

Article

Study of Local Convergence and Dynamics of a King-Like Two-Step Method with Applications

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Abstract: In this paper, we present the local results of the convergence of the two-step King-like method to approximate the solution of nonlinear equations. In this study, we only apply conditions to the first derivative, because we only need this condition to guarantee convergence. As a result, the applicability of the method is expanded. We also use different convergence planes to show family behavior. Finally, the new results are used to solve some applications related to chemistry.

Keywords: King-like iterative methods; local convergence; Lipschitz conditions; dynamics

1. Introduction

In this study, inspired in previous papers such as [1], we want to approximate a locally unique solution z^* of equation

$$G(z) = 0, \quad (1)$$

where $G : I \subset \mathbb{C} \rightarrow T$, $T = \mathbb{R}$ or $T = \mathbb{C}$, I is convex and G is a differentiable function.

As society changes, the way in which we understand science also changes, as can be seen in [2–6]. In the case of Mathematical Modelling [2–5] it is important to comment that it enables us to solve problems with the form of the Equation (1). Based on the results, researchers are working to use the same assumptions to increase the convergence domain of the iterative method because it is usually small, the final purpose is to estimate: $\|z_{n+1} - z_n\|$ and $\|z_n - z^*\|$.

The dynamic behavior of iterative methods provides key information about its reliability and stability. In the past ten years, some studies related to dynamic characteristics have been presented [3,4,6–25]. The main problem of this research is related to the fact that these families or methods have only one parameter, or even no parameters. In our case, we have more free parameters, so we need another tool to study it. The tools in [19,21], are very useful for studying the dynamic behavior of families with two different parameters or the actual dynamics in other interactive applications. In this article, we will study the convergence of:

$$w_n = z_n - \alpha \frac{G(z_n)}{G'(z_n)}$$
$$z_{n+1} = w_n - \frac{G(w_n)}{G'(z_n)} \frac{G(z_n) + \beta G(w_n)}{G(z_n) + (\beta - 2)G(w_n)},$$

Includes computable convergence radius and error estimates. In addition, we will show the behaviors related to the family by using the convergence planes.

We recognize local, semi-local and global convergence. In the case of local (convergence ball is centered at the solution) and semi-local (convergence ball is centered at the starting point x_0) the convergence is faster, and less expensive than global convergence. But global convergence provides all roots inside a domain. It is worth noticing that loocal convergence results are important because they demonstrate the degree of difficulty in choosing starters x_0 . There is a plethora of literature on solvers. See for example [26–30] and the references therein. In these references other techniques are used on polynomials (or not) considering proper intervals containing a root or all roots. These references do not provide computable bounds on $\|x_n - x_*\|$ or uniqueness of the solution x_* results. But we do. Moreover, the starter x_0 really a shot in the dark, whereas in our case it is picked up from a predetermined ball of convergence.

Consider the motivational example for $T = \mathcal{R}, I = [-\frac{1}{2}, \frac{3}{2}]$ where

$$f(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

Then, we have $x_* = 0$, and

$$f'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22.$$

Then, obviously, function $f'''(t)$ is unbounded on I . Hence, earlier results using the third (or higher order, that do not appear on (2), b ut used to compute the order of convergence) derivatives cannot guarantee convergence (2) to z^* ,

The convergence studies so far have shown convergence using high order derivatives that do not appear on the method. Hence, limiting the applicability of the method. But we show convergence using only the first derivative that actually appears on the method. Hence, we expand the applicability of the method. A radius of convergence, estimates on $\|x_n - x_*\|$ and uniqueness results not given before are also provided. Our technique can be used to expand the applicability of other methods along the same lines. The convergence order is determined using (COC) orr (ACOC) to be precised in Remark 2.

All this work is organized into different chapters: Next section, number 2, we display the method and the theorems that guarantee the local convergence of the method. In the next section, number 3, the authors study the convergence planes of the method of the previous section applied to a quadratic polynomial. In the next section we introduce some chemical applications that reinforce the applicability of the method and in the last section, number 5, we write the conclusions that complete the article.

2. Local Convergence Analysis

We start to introduce in this section the method we will study throughout the article. We study the local convergence analysis of the two-step King-like method defined as follow for each $n = 0, 1, \dots$ by

$$\begin{aligned} w_n &= z_n - \alpha \frac{G(z_n)}{G'(z_n)} \\ z_{n+1} &= w_n - \frac{G(w_n)}{G'(z_n)} \frac{G(z_n) + \beta G(w_n)}{G(z_n) + (\beta - 2)G(w_n)}, \end{aligned} \tag{2}$$

where z_0 is an initial point, $\alpha \in T$ and $\beta \in T$ are parameters. Some well-known methods, such as our King’s family methods or Traub-Ostrowski method, are special cases of this family.

We consider $N \geq 1, L_0, L > 0$ and $\alpha, \beta \in T$. We now want to demonstrate the local convergence of the method (2) for which we are going to use the following auxiliary functions and parameters defined as:

The first auxiliary function that we are going to define is called k_1 on the interval $[0, \frac{1}{L_0})$ by

$$k_1(t) = \frac{1}{2(1 - L_0t)}(Lt + 2N|1 - \alpha|)$$

and

$$r_1 = \frac{2(1 - N|1 - \alpha|)}{2L_0 + L}$$

$$r_A = \frac{2}{2L_0 + L}.$$

Next, taking into account the definition of N and α , we can write

$$|1 - \alpha|N < 1. \tag{3}$$

Based on the above, we obtain as a consequence

$$0 < r_1 < r_A < \frac{1}{L_0}, \tag{4}$$

$$k_1(r_1) = 1 \quad \text{and} \quad 0 \leq k_1(t) < 1 \quad \text{for each} \quad t \in [0, r_1).$$

Now consider the interval $[0, \frac{1}{L_0})$ and defining a new auxiliary functions

$$k_0(t) = \frac{L_0}{2}t + N|\beta - 2|k_1(t),$$

and other auxiliary function

$$p_0(t) = k_0(t) - 1,$$

Next, suppose that

$$N^2|(1 - \alpha)(2 - \beta)| < 1, \tag{5}$$

It clearly follows that

$$p_0(0) = N^2|(1 - \alpha)(\beta - 2)| - 1 < 0$$

and consequently,

$$p_0(t) \rightarrow +\infty$$

when

$$t \rightarrow \frac{1}{L_0}^-.$$

Using now the Intermediate Value Theorem, it is clear that there exist at least one root of p_0 located in the interval $(0, \frac{1}{L_0})$, we will call r_0 the smallest zero in that interval.

Now we can perform the same technique, for this we define on the interval $[0, r_0)$ a new auxiliary functions

$$k_2(t) = \left[1 + \frac{N^2(1 + |\beta|k_1(t))}{(1 - L_0t)(1 - k_0(t))} \right] k_1(t)$$

and

$$p_2(t) = k_2(t) - 1.$$

Similarly, suppose

$$\left[\frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} + 1 \right] |1 - \alpha|N < 1, \tag{6}$$

it is easy to see that

$$p_2(0) = \left[\frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} + 1 \right] |1 - \alpha|N - 1 < 0$$

and

$$p_2(t) \rightarrow \infty$$

when

$$t \rightarrow r_0^-.$$

So, it is clear that there exist at least one root of p_2 located in the interval $(0, r_0)$. We will call r_2 as the smallest zero in the interval.

We also define the value r as the minimum of the values between r_1 and r_2

$$r = \min\{r_1, r_2\}. \tag{7}$$

For $t \in [0, r)$, the following conditions are fulfilled by definition of the functions

$$0 \leq k_0(t) < 1, \tag{8}$$

$$0 \leq k_1(t) < 1 \tag{9}$$

and

$$0 \leq k_2(t) < 1. \tag{10}$$

From now on, we will also consider $U(v, \rho)$ as the open ball in T with center $v \in T$ and radius $\rho > 0$, as $\bar{U}(v, \rho)$ the closed one and $I_0 = I \cap U(z^*, \frac{1}{L_0})$, where L_0 is positive therefore we can present the following result.

Theorem 1. *Let $G : I \subset T \rightarrow T$ be a differentiable function. Suppose that there exists $z^* \in I$, $L_0, L > 0$, $N \geq 1$ and $\alpha, \beta \in T$ such that the following requirements are true*

$$G'(z^*) \neq 0, \quad G(z^*) = 0, \tag{11}$$

$$|(1 - \alpha)(2 - \beta)|N^2 < 1,$$

$$\left[\frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} \right] |1 - \alpha|N < 1,$$

$$\|G'(z^*)^{-1}(G'(z) - G'(z^*))\| \leq L_0\|z - z^*\|, \text{ for each } z \in I \tag{12}$$

$$\|G'(z^*)^{-1}(G'(z) - G'(w))\| \leq L\|z - w\|, \text{ for each } z, w \in I_0 \tag{13}$$

$$\|G'(z^*)^{-1}G'(z)\| \leq N, \text{ for each } z \in I_0 \tag{14}$$

and

$$\bar{U}(z^*, r) \subseteq I. \tag{15}$$

Then, the method (2) generate a well defined sequence $\{z_n\}$ for $z_0 \in U(z^*, r) \setminus \{z^*\}$, and $\{z_n\}$ remains in $U(z^*, r)$ for each $n = 0, 1, 2, \dots$ and moreover, the sequence converges to z^* .

Furthermore, the inequalities written below are also fulfilled

$$\|w_n - z^*\| \leq k_1(\|z_n - z^*\|)\|z_n - z^*\| < \|z_n - z^*\| \leq r, \tag{16}$$

and

$$\|z_{n+1} - z^*\| \leq k_2(\|z_n - z^*\|)\|z_n - z^*\| \leq \|z_n - z^*\|, \tag{17}$$

Moreover, if we take R in the form

$$R \in [r, \frac{2}{L_0})$$

the limit point z^* is the only solution of Equation (1) in the domain

$$I_1 := \bar{U}(z^*, R) \cap I.$$

Proof. To prove the Theorem we are going to use the mathematical induction method to show estimates (16) and (17). First of all, using

$$z_0 \in U(z^*, r) \setminus \{z^*\}$$

and (12) we get

$$\frac{\|(G'(z_0) - G'(z^*))\|}{\|G'(z^*)\|} \leq L_0 \|z_0 - z^*\| < L_0 r < 1. \tag{18}$$

Using this condition and the Banach lemma on invertible functions [2,3,5] it is obvious that

$$G'(z_0)$$

is not equal to zero and satisfied

$$\|G'(z_0)^{-1}G'(z^*)\| \leq \frac{1}{1 - L_0 \|z_0 - z^*\|} \leq \frac{1}{1 - L_0 r}. \tag{19}$$

As a consequence, w_0 is well defined. Now, using (11) we obtain

$$G(z_0) = G(z_0) - G(z^*) \text{ is equal to } \int_0^1 G'(z^* + \theta(z_0 - z^*))(z_0 - z^*)d\theta. \tag{20}$$

Notice that $\theta(z_0 - z^*) + z^* \in U(z^*, r)$ since $\|\theta(z_0 - z^*) - z^* + z^*\| = \theta \|z_0 - z^*\| < r$. Now, from (14) and (19), we obtain

$$\|G'(z^*)^{-1}G(z_0)\| \text{ is equal to } \left\| \int_0^1 G'(\theta(z_0 - z^*)z^*)(z_0 - z^*)d\theta \right\| \leq N \|z_0 - z^*\|. \tag{21}$$

From method (2) for $n = 0$, and the Equations (7), (8), (13), (19) and (22) we get

$$\begin{aligned} \|w_0 - z^*\| &\leq \|(z_0 - z^* - G'(z_0)^{-1}G(z_0)) + (1 - \alpha)G'(z_0)^{-1}G(z_0)\| \\ &\leq \|G'(z_0)^{-1}G(z^*)\| \left\| \int_0^1 G'(z^*)^{-1}G(z^* + \theta(z_0 - z^*) - G'(z_0))(z_0 - z^*)d\theta \right\| \\ &\quad + |1 - \alpha| \|G'(z_0)^{-1}G'(z^*)\| \|G'(z^*)^{-1}G(z_0)\| \\ &\leq \frac{L \|z_0 - z^*\|^2}{2(1 - L_0 \|z_0 - z^*\|)} + \frac{N |1 - \alpha| \|z_0 - z^*\|}{1 - L_0 \|z_0 - z^*\|} \\ &= k_1 (\|z_0 - z^*\|) \|z_0 - z^*\| \leq \|z_0 - z^*\| < r. \end{aligned} \tag{22}$$

As a result, it is satisfied (16) taking $n = 0$ and $y_0 \in U(x^*, r)$. Now, replace z_0 by w_0 in the previous inequality (22) we get

$$\|G'(z^*)^{-1}G(w_0)\| \leq N \|w_0 - z^*\|. \tag{23}$$

Now we must prove $G(z_0) + (\beta - 2)G(w_0) \neq 0$ for $z_0 \neq z^*$. From conditions (7), (9), (11), (12) and (22)–(23) it follows that

$$\begin{aligned}
 & \|G'(z_0)(z_0 - z^*)\|^{-1} [G(z_0) - G(z^*) - G'(z^*)(z_0 - z^*) + (\beta - 2)G(w_0)] \\
 & \leq \|z_0 - z^*\|^{-1} [\|G'(z^*)^{-1}(G(z_0) - G(z^*) - G'(z^*)(z_0 - z^*))\| + |2 - \beta|\|G'(z^*)^{-1}G(w_0)\|] \\
 & \leq \|z_0 - z^*\|^{-1} \left[\frac{L_0}{2}\|z_0 - z^*\|^2 + |2 - \beta|N\|w_0 - z^*\| \right] \tag{24} \\
 & \leq \frac{L_0}{2}\|z_0 - z^*\| + |2 - \beta|Nk_1(\|z_0 - z^*\|) \\
 & = k_0(\|z_0 - z^*\|) < k_0(r) < 1.
 \end{aligned}$$

Or what is the same, from (24) we obtain

$$\|(G(z_0) + (2 - \beta)G(w_0))^{-1}G'(z^*)\| \leq \frac{1}{(1 - k_0(\|z_0 - z^*\|))\|z_0 - z^*\|}. \tag{25}$$

Consequently, z_1 is well defined by the second substep of method (2) for $n = 0$.

Now, from (2) for $n = 0$, (7), (10), (19), (22) and (25) we get

$$\begin{aligned}
 \|z_1 - z^*\| & \leq \|w_0 - z^*\| + \|G'(z_0)^{-1}G'(z^*)\|\|G'(z^*)^{-1}G(y_0)\| \\
 & \times (\|G'(z^*)^{-1}G(z_0)\| + |\beta|\|G'(z^*)^{-1}G(w_0)\|)\|(G(z_0) + (\beta - 2)G(w_0))^{-1}G'(z^*)\| \\
 & \leq \|w_0 - z^*\| + \frac{N^2\|w_0 - z^*\|(\|z_0 - z^*\| + |\beta|\|w_0 - z^*\|)}{(1 - L_0\|z_0 - z^*\|)\|z_0 - z^*\|(1 - k_0(\|z_0 - z^*\|))} \\
 & = \left[1 + \frac{N^2(1 + |\beta|k_1(\|z_0 - z^*\|))}{(1 - L_0\|z_0 - z^*\|)(1 - k_0(\|z_0 - z^*\|))} \right] \|w_0 - z^*\| \\
 & \leq k_2(\|z_0 - z^*\|)\|z_0 - z^*\| \leq \|z_0 - z^*\| < r,
 \end{aligned}$$

which demonstrates the Equation (17) with $n = 0$ and $z_1 \in U(z^*, r)$.

Now substituting z_0, w_0, z_1 by z_k, w_k, z_{k+1} in the previous estimates we reach (16)–(18).

Next, from $\|z_{k+1} - z^*\| \leq c\|z_k - z^*\| < r$, where $c = k_2(\|z_0 - z^*\|) \in [0, 1]$, we obtain:

$$\lim_{k \rightarrow \infty} z_k = z^* \text{ and } z_{k+1} \in U(z^*, r).$$

Finally, we set

$$M = \int_0^1 G'(w^* + \theta(z^* - w^*))d\theta$$

for some $w^* \in \bar{U}(z^*, R)$ with $G(w^*) = 0$. From (12), we obtain

$$\begin{aligned}
 \|G'(z^*)^{-1}(M - G'(z^*))\| & \leq \int_0^1 L_0\|w^* + \theta(z^* - w^*)\|d\theta \\
 & = L_0 \int_0^1 (1 - \theta)\|w^* - z^*\| \leq \frac{L_0}{2}R < 1.
 \end{aligned} \tag{26}$$

It is clear that from (26), $M^{-1} \in L(T, T)$. Finally, using $0 = G(w^*) - G(z^*) = M(w^* - z^*)$, we get

$$z^* = w^*.$$

□

Remark 1.

- Moreover for the error bounds in practice we can use the computational order of convergence (COC) [31] using

$$\xi^* = \sup \frac{\ln \left(\frac{|x_{n+2}-x^*|}{|x_{n+1}-x^*|} \right)}{\ln \left(\frac{|x_{n+1}-x^*|}{|x_n-x^*|} \right)} \quad \text{for each } n = 0, 1, 2, \dots$$

- or the approximate computational order of convergence (ACOC) [31] using iterations instead of the exact solution

$$\zeta = \sup \frac{\ln \left(\frac{|x_{n+2}-x_{n+1}|}{|x_{n+1}-x_n|} \right)}{\ln \left(\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|} \right)} \quad \text{for each } n = 1, 2, \dots$$

Remark 2.

- Condition (11) and estimates (12)–(14) are natural Lipschitz-type conditions which are standard in the study of iterative methods. In fact (12)–(14) can be condensed in one condition. But used this way is better for reasons of precision.
- Concerning the second and the thirs conditions in Theorem 1, these were left as uncluttered as possible. One can easily see that there are infinitely many pair of values (α, β) satisfying these inequalities (but not as weak). Indeed, suppose that

$$|(1 - \alpha)(2 - \beta)| < \frac{1}{2N^2} \tag{27}$$

Then, the second inequality is satisfied provided that

$$2N^2(1 + |\beta|N|1 - \alpha|)|1 - \alpha|N < 1$$

or

$$|1 - \alpha| < \frac{1}{N^2\sqrt{N^2 + 2|\beta|}} \left(< \frac{1}{N} \right). \tag{28}$$

Then, the original inequalities certainly hold, if (27) and (28) hold. Moreover, (27) (in view of (28)) can ever be replaced by

$$|2 - \beta| < \frac{1}{2N^4\sqrt{N^2 + 2|\beta|}}. \tag{29}$$

However as noted earlier condtions (27) and (28) or (28) and (29) are stronger than the two inequalities appearing in the statemetn of Theorem 1 (and can certainly replace them). Either way, as claimed above there are infinitely many choices of (α, β) .

3. The Convergence Planes Applied to the Quadratic Polynomial $p(z) = z^2 - 1$ Using Method (2)

Once the convergence of the method has been demonstrated in the previous section, we will study the convergence planes. We should first study the parameter planes of the method (2), but the details of these parameter planes can be seen in [24]. We will show the convergence planes applied to $p(z) = z^2 - 1$ a polynomial of degree two, taking different values of the method parameters α and β . In this article, to draw the points, we iterate a maximum of 200 times and a tolerance of 10^{-3} . We will use different colors to differentiate the numerical behavior of the points when they are iterated:

- in yellow if the iteration of the initial point diverges to ∞ ,
- in magenta if the starting point converges to the first root of the quadratic polynomial, $z = 1$,
- in cyan if the iteration of the starting point converges to the other root of the quadratic polynomial $z = -1$,
- if the iteration of the initial point converges to any strange fixed points, it is colored in red,

- in other colors if the iteration of the initial point converge to different n -cycles ($n \leq 8$),
- in black if the iteration of the initial point has other behavior.

Consequently, we are only interested in points painted in cyan or magenta since it means that the selection of parameters α and β for that initial point z_0 in numerical terms is a good one.

In Figure 1 we see the convergence plane associated to the method (2) applied to the two order polynomial $p(z) = z^2 - 1$ with the initial point $z_0 = 0.5$ and in the region of the plane $(\alpha, \beta) \in [-5, 5] \times [-5, 5]$. and in Figure 2 with starting point $z_0 = -0.75$ and in the region of the plane $(\alpha, \beta) \in [-5, 5] \times [-5, 5]$

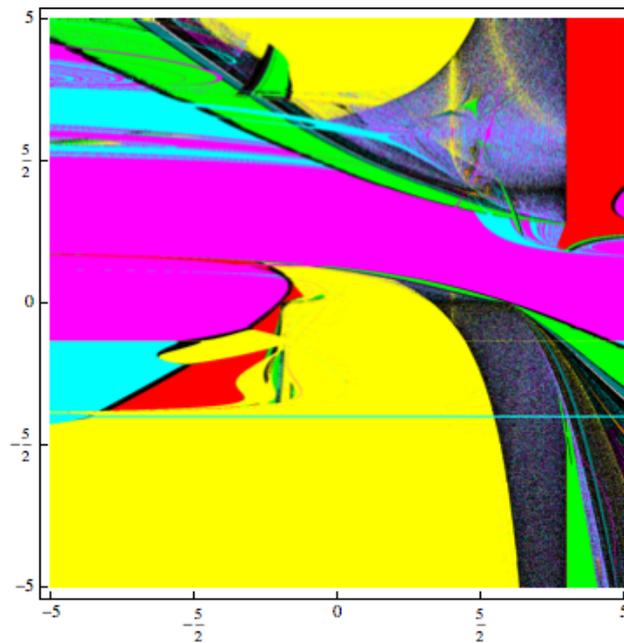


Figure 1. Convergence plane of method (2). The starting point is $z_0 = 0.5$.

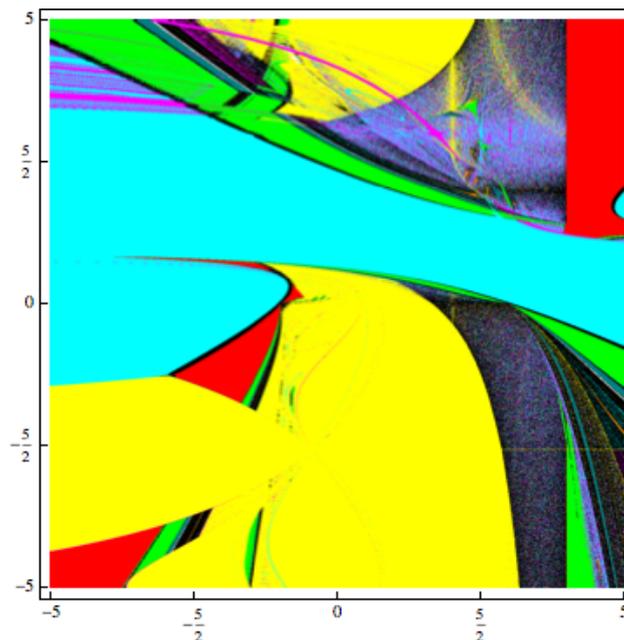


Figure 2. Convergence plane of method (2). The starting point is $z_0 = 0.75$.

4. Application Examples

To show the applicability of the theorems and results presented in this article, let's look at the Planck's radiation law problem, which can be obtained from the reference [32]:

$$\varphi(\lambda) = \frac{8\pi s P \lambda^{-5}}{e^{\frac{sP}{\lambda BK}} - 1} \tag{30}$$

which estimates the density of within an isothermal black body.

If we use a variable change:

$$x = \frac{sP}{\lambda BK'}$$

The expression which estimates the density of within an isothermal black body can be defined in the following terms:

$$1 - \frac{x}{5} = e^{-x} \tag{31}$$

And if we now define the function $h(x)$ as:

$$h(x) = e^{-x} - 1 + \frac{x}{5}. \tag{32}$$

As a result, if we are able to find the roots of Equation (32) can provide us the maximum wavelength of radiation (λ) through the expression:

$$\lambda \approx \frac{sP}{x^* BK}. \tag{33}$$

Let's consider the interval $D = [4, 6]$ and the solution $x^* = 4.965114 \dots$

Then, we will introduce the applicability of the three special cases introduced in previous Section 3.

Application 1. First we will consider the case $\alpha = 0.9$ and $\beta = 2.5$

We obtain the values for this case

$$L_0 = 0.060865 \dots,$$

$$L = 0.0948885 \dots$$

and

$$N = 1.02331 \dots$$

Moreover, we can conclude the following conditions

$$\left[\frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} + 1 \right] |1 - \alpha|N = 0.273262 \dots < 1.$$

$$N^2|(1 - \alpha)(2 - \beta)| = 0.0523577 \dots < 1,$$

are satisfied. Then, due to the definition of the "k" functions we can derive

$$r_0 = 9.71092 \dots,$$

$$r_1 = 9.23282 \dots$$

and

$$r_2 = 2.64413 \dots$$

Therefore, if we take the alpha and beta values as $\alpha = 0.9$, $\beta = 2.5$ and $r = r_2 = 2.64413 \dots$ we we can assure that the method described in (2) converges to the unique solution x^* of the function $h(x)$ by Theorem 1.

Application 2. Secondly we will consider the case $\alpha = 0.75$ and $\beta = 2.01$

We obtain the values for this case

$$L_0 = 0.060865 \dots,$$

$$L = 0.0948885 \dots$$

and

$$M = 1.02331 \dots$$

Moreover, we can conclude the following conditions

$$\left[1 + \frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} \right] |1 - \alpha|N = 0.867984 \dots < 1.$$

$$|(1 - \alpha)(2 - \beta)|N^2 = 0.00261788 \dots < 1,$$

are satisfied.

Now, applying the definition of the “k” functions we get

$$r_0 = 16.09389 \dots,$$

$$r_1 = 6.87083 \dots$$

and

$$r_2 = 1.18924 \dots$$

So, taking the values: $\alpha = 0.75$, $\beta = 2.01$ and $r = r_2 = 1.18924 \dots$ we can assure the convergence of the method described in (2) to the solution x^* of $h(x)$ applying Theorem 1.

Application 3. Finally we will consider the case $\alpha = \beta = 1$.

We obtain the values for this case

$$L_0 = 0.060865 \dots,$$

$$L = 0.0948885 \dots$$

and

$$N = 1.02331 \dots$$

Moreover, we can conclude the following conditions

$$\left[\frac{N^2(1 + |\beta|N|1 - \alpha|)}{1 - N^2|(1 - \alpha)(2 - \beta)|} + 1 \right] |1 - \alpha|N = 0 < 1.$$

$$|(1 - \alpha)(2 - \beta)|N^2 = 0 < 1,$$

are satisfied. Now, applying the definition of the “k” functions we get

$$r_0 = 9.71092 \dots,$$

$$r_1 = 9.23282 \dots$$

and

$$r_2 = 2.64413 \dots$$

So, taking the values: $\alpha = 0.9$, $\beta = 2.5$ and $r = r_2 = 2.64413 \dots$ we can assure the convergence of the method described in (2) to the solution x^* of $f(x)$ by Theorem 1.

5. Conclusions

In this paper, we have studied the local convergence of the two-step King-like method of Equation (2). The local convergence of the method has been proved giving the theorem that assure it, in addition to its proof. The advantage of this research is that it is only necessary to apply conditions to the first derivative to ensure local convergence, which can avoid the study of advanced derivatives that may not be simple, thus expanding the applicability of the method. In addition, the dynamics of the method for quadratic polynomials has been studied to prove the performance of the method. We finish by showing several examples using the new proposed method where we can assure the convergence.

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