated life model. It is useful to transform random lifetimes having different nature. For instance, consider the translated standard Brownian motion $W(t) = x + B_t$, $t \geq 0$, where $x > 0$ and B_t is the standard Brownian motion. Let*

$$\tau(x) = \inf\{t > 0 : W(t) \leq 0\}, \qquad A(x) = \int_0^{\tau(x)} W(t)\, \mathrm{d}t, \qquad W(0) = x,$$

be the first-passage time below zero and the first-passage area of $W(x)$, respectively (see Abundo and Del Vescovo [14] for a detailed analysis). Since the PDF of $A(x)$ is given by

$$f(t) = \frac{2^{1/3}}{2^{2/3}\Gamma(1/3)} \frac{x}{t^{4/3}} e^{-(2x^3)/(9t)}, \qquad t > 0,$$

where $\Gamma(\cdot)$ is the gamma function, one has $\mathbb{P}[A(x) < +\infty] = 1$ and $\mathbb{E}[A(x)] = +\infty$. It is worth noting that, for $\psi(x) = x^\alpha$, $\alpha > 0$, one has $\mathbb{P}[\psi(A(x)) < +\infty] = 1$ and

$$\mathbb{E}[\psi(A(x))] = \frac{2^\alpha}{3^{2\alpha}} x^{3\alpha} \frac{\Gamma(1/3 - \alpha)}{\Gamma(1/3)}, \qquad 0 < \alpha < \frac{1}{3}.$$

Hence, use of ψ allows for transforming a lifetime with infinite mean into a finite one, for $0 < \alpha < 1/3$.

To begin the analysis, from Equations (4) and (7), the following lemma is easily obtained.

Lemma 1. *For any $X \in \mathcal{S}(\mathbb{R}_+)$, we have*

$$m'_\psi(t) + \phi(t) = \lambda(t)\, m_\psi(t), \qquad t \geq 0. \tag{9}$$

Remark 2. *From Equations (7) and (9), it immediately follows that the weighted mean residual life is constant, say $m_\psi(t) = m$, if and only if the weight function and the hazard rate function are proportional, i.e., $\phi(t) = m\,\lambda(t)$ for all $t \geq 0$. The latter condition, which is equivalent to $\psi(t) = m\,\Lambda(t) \equiv -m \log \overline{F}(t)$ for all $t \geq 0$, clearly yields that $\psi(X)$ has exponential distribution.*

Example 1. *Let X have exponential distribution with parameter $\eta > 0$. For $\lambda, \mu > 0$, consider the weight function $\phi(x) = \frac{\lambda}{\mu} e^{\lambda x}$, $x \geq 0$, so that the survival function of $\psi(X) \equiv \frac{1}{\mu}(e^{\lambda X} - 1)$ is $\overline{F}_{\psi(X)}(x) = (1 + \mu x)^{-\eta/\lambda}$, $x \geq 0$. From (7), we thus have that the weighted mean residual life is finite when $0 < \lambda < \eta$, for*

$$m_\psi(t) = \frac{\lambda}{\mu} \frac{1}{\eta - \lambda} e^{\lambda t}, \qquad t \geq 0.$$

We remark that, if $\eta = \lambda$, then m_ψ is infinite, and $\psi(X)$ is distributed as the first arrival time of a Geometric counting process with intensity μ (see Di Crescenzo and Pellerey [15] for details).

Given a random lifetime $X \in \mathcal{S}(\mathbb{R}_+)$, we recall that it is said to have a new worse (better) than used in expectation distribution, i.e., X is NWUE (NBUE), if $m(t) \geq (\leq)\, m(0) = \mathbb{E}[X]$ for any $t > 0$.

Lemma 2. *Let us suppose that X is NWUE (NBUE). If $\psi(t)$ is increasing convex (concave) on $[0, \infty)$, then*

$$m_\psi(t) \geq (\leq)\, \psi(\mathbb{E}[X]), \tag{10}$$

for all $t \geq 0$.

Proof. By assumption, $\psi(t)$ is increasing convex (concave) on $[0, \infty)$, with $\psi(0) = 0$. Thus, $\psi(x)$ is superadditive (subadditive), i.e., $\psi(z + y) \geq (\leq)\, \psi(z) + \psi(y)$, $z, y \geq 0$. By substituting $z = t$ and $y = x - t$, $x \geq t$, we obtain $\psi(x) - \psi(t) \geq (\leq)\, \psi(x - t)$ for all $x, t \geq 0$. Recalling (7), for $t \geq 0$, we find that

$$\begin{aligned} m_\psi(t) &= \mathbb{E}[\psi(X) - \psi(t) | X > t], \\ &\geq (\leq) \;\; \mathbb{E}[\psi(X - t) | X > t], \\ &\geq (\leq) \;\; \psi(\mathbb{E}[X - t | X > t]) = \psi(m(t)). \end{aligned} \tag{11}$$

The last relation follows by Jensen's inequality. Recalling that $\psi(t)$ is an increasing function and by assumption $m(t) \geq (\leq)\, \mathbb{E}(X)$ for all $t \geq 0$, since X is NWUE (NBUE), this gives the desired result. $\square$

Equation (9) and Lemma 2 will be used to derive various results presented in the sequel. A typical problem in reliability theory is the analysis of the monotonicity of the mean residual life. Henceforth, we concentrate on the monotonicity of the weighted mean residual life function $m_\psi(t)$.

Definition 1. *A random variable $X \in \mathcal{S}(\mathbb{R}_+)$ is said to have increasing (decreasing) weighted mean residual life function, denoted by IWMRL (DWMRL), if $m_\psi(t)$ is an increasing (decreasing) function of $t \geq 0$.*

In the following two theorems, we provide sufficient conditions for the monotonicity of $m_\psi(t)$. First, we recall the concept of total positivity which is applied to demonstrate monotonicity results in the remainder of this paper. For two subsets of the real line A and B, a non-negative function $K(x, y)$ defined on $A \times B$ is said to be *totally positive of order 2 (regular of order 2)*, denoted by TP_2 (RR_2), if $K(x_1, y_1)K(x_2, y_2) \geq (\leq)\, K(x_1, y_2)K(x_2, y_1)$, for all $x_1 \leq x_2$ in A and $y_1 \leq y_2$ in B. For further details, we refer to Karlin [16].

The first result shows that the monotonicity of the weighted mean residual life function is obtained when the ratio between the function $\phi(x)$ and the hazard rate function is monotonic.

Theorem 1. *Let $\lambda(x)$ be the hazard rate function of $X \in \mathcal{S}(\mathbb{R}_+)$. If $\phi(x)/\lambda(x)$ is increasing (decreasing) in $x \geq 0$, then X is IWMRL (DWMRL).*

Proof. From (7), it is sufficient to assert that the function

$$m_\psi(t) = \frac{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}{\int_t^\infty f(x)\,\mathrm{d}x}, \tag{12}$$

is increasing (decreasing) in $t \geq 0$. We set

$$\Psi(i,t) = \int_0^\infty \nu(i,x)\eta(x,t)\,\mathrm{d}x, \qquad i = 1,2, \tag{13}$$

where

$$\nu(i,x) = \begin{cases} f(x), & i = 1, \\ \phi(x)\overline{F}(x), & i = 2, \end{cases}$$

and $\eta(x,t) = \mathbf{1}[x > t]$, such that $\mathbf{1}[\pi] = 1$ when π is true, and $\mathbf{1}[\pi] = 0$, otherwise. Due to an assumption, we observe that

$$\frac{\nu(2,x)}{\nu(1,x)} = \frac{\phi(x)}{\lambda(x)},$$

is increasing (decreasing) in x, i.e., $\nu(i,x)$ is TP_2 (RR_2) in (i,x) for $i = 1,2$, and $x \in (0,\infty)$. Moreover, it is not hard to find that $\eta(x,t)$ is TP_2 in (x,t) for $x > 0$ and $t > 0$. Consequently, the general composition theorem of Karlin [16] shows that $\Psi(i,t)$ is TP_2 (RR_2) in (i,t) for $i = 1,2$ and $t > 0$, i.e., $m_\psi(t)$ is increasing (decreasing) in t, due to (12). This gives the desired result. □

Remark 3. *The condition given in Theorem 1 that $\frac{\phi(x)}{\lambda(x)}$ is increasing (decreasing) in x, is ensured by the conditions that $\psi(x)$ is convex (concave) and X is DFR (IFR). (We remark that X is said to be, or to have, increasing (decreasing) failure rate, i.e., X is IFR (DFR), if $\lambda(x)$ is increasing (decreasing) in x.)*

Example 2. *Let $X \in \mathcal{S}(\mathbb{R}_+)$, with decreasing and differentiable PDF $f(x)$ such that $0 < f(0) < \infty$. We consider*

$$\psi(x) = -\log\frac{f(x)}{f(0)}, \qquad \phi(x) = -\frac{f'(x)}{f(x)} \geq 0, \qquad x > 0.$$

(The ratio $\frac{f'(x)}{f(x)}$ is termed the score function corresponding to f, cf. Kharazmi and Asadi [17] for instance.) After some calculations, from (7), one has that the WMRL function for $t > 0$ is

$$m_\psi(t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\log\frac{f(x)}{f(t)}\,\mathrm{d}x = H(t) + \log\lambda(t),$$

where

$$H(t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\log\frac{f(x)}{\overline{F}(t)}\,\mathrm{d}x, \qquad t > 0,$$

is the residual entropy of X. Namely, $H(t)$ is the entropy of the residual lifetime $[X - t|X > t]$; it has been first investigated by Ebrahimi [18], Ebrahimi and Pellerey [19], and Muliere et al. [20], and subsequently has been the object of many investigations. In this case, if

$$\frac{f'(x)\overline{F}(x)}{f^2(x)} \quad \text{is decreasing (increasing) in } x > 0,$$

then X is $IWMRL$ $(DWMRL)$ due to Theorem 1. The above condition can be also expressed as

$$\frac{\lambda'(x)}{\lambda^2(x)} \quad \text{is decreasing (increasing) in } x > 0. \tag{14}$$

Remark 4. *With reference to condition (14), it is worth noting that*

$$\frac{\lambda'(x)}{\lambda^2(x)} \quad \text{is constant in } x > 0$$

when X has the generalized Pareto distribution, with

$$\overline{F}(x) = \left(\frac{b}{ax+b}\right)^{\frac{1}{a}+1}, \qquad \lambda(x) = \frac{a+1}{ax+b}, \qquad x \geq 0 \ \text{ such that } x < -\frac{b}{a} \text{ if } a < 0,$$

where $a > -1$, $a \neq 0$, $b > 0$. Indeed, in this case, we have

$$\frac{\lambda'(x)}{\lambda^2(x)} = -\frac{a}{a+1}.$$

Clearly, for $a \to 0$, we obtain the exponential distribution. See, for instance, Arriaza et al. [21] for a recent characterization of the generalized Pareto distribution.

Hereafter, we provide different conditions such that $m_\psi(x)$ is monotonic. In this second result, we require that $m(x)$ and the function $\phi(x)$ are both monotonic. To this aim, recall that $X \in \mathcal{S}(\mathbb{R}_+)$ is increasing (decreasing) in mean residual life, i.e., IMRL (DMRL), if the MRL function $m(x)$ is increasing (decreasing).

Theorem 2. *If $\phi(x)$ is increasing (decreasing) in x, and if X is IMRL (DMRL), then X is IWMRL (DWMRL).*

Proof. Recalling (5) and (7), if $\phi(x)$ is increasing (decreasing), then $m_\psi(x)/m(x)$ is increasing (decreasing) in x. From this and from the assumption that the MRL function $m(x)$ is increasing (decreasing) in x, we conclude that the function

$$m_\psi(x) = m(x)\frac{m_\psi(x)}{m(x)}, \qquad x > 0,$$

is increasing (decreasing) in x, that is, X is IWMRL (DWMRL). □

Example 3. *Let us consider $\phi(t) = \Lambda(t) = -\log \overline{F}(t)$, and thus $\psi(t) = \int_0^t \Lambda(\tau)\,\mathrm{d}\tau$. In this case, from (7), we have*

$$m_\psi(t) = -\frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x) \log \overline{F}(x)\,\mathrm{d}x, \qquad t \geq 0.$$

Hence, making use of Equation (14) of Asadi and Zohrevand [22], one has

$$m_\psi(t) = \mathcal{E}(X;t) - m(t)\log \overline{F}(t), \qquad t \geq 0, \tag{15}$$

where

$$\mathcal{E}(X;t) = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{F}(x)}{\overline{F}(t)}\,\mathrm{d}x, \qquad t \geq 0,$$

is the dynamic cumulative residual entropy (DCRE) of X. Recalling Corollary 4.4 in Asadi and Zohrevand [22], we have that if X is IMRL, then $\mathcal{E}(X;t)$ is increasing in t, and thus from (15) we obtain that in this case X is IWMRL. This conclusion can also be obtained from Theorem 2.

Example 4. *Assume that the weight function ϕ is expressed as $\phi(t) = h(v(t))$, $t > 0$, where $v(t) := m(t)\overline{F}(t) \equiv \int_t^\infty \overline{F}(x)\,\mathrm{d}x$ and where h is a non-negative and differentiable function. Then, from (7), we get that the WMRL function of X is expressed as $m_\psi(t) = H(v(t))/\overline{F}(t)$, $t > 0$, where $H(t) := \int_0^t h(s)\,\mathrm{d}s$. Moreover,* *due to Theorem 2, one has that, if h is a decreasing (increasing) function and if X is IMRL (DMRL), then X is IWMRL (DWMRL).*

As reported in (6), the variance of X can be represented as the expectation of the squared MRL function evaluated at X. Similarly, in the main result of this section, hereafter we express the variance of $\psi(X)$ in terms of the WMRL function (7).

Theorem 3. *If the weighted mean residual life function evaluated at X has finite second moment, i.e., $\mathbb{E}[m_\psi^2(X)] < \infty$, then*

$$\sigma^2[\psi(X)] = \mathbb{E}[m_\psi^2(X)]. \tag{16}$$

Proof. Recalling (7), let us set

$$v_\psi(x) := \overline{F}(x) m_\psi(x) = \int_x^\infty \phi(u)\overline{F}(u)\,\mathrm{d}u, \qquad x > 0, \tag{17}$$

so that $v_\psi(x)\lambda(x) = f(x)m_\psi(x)$. Then, by applying Equation (9), we get

$$\begin{aligned} \mathbb{E}[m_\psi^2(X)] &= \textstyle\int_0^\infty f(x)m_\psi^2(x)\,\mathrm{d}x = \int_0^\infty v_\psi(x)\lambda(x)m_\psi(x)\,\mathrm{d}x \\ &= \textstyle\int_0^\infty v_\psi(x)\phi(x)\,\mathrm{d}x + \int_0^\infty v_\psi(x)m'_\psi(x)\,\mathrm{d}x. \end{aligned} \tag{18}$$

By noting that $\psi(0) = 0$, upon recalling (17) and (4), and using Fubini's theorem, we have

$$\begin{aligned} \textstyle\int_0^\infty v_\psi(x)\phi(x)\,\mathrm{d}x &= \textstyle\int_0^\infty \phi(x)m_\psi(x)\overline{F}(x)\,\mathrm{d}x = \int_0^\infty \int_x^\infty \phi(x)\phi(u)\overline{F}(u)\,\mathrm{d}u\,\mathrm{d}x \\ &= \textstyle\int_0^\infty \psi(u)\phi(u)\overline{F}(u)\,\mathrm{d}x = \int_0^\infty f(z)\int_0^z \psi(u)\phi(u)\,\mathrm{d}u\,\mathrm{d}z \\ &= \tfrac{1}{2}\,\mathbb{E}[\psi^2(X)], \end{aligned} \tag{19}$$

where last equality holds by noting that

$$\int_0^z \psi(u)\phi(u)\,\mathrm{d}u = \frac{1}{2}\,\psi^2(z), \qquad z > 0.$$

On the other hand, recalling (8) and (19), we have

$$\begin{aligned} \int_0^\infty v_\psi(x)m'_\psi(x)\,\mathrm{d}x &= \int_0^\infty m'_\psi(x)\int_x^\infty \phi(u)\overline{F}(u)\,\mathrm{d}u\,\mathrm{d}x = \int_0^\infty \phi(u)\overline{F}(u)\int_0^u m'_\psi(x)\,\mathrm{d}x\,\mathrm{d}u \\ &= \int_0^\infty \phi(u)\overline{F}(u)m_\psi(u)\,\mathrm{d}u - m_\psi(0)\int_0^\infty \phi(u)\overline{F}(u)\,\mathrm{d}u \\ &= \frac{1}{2}\,\mathbb{E}[\psi^2(X)] - (\mathbb{E}[\psi(X)])^2. \end{aligned}$$

Combining the latter result with Equations (18) and (19), we get

$$\mathbb{E}[m_\psi^2(X)] = \mathbb{E}[\psi^2(X)] - (\mathbb{E}[\psi(X)])^2 = \sigma^2[\psi(X)],$$

thus completing the proof. □

We note that condition $\psi(0) = 0$ in Theorem 3 is not compulsory. Indeed, by using similar arguments, we find

$$\sigma^2[g(X)] = \mathbb{E}[m_g^2(X)],$$

for every increasing function g, even if $g(0) \neq 0$, and where

$$m_g(t) = \frac{1}{\overline{F}(t)}\int_t^\infty g'(x)\overline{F}(x)\,\mathrm{d}x, \qquad t > 0.$$

The next example gives an application of (16) that involves the minimum of a random sample, which may be viewed as the lifetime of the series system.

Example 5. *Let $X_{1:m} = \min\{X_1, \dots, X_m\}$ denote the minimum of a random sample of continuous non-negative random variables $X_1, \dots, X_m$ having the CDF $F(x)$. Denote by $\overline{F}_{1:m}(x) = \mathbb{P}(X_{1:m} > x) = [\overline{F}(x)]^m$, $x \ge 0$, the survival function of $X_{1:m}$. Hence, by setting $\psi(t) = F(t)$, and thus $\phi(t) = f(t)$, from (7) we obtain, for $t > 0$,*

$$\begin{aligned} m_{\psi(X_{1:m})}(t) &= \frac{1}{\overline{F}_{1:m}(t)} \int_t^\infty f(x)\overline{F}_{1:m}(x)\,\mathrm{d}x \\ &= \frac{1}{[\overline{F}(t)]^m} \int_t^\infty f(x)[\overline{F}(x)]^m\,\mathrm{d}x = \frac{\overline{F}(t)}{m+1}. \end{aligned}$$

Thanks to Equation (6) and Theorem 3, thus the variance of the probability integral transformation $F(X_{1:m})$ can be obtained as

$$\begin{aligned} \sigma^2[F(X_{1:m})] &= m \int_0^\infty f(x)[\overline{F}(x)]^{m-1} \left[\frac{\overline{F}(x)}{m+1}\right]^2 \mathrm{d}x \\ &= \frac{m}{(m+1)^2(m+2)}. \end{aligned}$$

In the reminder of this section, we obtain some useful bounds to $\sigma^2[\psi(X)]$ and $\sigma[\psi(X)]$. In the following theorem, by making use of the above results, we first investigate the impact of the transformation $\psi(x)$ on the variance of the random lifetime X.

Theorem 4. *Suppose that $X \in \mathcal{S}(\mathbb{R}_+)$, with $\mathbb{E}[\psi^2(X)] < \infty$. If $\phi(x) \ge 1$ $(0 \le \phi(x) \le 1)$ for all $x \ge 0$, then*

$$\sigma^2[\psi(X)] \ge (\le)\, \sigma^2(X).$$

Proof. The desired result is easily obtained by recalling (6), (7), and (16). □

We are now able to give an upper bound for the standard deviation of $\psi(X)$ given by the cumulative weight function evaluated at the mean value of X under suitable conditions.

Theorem 5. *Let us suppose that X is NWUE (NBUE). If $\psi(t)$ is convex (concave) on $[0, \infty)$, with $\mathbb{E}[\psi^2(X)] < \infty$, then*

$$\sigma[\psi(X)] \ge (\le)\, \psi(\mathbb{E}[X]).$$

Proof. Thanks to Lemma 2 and Theorem 3, the proof can be simply obtained by relations (16) and (10). □

Recall that Rao et al. [23] (see Rao [24], too) proposed the *cumulative residual entropy* (CRE) of a non-negative random variable X by

$$\mathcal{E}(X) = -\int_0^\infty \overline{F}(x) \log \overline{F}(x)\,\mathrm{d}x = \int_0^\infty \overline{F}(x)\Lambda(x)\,\mathrm{d}x.$$

Another useful representation is in terms of (5) as follows:

$$\mathcal{E}(X) = \mathbb{E}[m(X)] = \int_0^\infty m(x) f(x)\,\mathrm{d}x. \tag{20}$$

Leser [25] applied the CRE for measuring the elasticity of life expectancy in life tables in the hypothesis of a proportional change of mortality. Several properties of CRE as well as its dynamic version are widely discussed in Asadi and Zohrevand [22], Navarro et al. [26], and references therein.

The cumulative residual entropy was successfully applied for measuring uncertainty of lifetime of systems by Toomaj et al. [27]. Now, we show a bound involving the standard deviation of cumulative weighted random variable and the cumulative residual entropy.

Theorem 6. *Let $\psi(x)$ be an increasing convex and differentiable function, such that $\mathbb{E}[\psi^2(X)] < \infty$. Then,*

$$\sigma[\psi(X)] \geq \psi(\mathcal{E}(X)).$$

Proof. Due to Equations (16) and (11), we obtain

$$\sigma^2[\psi(X)] = \int_0^\infty m_\psi^2(x) f(x)\,\mathrm{d}x \geq \int_0^\infty [\psi(m(x))]^2 f(x)\,\mathrm{d}x.$$

Hence, by applying twice Jensen's inequality, for $g(x) = x^2$ and $\psi(x)$, we have

$$\begin{aligned}\sigma^2[\psi(X)] &\geq \left[\int_0^\infty \psi(m(x)) f(x)\,\mathrm{d}x\right]^2 \\ &\geq \left[\psi\left(\int_0^\infty m(x) f(x)\,\mathrm{d}x\right)\right]^2 = \psi^2(\mathcal{E}(X)),\end{aligned}$$

where the last inequality is obtained by recalling (20). □

As an example, by taking $\psi(x) = x^2$, Theorem 5 implies

$$\sigma(X^2) \geq [\mathbb{E}(X)]^2,$$

provided that X is NWUE, with $\mathbb{E}[X^4] < \infty$. Moreover, Theorem 6 yields

$$\sigma(X^2) \geq [\mathcal{E}(X)]^2.$$

For instance, if X is exponential with parameter 1, then $\sigma(X^2) = 2\sqrt{5} > [\mathbb{E}(X)]^2 = [\mathcal{E}(X)]^2 = 1$.

3. Weighted Generalized Cumulative Residual Entropy

In this section, we develop suitable connections between the transformed GCRE, the epoch times of a NHPP, and the WMRL function. Specifically, consider a NHPP having intensity function $\lambda(x)$, $x \geq 0$. Henceforward, we denote its epoch times of by $0 \equiv X_0 \leq X_1 \leq X_2 \leq \cdots$ where X_1 is distributed identically as X. Moreover, let $T_n = X_n - X_{n-1}$, $n \in \mathbb{N}$, be the length of the n-th interepoch interval (or interoccurrence time). Denoting by $\overline{F}_{n+1}(x)$ the survival function of X_{n+1}, $n \in \mathbb{N}_0$, one has

$$\overline{F}_{n+1}(x) = \overline{F}(x) \sum_{k=0}^{n} \frac{\Lambda^k(x)}{k!}, \qquad x \geq 0, \tag{21}$$

where $\Lambda(x)= \int_0^x \lambda(t)\,\mathrm{d}t$ is the cumulative hazard function of X, so that the PDF of X_{n+1} is

$$f_{n+1}(x) = f(x)\frac{\Lambda^n(x)}{n!}, \qquad x \geq 0. \tag{22}$$

We recall that the GCRE of order n of X is obtained by taking $\phi(x) = 1$ in (3) and satisfies the following relation (see Psarrakos and Navarro [8]):

$$\mathcal{E}_n(X) = \frac{1}{n!}\int_0^\infty \Lambda^n(x)\overline{F}(x)\,\mathrm{d}x = \mathbb{E}[X_{n+1} - X_n], \tag{23}$$

for all $n \in \mathbb{N}_0$. Let us now provide a suitable extension of $\mathcal{E}_n(X)$. For any increasing non-negative and differentiable cumulative weight function ψ defined as in (4), we can introduce

$$\mathcal{E}_{\psi,n}(X) = \mathbb{E}[\psi(X_{n+1}) - \psi(X_n)] = \mathbb{E}\Big[\int_{X_n}^{X_{n+1}} \phi(x)\,\mathrm{d}x\Big], \tag{24}$$

for $n \in \mathbb{N}_0$. From (21) and (22), we thus obtain

$$\begin{aligned}
\mathcal{E}_{\psi,n}(X) &= \int_0^\infty \phi(x)[\overline{F}_{n+1}(x) - \overline{F}_n(x)]\,\mathrm{d}x \\
&= \int_0^\infty \phi(x)\frac{\Lambda^n(x)}{n!}\overline{F}(x)\,\mathrm{d}x = \mathbb{E}\left[\frac{\phi(X_{n+1})}{\lambda(X_{n+1})}\right],
\end{aligned} \tag{25}$$

for $n \in \mathbb{N}_0$. Hence, the function $\mathcal{E}_{\psi,n}(X)$ can be identified with the WGCRE introduced in (3). This measure extends the GCRE through a suitable ψ. For example, if we take $\psi(t) = t$, then the WGCRE coincides with the GCRE (see Psarrakos and Navarro [8], Psarrakos and Toomaj [28], and Toomaj and Di Crescenzo [13]). Moreover, if we take $\psi(t) = \frac{t^2}{2}$, it concurs with the weighted GCRE introduced by Kayal [9]. We note that $\mathcal{E}_{\psi,n}(X)$ can be viewed as the area between $\phi(x)\overline{F}_{n+1}(x)$ and $\phi(x)\overline{F}_n(x)$. Based on the proof of Proposition 2.1 of Toomaj and Di Crescenzo [13], in fact, (25) is equivalent to the GCRE of a cumulative weighted random variable $\psi(X)$, i.e., $\mathcal{E}_{\psi,n}(X) = \mathcal{E}_n(\psi(X))$ for all $n \in \mathbb{N}_0$.

In what follows, we provide another expression for the WGCRE in terms of the expectation of the WMRL function, defined in (7), evaluated at X_n.

Theorem 7. *For $X \in \mathcal{S}(\mathbb{R}_+)$, one has*

$$\mathcal{E}_{\psi,n}(X) = \mathbb{E}[m_\psi(X_n)], \qquad n \in \mathbb{N}, \tag{26}$$

where $m_\psi(x)$ is defined in (7).

Proof. For $n \in \mathbb{N}$, we have

$$\int_0^t \frac{\Lambda^{n-1}(x)}{(n-1)!}\lambda(x)\,\mathrm{d}x = \frac{\Lambda^n(t)}{n!}, \qquad t \geq 0,$$

From this, Equation (25) and using Fubini's theorem, since $\overline{F}(t)$ is the survival function of X, we obtain

$$\begin{aligned}
\mathcal{E}_{\psi,n}(X) &= \int_0^\infty \phi(t)\frac{\Lambda^n(t)}{n!}\overline{F}(t)\,\mathrm{d}t \\
&= \int_0^\infty \int_0^t \frac{\Lambda^{n-1}(x)}{(n-1)!}\lambda(x)\phi(t)\overline{F}(t)\,\mathrm{d}x\,\mathrm{d}t, \\
&= \int_0^\infty \frac{\Lambda^{n-1}(x)}{(n-1)!}\lambda(x)\int_x^\infty \phi(t)\overline{F}(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^\infty m_\psi(x)\frac{\Lambda^{n-1}(x)}{(n-1)!}f(x)\,\mathrm{d}x,
\end{aligned}$$

where the last equality is obtained from (7). Hence, Equation (26) now follows by recalling (22). □

Remark 5. *We point out that another weighted GCRE can be defined as follows:*

$$\begin{aligned}
\mathcal{E}_n^\psi(X) &= \mathbb{E}[\psi(T_{n+1})] = \mathbb{E}[\psi(X_{n+1} - X_n)] \\
&= \textstyle\int_0^\infty \phi(x)\overline{F}_{T_{n+1}}(x)\,\mathrm{d}x, \qquad n \in \mathbb{N}.
\end{aligned} \tag{27}$$

We recall that the survival function of the interepoch interval T_{n+1} is given by (see e.g., Theorem 4.1 of Nakagawa [29], page 96)

$$\overline{F}_{T_{n+1}}(x) = \int_0^\infty \lambda(t) \frac{[\Lambda(t)]^{n-1}}{(n-1)!} \overline{F}(x+t)\,\mathrm{d}t, \qquad n \in \mathbb{N}.$$

Hence, from (27) by Fubini's theorem for $n \in \mathbb{N}$, we get

$$\mathcal{E}_n^{\psi}(X) = \int_0^\infty \frac{[\Lambda(x)]^{n-1}}{(n-1)!} f(x) m^{\psi}(x)\,\mathrm{d}x = \mathbb{E}[m^{\psi}(X_n)],$$

where, for $t \geq 0$,

$$m^{\psi}(t) := \mathbb{E}[\psi(X-t)|X>t] = \frac{1}{\overline{F}(t)} \int_t^\infty \phi(x-t)\overline{F}(x)\,\mathrm{d}x. \tag{28}$$

Moreover, if $\psi(x)$ is increasing convex (concave) on $(0,\infty)$, using similar arguments of Lemma 2, one can conclude that, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{E}_{\psi,n}(X) = \mathbb{E}[\psi(X_{n+1}) - \psi(X_n)] &\geq (\leq)\ \mathbb{E}[\psi(X_{n+1} - X_n)] \\ &= \quad \mathcal{E}_n^{\psi}(X). \end{aligned}$$

In this paper, we just focus on $\mathcal{E}_{\psi,n}(X)$ since it has some meaningful properties about its connection with the variance of a transformed random variable.

Thus far, a lot of criteria in terms of the survival function uncertainty have been introduced in the literature; however, less attention was paid on its concept. In this case, we provide a counterexample that, for instance, the weighted GCRE is suitable for measuring uncertainty of square of a random variable X^2.

Example 6. *Let $X \in \mathcal{S}(\mathbb{R}_+)$ have Weibull survival function $\overline{F}(x) = e^{-x^{\alpha}}$, $x \geq 0$, $\alpha > 0$. Letting $\mathcal{E}_{k,n} = \mathcal{E}_{\psi,n}$ with $\psi(t) = t^k$, $k \in \mathbb{N}$, and taking into account (25), after some algebraic manipulations, we get*

$$\mathcal{E}_{k,n}(X) = \frac{k}{n!\alpha} \Gamma\left(n + \frac{k}{\alpha}\right), \qquad n \in \mathbb{N}.$$

Note that, for $k = 2$, $\mathcal{E}_{2,n}(X) = 2\mathcal{E}_n^{w}(X)$, where $\mathcal{E}_n^{w}(X)$ is known as the weighted GCRE studied in [9]. Moreover, the direct computation implies that

$$\mathbb{E}[X^k] = \Gamma\left(\frac{k}{\alpha} + 1\right), \qquad k \in \mathbb{N},$$

so that

$$\begin{aligned} \sigma(X^k) &= \{\mathbb{E}[X^{2k}] - (\mathbb{E}[X^k])^2\}^{1/2} \\ &= \left\{\Gamma\left(\frac{2k}{\alpha} + 1\right) - \left[\Gamma\left(\frac{k}{\alpha} + 1\right)\right]^2\right\}^{1/2}. \end{aligned}$$

It is clear that $\mathcal{E}_{k,n}(X)$ can be written in terms of $\sigma(X^k)$, as

$$\mathcal{E}_{k,n}(X) = r_n\left(\frac{k}{\alpha}\right) \sigma(X^k), \qquad k, n \in \mathbb{N}, \tag{29}$$

where

$$r_n(\beta) := \frac{\beta\,\Gamma(\beta+n)}{n!\left\{\Gamma(2\beta+1)-[\Gamma(\beta+1)]^2\right\}^{1/2}}.$$

Note that, if $\alpha = k$, *then* $r_n(1) = 1$, *and thus, from (29), one has*

$$\mathcal{E}_{k,n}(X) = \sigma(X^k), \qquad k, n \in \mathbb{N}. \tag{30}$$

It is worth pointing out that, if X has a Weibull distribution with shape parameter k and scale parameter 1, then X^k *is exponentially distributed. Hence, equality (30) is not surprising. Generally, closed forms for* $r_n(\beta)$ *are not feasible, and thus we are enforced to go through numerical computations. Indeed, in Figure 1, we display* $r_n(\beta)$ *with respect to* β*, and for some choices of n.*

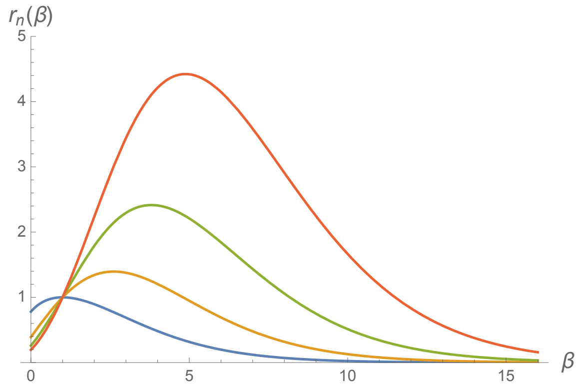

Figure 1. The function $r_n(\beta)$ of the Weibull distribution for $n = 1, 2, 3, 4$ from bottom to top near $\beta = 5$.

Remark 6. *We note that, for any non-negative increasing function* ψ *as given in (4), the differential entropy of* $\psi(X)$ *can be expressed as (see e.g., Equation (7) of Ebrahimi et al. [30]):*

$$\begin{aligned} H[\psi(X)] &= H(X) + \mathbb{E}\left[\log \tfrac{\mathrm{d}}{\mathrm{d}X}\psi(X)\right] \\ &= H(X) + \mathbb{E}[\log \phi(X)]. \end{aligned} \tag{31}$$

Thus, from Theorem 2.9 of Tahmasebi [7], one can get

$$\mathcal{E}_{\psi,n}(X) \geq \frac{1}{n!}\exp\{C_n^\star H[\psi(X)]\},$$

where $C_n^\star = \exp\left(\int_0^1 \log(u[-\log u]^n)\,\mathrm{d}u\right)$, $n \in \mathbb{N}$.

We are now able to prove a monotonicity result that involves the GCRE given in (23). We recall that, if $X \leq_{st} Y$, (the term $\leq_{st}$ stands for the usual stochastic order), then

$$\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)], \tag{32}$$

for all increasing functions $h(\cdot)$, cf. Shaked and Shanthikumar [31].

Theorem 8. *If the cumulative weight function* $\psi(x)$ *is increasing convex (concave) on* $(0, \infty)$*, then* $\mathcal{E}_{\psi,n}(X)/\mathcal{E}_n(X)$ *is increasing (decreasing) in* $n \in \mathbb{N}$.

Proof. From (26), one has

$$\begin{aligned} \frac{\mathcal{E}_{\psi,n}(X)}{\mathcal{E}_n(X)} &= \frac{\mathbb{E}[m_\psi(X_n)]}{\mathcal{E}_n(X)} = \int_0^\infty \frac{m_\psi(x)}{m(x)} \frac{m(x) f_n(x)}{\mathcal{E}_n(X)}\,\mathrm{d}x \\ &= \mathbb{E}[\Psi(Z_n)], \end{aligned} \tag{33}$$

where

$$\Psi(t) := \frac{m_\psi(t)}{m(t)} = \frac{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}{\int_t^\infty \overline{F}(x)\,\mathrm{d}x}, \qquad t > 0,$$

is easily seen to be increasing (decreasing) in $t > 0$, since $\psi(t)$ is increasing convex (concave) on $(0, \infty)$. Moreover, for all $n \in \mathbb{N}$, the random variable Z_n has the following PDF:

$$f_{Z_n}(x) = \frac{m(x) f_n(x)}{\mathcal{E}_n(X)}, \qquad x > 0.$$

Indeed, the latter is a *bona fide* density, since the function (23) satisfies the relation (cf. [13])

$$\mathcal{E}_n(X) = \mathbb{E}[m(X_n)], \qquad n \in \mathbb{N}. \tag{34}$$

Moreover, by Equation (22), it holds that, for $x > 0$,

$$\frac{f_{Z_{n+1}}(x)}{f_{Z_n}(x)} = \frac{\mathcal{E}_n(X)}{\mathcal{E}_{n+1}(X)} \frac{f_{n+1}(x)}{f_n(x)} = \frac{\mathcal{E}_n(X)}{\mathcal{E}_{n+1}(X)} \frac{\Lambda(x)}{n+1}.$$

Thus, since the latter ratio is increasing in $x > 0$, we have $Z_n \leq_{lr} Z_{n+1}$ where $\leq_{lr}$ stands for the likelihood ratio order; see [31] for details. Consequently, one has $Z_n \leq_{st} Z_{n+1}$, and thus $\mathbb{E}[\Psi(Z_n)]$ is increasing (decreasing) in $n \in \mathbb{N}$, since $\Psi(t)$ is increasing in $t > 0$ under the given assumptions on $\psi(x)$ by recalling (32). The thesis then follows from (33). □

We next give some results on $\mathcal{E}_{\psi,n}(X)$. The following theorem is stated under the same conditions of Lemma 2.

Theorem 9. *Let us suppose that X is NWUE (NBUE). If $\psi(t)$ is convex (concave) on $[0, \infty)$, then*

$$\mathcal{E}_{\psi,n}(X) \geq (\leq)\, \psi(\mathbb{E}[X]), \qquad n \in \mathbb{N}.$$

Proof. The proof can be obtained simply from relations (10) and (26). □

Remark 7. *We point out that the preceding theorem is sharp, i.e., the equality holds if $\psi(t) = t$ and X has the exponential distribution.*

Another useful theorem is given below concerning a connection between the weighted GCRE and the generalized cumulative residual entropy.

Theorem 10. *If $\psi(x)$ is increasing convex (concave) on $[0, \infty)$, then*

$$\mathcal{E}_{\psi,n}(X) \geq (\leq)\, \psi(\mathcal{E}_n(X)),$$

for all $n \in \mathbb{N}$.

Proof. Recalling (11), from the assumptions on ψ one has $m_\psi(t) \ge (\le)\, \psi(m(t))$, $t > 0$. By this, due to (26), it holds that

$$\begin{aligned}\mathcal{E}_{\psi,n}(X) &= \int_0^\infty m_\psi(x) f_n(x)\,\mathrm{d}x \\ &\ge (\le) \int_0^\infty \psi(m(x)) f_n(x)\,\mathrm{d}x \\ &\ge (\le)\ \psi\left(\int_0^\infty m(x) f_n(x)\,\mathrm{d}x\right) = \psi(\mathcal{E}_n(X)),\end{aligned}$$

the last inequality following by Jensen's inequality, since ψ is a convex (concave) function, and the last equality being due to (34). □

We are now able to provide a recurrence relation for $\mathcal{E}_{\psi,n}(X)$.

Theorem 11. *Under the assumption of Theorem 7, we have*

$$\mathcal{E}_{\psi,n}(X) = \mathcal{E}_{\psi,n-1}(X) + \frac{1}{(n-1)!}\mathbb{E}[h_{\psi,n}(X)], \qquad n \in \mathbb{N}, \tag{35}$$

where

$$h_{\psi,n}(u) := \int_0^u m'_\psi(x)\Lambda^{n-1}(x)\,\mathrm{d}x, \qquad n \in \mathbb{N}.$$

Proof. By virtue of Equations (22), (25), and (26), and $m_\psi(x)\lambda(x) = \phi(x) + m'_\psi(x)$, one has

$$\begin{aligned}\mathcal{E}_{\psi,n}(X) &= \int_0^\infty m_\psi(x)\lambda(x)\overline{F}(x)\frac{\Lambda^{n-1}(x)}{(n-1)!}\,\mathrm{d}x \\ &= \mathcal{E}_{\psi,n-1}(X) + \int_0^\infty m'_\psi(x)\overline{F}(x)\frac{\Lambda^{n-1}(x)}{(n-1)!}\,\mathrm{d}x,\end{aligned}$$

for all $n \in \mathbb{N}$. Using Fubini's theorem, we obtain

$$\begin{aligned}&\int_0^\infty m'_\psi(x)\overline{F}(x)\frac{\Lambda^{n-1}(x)}{(n-1)!}\,\mathrm{d}x \\ &\quad= \int_0^\infty m'_\psi(x)\int_x^\infty f(u)\frac{\Lambda^{n-1}(x)}{(n-1)!}\,\mathrm{d}u\,\mathrm{d}x \\ &\quad= \int_0^\infty f(u)\int_0^u m'_\psi(x)\frac{\Lambda^{n-1}(x)}{(n-1)!}\,\mathrm{d}x\,\mathrm{d}u.\end{aligned}$$

The thesis thus follows. □

Note that, when X is IWMRL (DWMRL), since $m'_\psi(x) \ge (\le)\, 0$, we have the following immediate consequence from Equation (35):

$$\mathcal{E}_{\psi,n}(X) \ge (\le)\, \mathcal{E}_{\psi,n-1}(X),$$

for all $n \in \mathbb{N}$. Here, we obtain an upper bound for the WGCRE in terms of the variance of the transformed random variable $\psi(X)$, if existing. The proof is omitted, being analogous to that of Theorem 8 of [13].

Theorem 12. *For $X \in \mathcal{S}(\mathbb{R}_+)$ with finite $\sigma[\psi(X)]$, the WGCRE function $\mathcal{E}_{\psi,n}(X)$ satisfies*

$$\mathcal{E}_{\psi,n}(X) \le \frac{\sqrt{[2(n-1)]!}}{(n-1)!}\sigma[\psi(X)], \tag{36}$$

for all $n \in \mathbb{N}$.

In the next theorem, we provide two different probabilistic meanings for the WGCRE. In fact, we deal with a suitable mean and with the covariance of the transformation of the n-th epoch time and the random variable $\Lambda(X_n)$.

Theorem 13. *For all $n \in \mathbb{N}$, it holds that*

(i) $\frac{1}{n}\,\mathbb{E}\left[\frac{\phi(X_n)\Lambda(X_n)}{\lambda(X_n)}\right] = \mathcal{E}_{\psi,n}(X)$;

(ii) $\frac{1}{n}\,\mathrm{Cov}[\psi(X_n), \Lambda(X_n)] = \mathcal{E}_{\psi,n}(X)$.

Proof. (i) Recalling (22), we have

$$\begin{aligned}\frac{1}{n}\,\mathbb{E}\left[\frac{\phi(X_n)\Lambda(X_n)}{\lambda(X_n)}\right] &= \frac{1}{n}\int_0^\infty \frac{\phi(x)\Lambda(x)}{\lambda(x)} f_n(x)\,\mathrm{d}t \\ &= \int_0^\infty \phi(x)\frac{\Lambda^n(x)}{n!}\overline{F}(x)\,\mathrm{d}x,\end{aligned}$$

this giving the desired result due to (25).
(ii) By the definition, we have

$$\mathrm{Cov}[\psi(X_n), \Lambda(X_n)] = \mathbb{E}[\psi(X_n)\,\Lambda(X_n)] - \mathbb{E}[\psi(X_n)]\,\mathbb{E}[\Lambda(X_n)].$$

Since $\Lambda(X_n) = -\log \overline{F}(X_n)$, one has $\mathbb{P}(\Lambda(X_n) \le x) = 1 - \overline{F}_n(\overline{F}^{-1}(e^{-x}))$, so that, from Equation (21), we get $\Lambda(X_n) \sim \mathrm{Gamma}(n, 1)$, and thus $\mathbb{E}[\Lambda(X_n)] = n$. Moreover, from (22), one can easily obtain

$$\begin{aligned}\mathbb{E}[\psi(X_n)\,\Lambda(X_n)] &= \int_0^\infty \psi(x)\,\Lambda(x)\,f_n(x)\,\mathrm{d}x \\ &= n\int_0^\infty \psi(x)\,\frac{\Lambda^n(x)}{n!}\,f(x)\,\mathrm{d}x \\ &= n\,\mathbb{E}[\psi(X_{n+1})],\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{Cov}[\psi(X_n), \Lambda(X_n)] &= n\,\{\mathbb{E}[\psi(X_{n+1})] - \mathbb{E}[\psi(X_n)]\} \\ &= n\,\mathcal{E}_{\psi,n}(X),\end{aligned}$$

due to (24). The proof is thus complete. □

4. Ordering Results

Hereafter, we discuss various ordering results of variance and WGCRE. First, we recall some stochastic orders.

Let $X, Y \in \mathcal{S}(\mathbb{R}_+)$, with survival functions $\overline{F}(t)$ and $\overline{G}(t)$, and mean residual life functions $m_X(t)$ and $m_Y(t)$, respectively. The following conditions define useful stochastic orders:
– hazard rate order ($X \le_{hr} Y$):

$$\overline{G}(t)/\overline{F}(t) \quad \text{is increasing in } t > 0;$$

– mean residual lifetime order ($X \le_{mrl} Y$):

$$\int_t^\infty \overline{G}(x)\,\mathrm{d}x \Big/ \int_t^\infty \overline{F}(x)\,\mathrm{d}x \quad \text{is increasing in } t > 0;$$

– dispersive order ($X \leq_{disp} Y$):

$$G^{-1}(F(t)) - t \quad \text{is increasing in } t > 0,$$

where, for $u \in [0,1]$, $G^{-1}(u) = \inf\{x \in \mathbb{R}^+ : G(x) \geq u\}$ denotes the left-continuous quantile function of $G(x)$.

Now, we define a new stochastic order that involves the weighted mean residual life function.

Definition 2. *For a given non-negative weight function $\phi(\cdot)$, let $X, Y \in \mathcal{S}(\mathbb{R}_+)$ have weighted mean residual life functions $m_{\psi(X)}(t)$ and $m_{\psi(Y)}(t)$, respectively. Then, X is smaller than Y in the weighted mean residual lifetime with respect to the weight function $\phi(x)$, denoted by $X \leq^{\phi}_{wmrl} Y$, if $m_{\psi(X)}(t) \leq m_{\psi(Y)}(t)$, for all $t \geq 0$.*

The next theorem provides equivalent statements for the above defined stochastic order.

Theorem 14. *For $X, Y \in \mathcal{S}(\mathbb{R}_+)$ and any non-negative weight function $\phi(\cdot)$, the following statements are equivalent:*

(i) $X \leq^{\phi}_{wmrl} Y$;

(ii) $\frac{\int_t^\infty \phi(x)\overline{G}(x)\,\mathrm{d}x}{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}$ *is increasing in $t > 0$;*

(iii) $\mathbb{E}[\psi(X)|X > t] \leq \mathbb{E}[\psi(Y)|Y > t]$ *for all $t > 0$.*

Proof. In this case, for $t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_t^\infty \phi(x)\overline{G}(x)\,\mathrm{d}x}{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x} = \frac{\phi(t)\int_t^\infty \phi(x)\left[\overline{G}(x)\overline{F}(t) - \overline{G}(t)\overline{F}(x)\right]\mathrm{d}x}{\left[\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x\right]^2}.$$

Recalling (7) and after some calculations, one has $X \leq^{\phi}_{wmrl} Y$ if and only if $\int_t^\infty \phi(x)\left[\overline{G}(x)\overline{F}(t) - \overline{G}(t)\overline{F}(x)\right]\mathrm{d}x \geq 0$ for all $t > 0$. This proves the equivalence between (i) and (ii). Finally, the equivalence of statements (i) and (iii) is clear. □

Recall that the limiting distribution of the excess time (or the forward recurrence time) in a renewal process (or in shock models) gives the so-called equilibrium distribution. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and non-negative random variables describing the interarrivals between shocks. Suppose further that such random variables have identical distribution $F(t)$, with finite mean μ. Moreover, X_1 has a possible different distribution $F_1(t)$ with finite mean μ_1. Both distribution functions $F_1(t)$ and $F(t)$ are not degenerate at $t = 0$, and $F_1(0) = F(0) = 0$. For $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$, and $S_0 \equiv 0$, let $N(t) = \max\{n : S_n \leq t\}$ represent the number of renewals during $(0, t]$. Hence, the expected number of renewals in $(0, t]$ can be obtained as

$$M(t) = \mathbb{E}[N(t)] = \int_0^t [1 + M_0(t-u)]\,\mathrm{d}F_1(u),$$

where $M_0(t)$ is the renewal function of an ordinary renewal process with distribution F, i.e., $M_0(t) = \sum_{k=1}^n F^{(k)}(t)$, where $F^{(k)}$ is the k-fold Stieltjes convolution of F. Let $\gamma(t)$ be the excess time in a stochastic process or residual lifetime in reliability theory at time t, that is, $\gamma(t) = S_{N(t)+1} - t$. By the elementary renewal theorem, the distribution of the equilibrium random variable $\widetilde{X}$ is known as

$$\widetilde{F}(x) = \lim_{t\to\infty} \mathbb{P}[\gamma(t) \leq x] = \frac{1}{\mu}\int_0^x \overline{F}(u)\,\mathrm{d}u, \qquad x > 0.$$

For further details, see, e.g., Nakagawa [29].

The MRL order is characterized via the hazard rate order as (cf. Hu et al. [32])

$$X \leq_{mrl} Y \quad \Longleftrightarrow \quad \widetilde{X} \leq_{hr} \widetilde{Y}. \tag{37}$$

Weighted distributions are adopted in many problems to model different sampling rules; see, e.g., Misagh and Yari [3] and Nanda and Jain [33] and references therein. For a given $X \in \mathcal{S}(\mathbb{R}_+)$ with PDF f, and a non-negative real function w, let

$$f^w(x) = \frac{w(x)f(x)}{\mathbb{E}[w(X)]}, \qquad x > 0, \tag{38}$$

be the PDF of the related weighted random variable X^w, provided that $\mathbb{E}[w(X)]$ is positive and finite. Note that the equilibrium random variable $\widetilde{X}$ is the weighted random variable obtained from X and $w(x) = 1/\lambda(x)$, where λ is the failure rate function of X. Denote by $\widetilde{X}_\phi$ and $\widetilde{Y}_\phi$ the weighted versions of $\widetilde{X}$ and $\widetilde{Y}$, respectively; they have PDFs

$$\widetilde{f}_\phi(x) = \frac{\phi(x)\overline{F}(x)}{\mathbb{E}[\psi(X)]}, \quad \text{and} \quad \widetilde{g}_\phi(x) = \frac{\phi(x)\overline{G}(x)}{\mathbb{E}[\psi(Y)]}, \tag{39}$$

for $x > 0$. Let $\lambda_{\widetilde{X}_\phi}(t) = \frac{\widetilde{f}_\phi(t)}{\int_t^\infty \widetilde{f}_\phi(x)\mathrm{d}x}$ and $\lambda_{\widetilde{Y}_\phi}(t) = \frac{\widetilde{g}_\phi(t)}{\int_t^\infty \widetilde{g}_\phi(x)\mathrm{d}x}$ denote the hazard rate functions of $\widetilde{X}_\phi$ and $\widetilde{Y}_\phi$, respectively. Now, to compare $\widetilde{X}_\phi$ and $\widetilde{Y}_\phi$ by the hazard rate order, we have

$$\widetilde{X}_\phi \leq_{hr} \widetilde{Y}_\phi \quad \Longleftrightarrow \quad \frac{\widetilde{f}_\phi(t)}{\int_t^\infty \widetilde{f}_\phi(x)\mathrm{d}x} \geq \frac{\widetilde{g}_\phi(t)}{\int_t^\infty \widetilde{g}_\phi(x)\mathrm{d}x} \quad \text{for all } t \geq 0,$$

which is equivalent to

$$\widetilde{X}_\phi \leq_{hr} \widetilde{Y}_\phi \quad \Longleftrightarrow \quad \frac{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)} \leq \frac{\int_t^\infty \phi(x)\overline{G}(x)\,\mathrm{d}x}{\overline{G}(t)} \quad \text{for all } t \geq 0,$$

due to (39). From the above results, the weighted mean residual lifetime order is related to the hazard rate order as

$$X \leq_{wmrl}^{\phi} Y \quad \Longleftrightarrow \quad \widetilde{X}_\phi \leq_{hr} \widetilde{Y}_\phi.$$

By assuming that $\phi(x)$ weight function, then (see e.g., Kayid et al. [34] for details)

$$\widetilde{X} \leq_{hr} \widetilde{Y} \quad \Longrightarrow \quad \widetilde{X}_\phi \leq_{hr} \widetilde{Y}_\phi. \tag{40}$$

It is worthwhile to mention that the choice $\phi(x) = x$ in (40) motivated Kayid et al. [34] to define the combination mean residual life function (CMRL). Furthermore, they proposed the CMRL order by assuming $\phi(x) = x$ in Definition 2. If $\psi(x)$ is an increasing convex function, from (37) and (40), we get

$$X \leq_{mrl} Y \quad \Longrightarrow \quad X \leq_{wmrl}^{\phi} Y. \tag{41}$$

By making use of Proposition 2.3 of Nanda and Jain [33], for any increasing cumulative weight function $\psi(\cdot)$, it holds that

$$X \leq_{hr} Y \quad \Longrightarrow \quad X \leq_{wmrl}^{\phi} Y. \tag{42}$$

The next theorem shows an ordering result based on the dispersive order. We first consider the following remark.

Remark 8. *Consider two cumulative weighted random variables $\psi(X)$ and $\psi(Y)$ with cumulative distribution functions H and Q, respectively. On the other hand, assume that $\psi(X_n)$ and $\psi(Y_n)$ have cumulative distribution functions H_n and Q_n, respectively. We note that the distribution function of $\psi(X_n)$, $n \in \mathbb{N}_0$, can be written as*

$$H_n(x) = 1 - (1 - H(x)) \sum_{k=0}^{n-1} \frac{(-\log(1 - H(x)))^k}{k!} = 1 - g_n(1 - H(x)) = g_n^\star(H(x)), \qquad x \geq 0,$$

where

$$g_n(x) = x \sum_{k=0}^{n-1} \frac{(-\log x)^k}{k!}, \qquad g_n^\star(x) = 1 - g_n(1 - x), \qquad x \geq 0,$$

are increasing functions. The same results also hold for the random variable $\psi(Y_n)$. Now, for all x, we have

$$Q_n^{-1} H_n(x) = (g_n^\star(Q))^{-1} g_n^\star(H(x)) = Q^{-1} (g_n^\star)^{-1} g_n^\star(H(x)) = Q^{-1} H(x).$$

Theorem 15. *Let X_n and Y_n be the epoch times of two nonhomogeneous Poisson processes associated with X and Y, respectively. If $\psi(X) \leq_{disp} \psi(Y)$, then for all $n \in \mathbb{N}$*

$$\psi(X_{n+1}) - \psi(X_n) \leq_{st} \psi(Y_{n+1}) - \psi(Y_n). \tag{43}$$

Proof. Let us consider two cumulative weighted random variables $\psi(X)$ and $\psi(Y)$ with cumulative distribution functions H and Q, respectively. On the other hand, assume that $\psi(X_n)$ and $\psi(Y_n)$ have cumulative distribution functions H_n and Q_n, respectively. Note that $\psi(X) \leq_{disp} \psi(Y)$ implies $\psi(X_n) \leq_{disp} \psi(Y_n)$ by noting that $Q^{-1} H = Q_n^{-1} H_n$ for all $n \in \mathbb{N}$, as shown in Remark 8. Now, we have

$$\psi(X_n) \stackrel{d}{=} H^{-1} Q(\psi(Y_n)), \qquad n \in \mathbb{N}, \tag{44}$$

where $\stackrel{d}{=}$ means equality in distribution and $H^{-1}Q$ is an increasing function. One clearly has $Y_n \leq Y_{n+1}$, $n \in \mathbb{N}$, almost surely and, since $\psi(\cdot)$ is increasing, $\psi(Y_n) \leq \psi(Y_{n+1})$, $n \in \mathbb{N}$, almost surely. From this and by noting that $\psi(X_n) \leq_{disp} \psi(Y_n)$, using Equation (3.B.15) in Shaked and Shanthikumar [31] we obtain that the following relation holds almost surely:

$$H^{-1} Q(\psi(Y_{n+1})) - H^{-1} Q(\psi(Y_n)) \leq \psi(Y_{n+1}) - \psi(Y_n),$$

for $n \in \mathbb{N}$. This implies, by recalling (44), the validity of (43). The proof is thus complete. □

Remark 9. *With reference to Theorem 15, we note that X and Y are independent and hence so also X_n and Y_n. However, (X_{n+1}, X_n) and (Y_{n+1}, Y_n) have the same copula since two random vectors of record values, possessing the same set of parameters with possibly different distributions, have the same copula (cf. Belzunce et al. [35]). Since the copula is invariant under strictly increasing transformations $\psi(\cdot)$, we conclude that $(\psi(X_{n+1}), \psi(X_n))$ and $(\psi(Y_{n+1}), \psi(Y_n))$ have the same copula. Hence, for an increasing function $H^{-1}Q$, we get $(\psi(X_{n+1}), \psi(X_n)) \stackrel{d}{=} (H^{-1}Q(\psi(Y_{n+1})), H^{-1}Q(\psi(Y_n)))$, where we can write $\psi(X_{n+1}) - \psi(X_n) \stackrel{d}{=} H^{-1}Q(\psi(Y_{n+1})) - H^{-1}Q(\psi(Y_n))$ using Theorem 2.4.3 of Nelsen [36]. From this note, one has an alternative way to prove Theorem 15 by considering $H^{-1}Q = \varphi$ in Equation (3.B.15) of Shaked and Shanthikumar [31].*

By making use of Theorem 15, we can prove the following result, which allows for comparing WGCRE and variance of weighted cumulative of two random variables under the dispersive ordering.

Theorem 16. *Let us assume that $X \leq_{disp} Y$. Then,*

(i) If $\psi(\cdot)$ is convex, then $\sigma[\psi(X)] \leq \sigma[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \leq \mathcal{E}_{\psi,n}(Y)$ for all $n \in \mathbb{N}$.
(ii) If $\psi(\cdot)$ is concave, then $\sigma[\psi(X)] \geq \sigma[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \geq \mathcal{E}_{\psi,n}(Y)$ for all $n \in \mathbb{N}$.

Proof. (i) Let $X \le_{disp} Y$. Since $\psi(\cdot)$ is a convex function, from Theorem 3.B.10 in Shaked and Shanthikumar [31], it holds that $\psi(X) \le_{disp} \psi(Y)$ which yields that $\sigma^2[\psi(X)] \le \sigma^2[\psi(Y)]$. On the other hand, due to Theorem 15, one has relation (43). Recalling (24), we have the stated result. The case (ii) is similar. □

In the following theorem, we show that, if X and Y are ordered in the weighted mean residual lifetime, then their corresponding variance and WGCRE will be ordered too, under suitable assumptions on the distributions of X or Y.

Theorem 17. *Let $X, Y \in \mathcal{S}(\mathbb{R}_+)$, with weighted mean residual life functions $m_{\psi(X)}(x)$ and $m_{\psi(Y)}(x)$, respectively, and such that $X \le_{st} Y$. Then,*

(i) If $X \le^{\phi}_{wmrl} Y$ and either X or Y is IWMRL, then $\sigma^2[\psi(X)] \le \sigma^2[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \le \mathcal{E}_{\psi,n}(Y)$, for all $n \in \mathbb{N}$.
(ii) If $X \ge^{\phi}_{wmrl} Y$ and either X or Y is DWMRL, then $\sigma^2[\psi(X)] \ge \sigma^2[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \ge \mathcal{E}_{\psi,n}(Y)$, for all $n \in \mathbb{N}$.

Proof. Let Y be IWMRL. From (16), we get

$$\begin{aligned}\sigma^2[\psi(X)] = \mathbb{E}[m^2_{\psi(X)}(X)] &\le \mathbb{E}[m^2_{\psi(Y)}(X)] \\ &\le \mathbb{E}[m^2_{\psi(Y)}(Y)] = \sigma^2[\psi(Y)].\end{aligned}$$

The first inequality follows from the hypothesis $X \le^{\phi}_{wmrl} Y$ while the last inequality is obtained due to (32). Let X_n and Y_n denote the n-th successive epoch times of two non-homogeneous Poisson processes with PDFs $f_n(x)$ and $g_n(x)$, respectively. Then, $X \le_{st} Y$ implies $X_n \le_{st} Y_n$ for all $n \in \mathbb{N}$. From this and recalling (26), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathcal{E}_{\psi,n}(X) = \mathbb{E}[m_{\psi(X)}(X_n)] &\le \mathbb{E}[m_{\psi(Y)}(X_n)] \\ &\le \mathbb{E}[m_{\psi(Y)}(Y_n)] = \mathcal{E}_{\psi,n}(Y).\end{aligned}$$

The first inequality comes from the assumption $X \le^{\phi}_{wmrl} Y$ while the last inequality derives from (32) by noting that $X_n \le_{st} Y_n$. Now, let X be IWMRL. Then, we similarly have

$$\begin{aligned}\sigma^2[\psi(X)] = \mathbb{E}[m^2_{\psi(X)}(X)] &\le \mathbb{E}[m^2_{\psi(X)}(Y)] \\ &\le \mathbb{E}[m^2_{\psi(Y)}(Y)] = \sigma^2[\psi(Y)],\end{aligned}$$

and analogously

$$\begin{aligned}\mathcal{E}_{\psi,n}(X) = \mathbb{E}[m_{\psi(X)}(X_n)] &\le \mathbb{E}[m_{\psi(X)}(Y_n)] \\ &\le \mathbb{E}[m_{\psi(Y)}(Y_n)] = \mathcal{E}_{\psi,n}(Y).\end{aligned}$$

which the results stated in (i) follow. The proof of (ii) is similar. □

The next theorem leads to the same results under slightly different assumptions. The proof is similar and then is omitted.

Theorem 18. *Let $X, Y \in \mathcal{S}(\mathbb{R}_+)$ have weighted mean residual life functions $m_{\psi(X)}(t)$ and $m_{\psi(Y)}(t)$, respectively. If $X \le_{hr} Y$, and X or Y is IWMRL, then*

(i) $\sigma^2[\psi(X)] \le \sigma^2[\psi(Y)]$.
(ii) $\mathcal{E}_{\psi,n}(X) \le \mathcal{E}_{\psi,n}(Y)$, for all $n \in \mathbb{N}$.

The following corollary is concerned with the mean residual lifetime order. It immediately follows from (41).

Corollary 1. *Let $X, Y \in \mathcal{S}(\mathbb{R}_+)$, with mean residual life functions $m_X(x)$ and $m_Y(x)$, such that $X \leq_{st} Y$, and assuming that $\psi(x)$ is an increasing convex function. Then,*

(i) If $X \leq_{mrl} Y$ and either X or Y is IWMRL, then $\sigma^2[\psi(X)] \leq \sigma^2[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \leq \mathcal{E}_{\psi,n}(Y)$, for all $n \in \mathbb{N}$.

(ii) If $X \geq_{mrl} Y$ and either X or Y is DWMRL, then $\sigma^2[\psi(X)] \geq \sigma^2[\psi(Y)]$ and $\mathcal{E}_{\psi,n}(X) \geq \mathcal{E}_{\psi,n}(Y)$, for all $n \in \mathbb{N}$.

5. Connection with the Excess Wealth Transform

In several applied contexts, numerous variability quantities such as variance, standard deviation and some other dispersion measures are widely used since their comparisons are based only on summary statistics. Notwithstanding, they are often noninformative being their comparisons based only on single numbers. Additionally, the standard deviations of some distributions may not exist or, in some situations, they may not be the appropriate quantities to compare. Therefore, transforms and stochastic orders for comparing their variabilities have been defined and widely investigated in the literature; for a comprehensive discussion, see Shaked and Shanthikumar [31]. One such order is the excess wealth order (or right spread order) as a measure of spread. For a non-negative random variable X with cumulative distribution function F, the left continuous inverse (quantile function) is given by

$$F^{-1}(p) = \inf\{x \in \mathbb{R}^+ : F(x) \geq p\}, \qquad p \in (0,1).$$

Moreover, if X has finite mean, then the excess wealth transform (or right spread function), for $p \in (0,1)$ is

$$\begin{aligned} W_X(p) &= \mathbb{E}[(X - F^{-1}(p))^+] = \int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,\mathrm{d}x \\ &= \int_p^1 (F^{-1}(q) - F^{-1}(p))\,\mathrm{d}q, \end{aligned} \tag{45}$$

where, as usual, $(Z)^+$ represents the positive part of Z, i.e., $(Z)^+ = Z$ if $Z \geq 0$, and $(Z)^+ = 0$ otherwise. It is worth to mentioning that X does not need to be non-negative for the validity of (45), but it just needs to have a finite mean. In addition, the excess wealth transform and the mean residual life function are related by

$$m_X(F^{-1}(p)) = \frac{W_X(p)}{1-p}, \qquad 0 < p < 1. \tag{46}$$

Fernández-Ponce et al. [37] obtained an expression for the variance of X in terms of the quantity given in Equation (46) as

$$\sigma^2(X) = \int_0^1 [m_X(F^{-1}(p))]^2\,\mathrm{d}p.$$

Hereafter, we obtain expressions for the variance of cumulative weighted random variable and weighted GCRE in terms of transformed excess wealth function. For $X \in \mathcal{S}(\mathbb{R}_+)$ with CDF $F(x)$, assume that ψ is given in (4). We define the *transformed (or weighted) excess wealth function* of X as

$$\begin{aligned} W_{\psi(X)}(p) &= \mathbb{E}[(\psi(X) - \psi(F^{-1}(p)))^+] \\ &= \int_{F^{-1}(p)}^{\infty} \phi(x)\overline{F}(x)\,\mathrm{d}x \\ &= \int_p^1 \left[\psi(F^{-1}(q)) - \psi(F^{-1}(p))\right]\,\mathrm{d}q, \end{aligned} \tag{47}$$

for all $p \in (0,1)$. When $\psi(t) = t$, then (47) coincides with (45). This function, due to (46), satisfies the following relation:

$$m_{\psi(X)}(F^{-1}(p)) = \frac{W_{\psi(X)}(p)}{1-p}, \qquad 0 < p < 1. \tag{48}$$

The following theorem gives expressions for both the variance of a transformed random variable and the weighted GCRE in terms of $m_{\psi(X)}(F^{-1}(p))$ given in (48). The proof is omitted, since it can be easily derived from Theorems 3 and 7.

Theorem 19. *If $X \in \mathcal{S}(\mathbb{R}_+)$ has CDF F, then*

(i) $\sigma^2[\psi(X)] = \int_0^1 \left[m_{\psi(X)}(F^{-1}(p))\right]^2 \mathrm{d}p,$

(ii) $\mathcal{E}_{\psi,n}(X) = \frac{1}{(n-1)!}\int_0^1 m_{\psi(X)}(F^{-1}(p))[-\log(1-p)]^{n-1}\,\mathrm{d}p, \quad$ *for all* $n \in \mathbb{N}$.

Let X and Y be absolutely continuous nonnegative random variables with CDFs F and G, respectively. Then, X is said to be smaller than Y in the excess wealth order, denoted as $X \le_{ew} Y$, when $W_X(p) \le W_Y(p)$ for all $p \in (0,1)$. We recall that the excess wealth order is preserved under monotone convex transformations. In other words, if $X \le_{ew} Y$, then $h(X) \le_{ew} h(Y)$ for all increasing convex function $h(x)$. Using this property, from Theorem 19, we immediately obtain the following theorem. We omit the proof, being straightforward.

Theorem 20. *Let $X, Y \in \mathcal{S}(\mathbb{R}_+)$, such that $X \le_{ew} Y$. If $\psi(x)$ is increasing convex, then*

(i) $\sigma^2[\psi(X)] \le \sigma^2[\psi(Y)]$;

(ii) $\mathcal{E}_{\psi,n}(X) \le \mathcal{E}_{\psi,n}(Y), \quad$ *for all* $n \in \mathbb{N}$.

Proof. Since $\psi(x)$ is an increasing convex function, the assumption $X \le_{ew} Y$ implies $\psi(X) \le_{ew} \psi(Y)$. This means that $m_{\psi(X)}(F^{-1}(p)) \le m_{\psi(Y)}(F^{-1}(p))$ for all $p \in (0,1)$. Upon using Theorem 19, the results are obtained. □

6. Applications to System Reliability

This section is devoted to discuss various relevant applications of the weighted mean residual lifetime order in survival analysis and reliability theory.

6.1. Application to Shock Models

Assume that a one-unit system is able to withstand a random number of shocks. As customary, the shocks are governed by a non-homogeneous Poisson process. We assume independence between the number of shocks and the interarrival (or successive) times of shocks. Moreover, $N \in \mathcal{S}(\mathbb{N})$ denotes the random number of shocks survived by the system, with discrete survival function $\overline{P}(k) = \mathbb{P}(N > k)$, $k \in \mathbb{N}_0$. If X_j denotes the interarrival time between the $(j-1)$-th and j-th shocks, then $T = \sum_{j=1}^{N} X_j$ gives the lifetime of the system. Furthermore, let the renewal process describing the number of shocks have cumulative intensity function $\Lambda(t) = -\log \overline{F}(t)$, $t \ge 0$. Then, T has survival function

$$\overline{F}_T(t) = \sum_{k=0}^{\infty} \overline{P}(k)\frac{\Lambda^k(t)}{k!}\overline{F}(t), \qquad t > 0. \tag{49}$$

Relation (49) also holds for a repairable system which is discussed by Chahkandi et al. [38]. The following theorem gives sufficient conditions for ordering of weighted mean residual lifetime of two systems. Let $N_i \in \mathcal{S}(\mathbb{N})$ have survival function $\overline{P}_i(k)$, $i = 1,2$. We recall that $N_1 \le_{hr} N_2$ if $\overline{P}_2(k)/\overline{P}_1(k)$ is increasing in $k \in \mathbb{N}_0$.

Theorem 21. *Let T_1 and T_2 be the random lifetimes of devices subject to shocks governed by a non-homogeneous Poisson process having cumulative intensity function $\Lambda(t)$. Let N_1 and N_2 be respectively the number of survived shocks of the devices. If $N_1 \le_{hr} N_2$, then $T_1 \le_{wmrl}^{\phi} T_2$.*

Proof. Let us set, for all $t > 0$,

$$g(i,t) = \sum_{k=0}^{\infty} \overline{P}_i(k) \int_t^{\infty} \phi(x) \frac{\Lambda^k(x)}{k!} \overline{F}(x)\,\mathrm{d}x, \qquad i = 1,2.$$

To prove $T_1 \leq_{wmrl}^{\phi} T_2$, we need to show that $g(2,t)/g(1,t)$ is increasing in t, due to Theorem 14, or equivalently $g(i,t)$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbb{R}^+$. By assumption $N_1 \leq_{hr} N_2$, which means that $\overline{P}_i(k)$ is TP_2 in $(i,k) \in \{1,2\} \times \mathbb{N}_0$. Moreover, the function

$$\int_t^{\infty} \phi(x) \frac{\Lambda^k(x)}{k!} \overline{F}(x)\,\mathrm{d}x = \int_0^{\infty} \phi(x) \frac{\Lambda^k(x)}{k!} \overline{F}(x) I_{[x \geq t]}\,\mathrm{d}x,$$

is TP_2 in $(t,k) \in \mathbb{R}^+ \times \mathbb{N}_0$ due to the general composition theorem of Karlin [16], since $\phi(x)\frac{\Lambda^k(x)}{k!}\overline{F}(x)$ is TP_2 in $(x,k) \in \mathbb{R}^+ \times \mathbb{N}_0$ and the indicator function $I_{[x \geq t]}$ is TP_2 in $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Thus, again, the general composition theorem provides that $g(i,t)$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbb{R}^+$, which completes the proof. □

Remark 10. *A-Hameed and Proschan [39] investigated some aging properties of the shock model (49). They proved that, if the parent distribution function $F(x)$ is IFR (DFR) and N is discrete DMRL (IMRL), then the system with lifetime T is DMRL (IMRL). In the present case, the following similar results hold, due to Theorem 2.*
(i) Let $\phi(x)$ be decreasing in x. If X is IFR and N is discrete DMRL, then T is DWMRL.
(ii) Let $\phi(x)$ be increasing in x. If X is DFR and N is discrete IMRL, then T is IWMRL.

Using the above results, we can now compare the variance and WGCRE of two systems under different shocks.

Theorem 22. *Under the same assumptions of Theorem 21, let X be DFR and $\phi(x)$ be increasing in x. If $N_1 \leq_{hr} N_2$ and either N_1 or N_2 is discrete IMRL, then $\sigma^2[\psi(T_1)] \leq \sigma^2[\psi(T_2)]$ and $\mathcal{E}_{\psi,n}(T_1) \leq \mathcal{E}_{\psi,n}(T_2)$, for all $n \in \mathbb{N}$.*

Proof. Let N_1 be discrete IMRL. Since X is DFR and $\phi(x)$ is increasing, point (ii) of Remark 10 implies that T_1 is IWMRL. On the other hand, $N_1 \leq_{hr} N_2$ gives $T_1 \leq_{wmrl}^{\phi} T_2$ due to Theorem 21. Moreover, $N_1 \leq_{hr} N_2$ yields $N_1 \leq_{st} N_2$ which ensures that $T_1 \leq_{st} T_2$, (see e.g., Theorem 4.2 of Chahkandi et al. [38]). Thus, from Theorem 17, we have the results. When N_2 is discrete IMRL, the proof is similar. □

In the special case in which the interarrival times are independent and identically exponentially distributed, one clearly has that $\Lambda^k(t) = (\lambda t)^k$ on the right-hand-side of the survival function (49).

Let us consider the cumulative weight function $\psi(x) = x^r$, or weight function $\phi(x) = rx^{r-1}$, for all $r \in \mathbb{N}$.

Theorem 23. *Let T_1 and T_2 be the random lifetimes of two devices subject to shocks governed by a homogeneous Poisson having intensity λ, and let N_i, $i = 1,2$, be the random number of shocks survived by the i-th device, with $\overline{P}_i(k) = \mathbb{P}(N > k)$, $k \in \mathbb{N}_0$. If, for $r \in \mathbb{N}$, one has that*

$$\frac{\sum_{j=k+r-1}^{\infty} \binom{j}{r-1} \overline{P}_2(j-r+1)}{\sum_{j=k+r-1}^{\infty} \binom{j}{r-1} \overline{P}_1(j-r+1)} \quad \text{is increasing in } k \in \mathbb{N}, \tag{50}$$

then $T_1 \leq_{wmrl}^{\phi} T_2$, for the cumulative weight function $\psi(x) = x^r$.

Proof. It is known that the survival function of T_i, $i = 1, 2$, is given by

$$\overline{H}_{T_i}(x) = \sum_{k=0}^{\infty} \overline{P}_i(k) \frac{e^{-\lambda x}(\lambda x)^k}{k!}, \qquad x \geq 0. \tag{51}$$

Let us consider the following well-known relation

$$\int_t^{\infty} e^{-\lambda x} \frac{\lambda^{k+1} x^k}{k!} \, dx = \sum_{j=0}^{k} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \qquad k \in \mathbb{N}_0,$$

for all $t > 0$. Recalling (51) and using the aforementioned equation, after some manipulations, we get, for $r \in \mathbb{N}$ and $i = 1, 2$,

$$\begin{aligned}
\int_t^{\infty} r x^{r-1} \overline{H}_{T_i}(x) \, dx &= \int_t^{\infty} r x^{r-1} \sum_{k=0}^{\infty} \overline{P}_i(k) \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dx \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} \overline{P}_i(k) \binom{r+k-1}{k} \int_t^{\infty} e^{-\lambda x} \frac{\lambda^{k+r} x^{k+r-1}}{(k+r-1)!} \, dx \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} \overline{P}_i(k) \binom{r+k-1}{k} \sum_{j=0}^{k+r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \sum_{j=k+r-1}^{\infty} \binom{j}{r-1} \overline{P}_i(j-r+1).
\end{aligned}$$

Since $e^{-\lambda t}(\lambda t)^k / k!$ is TP_2 in $(k, t) \in \mathbb{N} \times \mathbb{R}^+$, and recalling relation (50), the general composition theorem of Karlin [16] implies that $\int_t^{\infty} r x^{r-1} \overline{H}_{T_i}(x) \, dx$ is TP_2 in $(i, t) \in \{1, 2\} \times \mathbb{R}^+$. This is equivalent to state that $T_1 \leq_{wmrl}^{\phi} T_2$ for $\psi(x) = x^r$. □

The case concerning the weight function $\phi(x) = x$ is considered in Theorem 4.1 of Kayid and Izadkhah [34]. From Theorem 23, we immediately obtain the following result.

Theorem 24. *Under the conditions of Theorem 23, if $N_1 \leq_{st} N_2$, and either N_1 or N_2 is IMRL, then $\sigma^2(T_1^r) \leq \sigma^2(T_2^r)$ and $\mathcal{E}_{r,n}(T_1) \leq \mathcal{E}_{r,n}(T_2)$, for all $n, r \in \mathbb{N}$.*

Proof. Let N_1 be discrete IMRL. Since $\phi(x) = r x^{r-1}$ is increasing in x for all $r \in \mathbb{N}$, then N_1 is IWMRL due to Remark 10. Moreover, recalling Theorem 23, it holds that $T_1 \leq_{wmrl}^{\phi} T_2$. Use of Theorem 17 thus completes the proof. When N_2 is discrete IMRL, the proof is similar. □

6.2. Application to Random Minima

Now, we consider an application to random minima. Let us consider a sequence of random variables $X_1, X_2, \ldots$ which is independent of a discrete non-negative random variable N. The minimum extreme order statistics is defined by

$$X_{1:N} = \min\{X_1, X_2, \ldots, X_N\}.$$

As is well known, it can be viewed as the random lifetime of a series system consisting of a random number of components with i.i.d. lifetimes $X_1, X_2, \ldots, X_N$. In case of life testing, if a random censoring is pursued, the completely observed data make a sample of random size N. For each $i = 1, 2$, let $X_{1:N_i}$ denote the minimum order statistic among $X_1, X_2, \ldots, X_{N_i}$, where N_i is a positive integer-valued random variable which is independent from $\{X_n\}$. For the concept of likelihood ratio order in the discrete case, see Shaked and Shanthikumar [31].

Theorem 25. *Let $\phi(x)$ be increasing in x. If $N_1 \leq_{lr} N_2$ and X is DFR, then $Var[\psi(X_{1:N_1})] \geq Var[\psi(X_{1:N_2})]$ and $\mathcal{E}_{\psi,n}(X_{1:N_1}) \geq \mathcal{E}_{\psi,n}(X_{1:N_2})$ for all $n \in \mathbb{N}$.*

Proof. From Theorem 2.4 of Shaked and Wong [40], we have $X_{1:N_1} \geq_{lr} X_{1:N_2}$ which implies that $X_{1:N_1} \geq_{hr} X_{1:N_2}$. On the other hand, since X is DFR, then either $X_{1:N_1}$ or $X_{1:N_2}$ is DFR due to Shaked [41] and thus it is IMRL. Because $\phi(x)$ is increasing in x, due to Theorem 2, then $X_{1:N_1}$ or $X_{1:N_2}$ is IWMRL. Thus, Theorem 18 completes the proof. □

7. Conclusions

An eligible measure of uncertainty in some practical situations in neurobiology and reliability is the weighted shift-dependent measure as pointed out by [2]. Objects in a scale and translation-invariant manner can be recognized by the human visual system. However, this important feature using biologically realistic networks is a challenge. Thus, the weighted entropy can be used in such situations [3]. Thus, the differential entropy is one such pioneer measure. Accordingly, the weighted generalized cumulative residual entropy is also introduced. Various papers investigated some properties of the mentioned measure. Such properties however are not exhaustive and hence we tried to continue this line of research with a different point of view. Indeed, we first considered the cumulative weight function which can be related to a weighted random variable. For instance, when the weight function is the hazard rate function, then we have the cumulative hazard function as a special case. By using this function, we defined the weighted mean residual life function which was successfully applied to provide expressions for the variance of transformed random variable and WGCRE. We remark that the variance of transformed random variable and WGCRE can be used as dispersion measures and uncertainties. Indeed, by virtue of Theorem 7, the WGCRE is a suitable alternative to the standard deviation of a transformed random variable when the latter is not existing. This motivated us to study both measures at the same time. Moreover, the new expressions obtained in this paper enabled us to provide several results including bounds, stochastic comparisons based on various stochastic orders, and specifically their connections with the transformed excess wealth order. In fact, the latter result seems to be novel for the WGCRE in the sense that it is connected with the excess wealth transform being a well-known concept in survival analysis, reliability theory, risk theory, actuarial science, and other applied areas. We finally illustrated various applications of the given results by comparing variance of transformed random lifetime and WGCRE of two shock models and of random minima.

Several results obtained in this paper involve integrals of the form $\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x$, as for the WMRL function introduced in (7). However, as pointed out in Remark 5 for the weighted GCRE, other developments can be performed by considering the form $\int_t^\infty \phi(x-t)\overline{F}(x)\,\mathrm{d}x$. In other terms, the two approaches allow for dealing with different expectations as $\mathbb{E}[\psi(X)-\psi(t)|X>t]$ and $\mathbb{E}[\psi(X-t)|X>t]$. Indeed, in the second case, a suitable choice of the function ψ addresses toward distributional results based on the Weyl fractional-order integral operator (see, for instance, Pakes and Navarro [42]). In particular, when $\psi(x)=x^\alpha/\alpha$, $\alpha>0$, Equation (28) yields

$$m^\psi(t) = \frac{\mathbb{E}[X^\alpha]}{\alpha}\,\frac{\overline{F}^\alpha(t)}{\overline{F}(t)}, \qquad t \geq 0,$$

where $\overline{F}(t)$ is the survival function of X, and $\overline{F}^\alpha(t)$ is the survival function of the fractional equilibrium distribution of X, which plays a relevant role in the fractional probabilistic Taylor's and mean value theorems (see Di Crescenzo and Meoli [43]). Hence, in analogy with the given results, future investigations can be based on expectations of the form $\mathbb{E}[\psi(X-t)|X>t]$ and can be related to notions of fractional calculus.

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