mathematics

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Fuzzy Stability Results of Finite Variable Additive Functional Equation: Direct and Fixed Point Methods

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Received: 3 June 2020; Accepted: 16 June 2020; Published: 30 June 2020

Abstract: In this current work, we introduce the finite variable additive functional equation and we derive its solution. In fact, we investigate the Hyers–Ulam stability results for the finite variable additive functional equation in fuzzy normed space by two different approaches of direct and fixed point methods.

Keywords: additive functional equation; fixed point; Hyers–Ulam Stability; fuzzy normed space

1. Introduction and Preliminaries

Sometimes in modeling solved problems there can be a degree of uncertainty in the limitations used within the model or a few capacities can be vague. Because of such capabilities, we are interested to keep in mind the regard of functional equations within the fuzzy placing. In 1965, the knowledge of fuzzy sets developed first with the aid of Zadeh [1], which is an effective tool set for modelling indecision and elusiveness in numerous issues springing up inside the field of technology. For the past four decades, the fuzzy principle has become a very lively area of studies and plenty of developments have been made within the concept of fuzzy sets to find the fuzzy analogues of the classical set theory. Functional equations are also used to establish the distance formula in non-Euclidean geometries. Other applications include non-Euclidean geometry, which are also related to problems in mechanics, and one related also to non-Euclidean theory of relativity.

In 1940, Ulam [2] raised the subsequent query. Under what conditions does there exist an additive mapping close to an approximate expansion mapping? The case of approximate additive function capacities got explained by Hyers [3] under certain suppositions.

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called Cauchy additive functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional Equation (1) is called an additive function.

In 1978, a generalized model of the concept of Hyers for approximate linear mapping was given via Rassias [4]. Some mathematicians have been attracted by the end result of Rassias. The stability idea that was proposed and researched via Rassias is known as the Hyers–Ulam–Rassias stability.

Over recent years, the stability issues of numerous functional equations were significantly investigated through a number of authors (c.f. [5–22] and references therein). Katsaras [23] described a fuzzy norm on a vector space to build a fuzzy vector topological structure on the space. Few mathematicians have depicted fuzzy norms on a vector space from different points of view [24–26].

Especially, Bag and Samanta [27], following Cheng and Mordeson [28], proposed a fuzzy norm such that the corresponding fuzzy metric is of the Kramosil and Michalek kind [29]. They set up a decomposition theorem of a fuzzy norm into a group of crisp norms and researched a few properties of fuzzy normed spaces [30]. Moreover, the following works help us to develop this paper such as [31–42].

We utilize the notions of fuzzy normed spaces given in [27,43,44] to explore a fuzzy version of the generalized Hyers–Ulam stability for the finite variable additive functional equation

$$\sum_{a=1}^{l} \Psi\left(-v_a + \sum_{b=1;b\neq a}^{l} v_b\right) = (l-2)\sum_{a=1}^{l} \Psi\left(v_a\right) \tag{2}$$

where l is a positive integer with $l \geq 3$ by two different approaches of direct method and fixed point method.

This paper is organized as follows: In Section 2, authors obtain the general solution for (2). In Section 3, authors investigate the stability results for (2) in fuzzy normed spaces by means of direct method. In Section 4, authors investigate the stability results for (2) in fuzzy normed spaces by means of fixed point method. Finally, we examine the nonstability for (2) by a counter example.

Definition 1 ([27,43]). *Let E be a real vector space. A function $N_n : E \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on E if for all $a, b \in E$ and all $p, q \in \mathbb{R}$,*

(N_1) $N_n(a,q) = 0$ *for* $q \leq 0$;
(N_2) $a = 0$ *iff* $N_n(a,q) = 1$ *for all* $q > 0$;
(N_3) $N_n(\alpha a, q) = N_n(a, \frac{q}{|\alpha|})$ *if* $\alpha \neq 0$;
(N_4) $N_n(a+b, p+q) \geq min\{N_n(a,p), N_n(b,q)\}$;
(N_5) $N_n(a,\cdot)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{q\to+\infty} N_n(a,q) = 1$;
(N_6) *for* $a \neq 0$, $N_n(a,\cdot)$ *is continuous on* $\mathbb{R}$.

The pair (E, N_n) is called a *fuzzy normed vector space*.

We will utilize the subsequent essential result in fixed point theory.

Theorem 1 ([45]). *Let (E,d) be a generalized complete metric space and $\Lambda : E \to E$ be a strictly contractive function with the Lipschitz constant $L < 1$. Suppose that for a given element $a \in E$ there exists a positive integer k such that $d(\Lambda^{k+1}a, \Lambda^k a) < +\infty$. Then*

(i) *the sequence* $\{\Lambda^n a\}_{n=1}^{+\infty}$ *converges to a fixed point* $b \in E$ *of* Λ;
(ii) b *is the unique fixed point of* Λ *in the set* $Y = \{x \in E : d(\Lambda^k a, x) < +\infty\}$;
(iii) $d(x,b) \leq \frac{1}{1-L} d(x, \Lambda x)$ *for all* $x \in Y$.

2. General Solution

In this section, we obtain the general solution for the finite variable additive functional Equation (2).

Theorem 2. *Let E and F be real vector spaces. The mapping $\Psi : E \to F$ fulfils (2) for all $v_1, v_2, \cdots, v_l \in E$, then Ψ is additive.*

Proof. Suppose that the mapping $\Psi : E \to F$ fulfils (2) for all $v_1, v_2, \cdots, v_l \in E$. Considering $v_1 = v_2 = \cdots = v_l = 0$ in (2), we obtain $\Psi(0) = 0$. Now, replacing $(v, 0, \cdots, 0)$ in (2), we attain $\Psi(-v) = -\Psi(v)$ for all $v \in E$. Therefore, Ψ is odd. Switching $(v, v, 0, \cdots, 0)$ in (2), we reach

$$\Psi(2v) = 2\Psi(v) \tag{3}$$

for all $v \in E$. Substituting v by $2v$ in (3), we get

$$\Psi(2^2 v) = 2^2 \Psi(v) \tag{4}$$

for all $v \in E$. Interchanging v by $2v$ in (4),we reach

$$\Psi(2^3 v) = 2^3 \Psi(v) \tag{5}$$

for all $v \in E$. From (3), (4) and (5), we can conclude for a positive integer l, we have

$$\Psi(2^l v) = 2^l \Psi(v) \tag{6}$$

for all $v \in E$. Similarly, replacing v by $\frac{v}{2^l}$ in (6), we get

$$\Psi\left(\frac{v}{2^l}\right) = \frac{1}{2^l}\Psi(v) \tag{7}$$

for all $v \in E$. Substituting $(v_1, v_2, \cdots, v_l)$ by $(u, v, 0, \cdots, 0)$ in (2) and utilizing the oddness of Ψ and (6), we reach

$$\Psi(u + v) = \Psi(u) + \Psi(v) \tag{8}$$

for all $u, v \in E$. Therefore, Ψ is additive. □

Remark 1. *Let F be a linear space and $\Psi : E \to F$ be a function fulfils (2). Then the upcoming two claims hold:*

(1) $\Psi(r^k v) = r^k \Psi(v)$ for all $v \in \mathbb{R}, r \in \mathbb{Q}$, k integers.
(2) $\Psi(v) = v\Psi(1)$ for all $v \in \mathbb{R}$ if Ψ is continuous.

In upcoming sections, let us take E, (Z, N_n) and (F, N_b) as linear space, fuzzy normed space and fuzzy Banach space, respectively. We define a function Ψ from E to F by

$$D\Psi\left(v_1, v_2, \cdots, v_l\right) = \sum_{a=1}^{l} \Psi\left(-v_a + \sum_{b=1; b\neq a}^{l} v_b\right) - (l-2)\sum_{a=1}^{l} \Psi\left(v_a\right)$$

for all $v_1, v_2, \cdots, v_l \in E$.

3. Result and Discussion: Direct Method

In this section, we investigate the stability results for (2) in fuzzy normed spaces by means of direct method.

Theorem 3. *Let $\phi : E^l \to Z$ be a mapping with $\varrho > 0$ and $\varrho < 2$*

$$N_n\left(\phi\left(2v, 2v, 0, \cdots, 0\right), \alpha\right) \geq N_n\left(\varrho\phi\left(v, v, 0, \cdots, 0\right), \alpha\right) \tag{9}$$

for all $v \in E$and all $\alpha > 0$, and

$$\lim_{r\to+\infty} N_n\left(\phi\left(2^r v_1, 2^r v_2, \cdots, 2^r v_l\right), 2^r \alpha\right) = 1 \tag{10}$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$. Suppose an odd mapping $\Psi : E \to F$ fulfils

$$N_b\left(D\Psi\left(v_1, v_2, \cdots, v_l\right), \alpha\right) \geq N_n\left(\phi\left(v_1, v_2, \cdots, v_l\right), \alpha\right) \tag{11}$$

for all $v_1, v_2, \cdots, v_l \in X$ and all $\alpha > 0$. Then the limit

$$A_1(v) = N_b - \lim_{r \to +\infty} \frac{\Psi\left(2^r v\right)}{2^r} \tag{12}$$

exists for every v in E and $A_1 : E \to F$ is the unique additive mapping such that

$$N_b\left(\Psi(v) - A_1(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), (l-2)\alpha(2-\varrho)\right) \tag{13}$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Switching $(v_1, v_2, \cdots, v_l)$ by $(v, v, 0, \cdots, 0)$ in (11), we obtain

$$N_b\left((l-2)\Psi(2v) - 2(l-2)\Psi(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{14}$$

for all $v \in E$ and all $\alpha > 0$. From (14), we attain

$$N_b\left(\frac{\Psi(2v)}{2} - \Psi(v), \frac{\alpha}{2(l-2)}\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{15}$$

for all $v \in E$ and all $\alpha > 0$. Interchanging v by $2^l v$ in (15), we reach

$$N_b\left(\frac{\Psi(2^{l+1}v)}{2} - \Psi(2^l v), \frac{\alpha}{2(l-2)}\right) \geq N_n\left(\phi(2^l v, 2^l v, 0, \cdots, 0), \alpha\right) \tag{16}$$

for all $v \in E$ and all $\alpha > 0$. Utilizing (9) and N_3 in (17), we get

$$N_b\left(\frac{\Psi(2^{l+1}v)}{2^{(l+1)}} - \frac{\Psi(2^l v)}{2^l}, \frac{\alpha}{2^{(l+1)}(l-2)}\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \frac{\alpha}{\varrho^l}\right) \tag{17}$$

for all $v \in E$, $\alpha > 0$. Setting α by $\varrho^l \alpha$ in (17), we attain

$$N_b\left(\frac{\Psi(2^{l+1}v)}{2^{(l+1)}} - \frac{\Psi(2^l v)}{2^l}, \frac{\varrho^l \alpha}{2^{(l+1)}(l-2)}\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{18}$$

for all $v \in E$ and all $\alpha > 0$. It follows from that

$$\frac{\Psi(2^l v)}{2^l} - \Psi(v) = \sum_{j=0}^{l-1} \frac{\Psi(2^{j+1}v)}{2^{(j+1)}} - \frac{\Psi(2^j v)}{2^j} \tag{19}$$

and from (18) and (19), we get

$$\begin{aligned} N_b\left(\tfrac{\Psi(2^l v)}{2^l} - \Psi(v), \textstyle\sum_{j=0}^{l-1} \tfrac{\varrho^j \alpha}{2^{(j+1)}(l-2)}\right) &\geq \min \cup_{j=1}^{l-1}\left\{N_b\left(\tfrac{\Psi(2^{(j+1)}v)}{2^{(j+1)}} - \tfrac{\Psi(2^j v)}{2^j}, \tfrac{\varrho^j \alpha}{(l-2)2^{(j+1)}}\right)\right\} \\ &\geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \end{aligned} \tag{20}$$

for all $v \in E$ and all $\alpha > 0$. Switching v by $2^n v$ in (20) and utilizing (9), N_3, we get

$$N_b\left(\frac{\Psi(2^{(l+n)}v)}{2^{(l+n)}} - \frac{\Psi(2^n v)}{2^n}, \sum_{j=n}^{l+n-1} \frac{\varrho^j \alpha}{2^{(j+1)}(l-2)}\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{21}$$

for all $v \in E$, $\alpha > 0$ and all $l, n \geq 0$. Replacing α by $\frac{\alpha}{\sum_{j=n}^{l+n-1} \frac{\varrho^j}{2^{(j+1)}(l-2)}}$ in (21), we get

$$N_b\left(\frac{\Psi(2^{(l+n)}v)}{2^{(l+n)}} - \frac{\Psi(2^n v)}{2^n}, \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \frac{\alpha}{\sum_{j=n}^{l+n-1} \frac{\varrho^j}{2^{(j+1)}(l-2)}}\right) \tag{22}$$

for every $v \in E$, $\alpha > 0$ and each $l, n \geq 0$. As, $0 < \varrho < 2$ with $\sum_{j=0}^{+\infty} \left(\frac{\varrho}{2}\right)^j < +\infty$, the Cauchy criterion for convergence and N_5 towards that $\left\{\frac{\Psi(2^l v)}{2^l}\right\}$ is a Cauchy sequence in (F, N_b) is a fuzzy Banach space, $\left\{\frac{\Psi(2^l v)}{2^l}\right\}$ converges to a point $A_1(v) \in F$. Define the mapping $A_1 : E \to F$ by

$$A_1(v) = N_b - \lim_{r \to +\infty} \frac{\Psi\left(2^r v\right)}{2^r}$$

for all $v \in E$. Since Ψ and A_1 are odd. Taking $n = 0$ and passing the limit $l \to +\infty$ in (22) with utilizing N_6, we obtain

$$N_b\left(\Psi(v) - A_1(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), (l-2)\alpha(2-\varrho)\right)$$

for all $v \in E, \alpha > 0$. Next, to show that A_1 is additive. Switching $(v_1, v_2, \cdots, v_l)$ by $(2^r v_1, 2^r v_2, \cdots, 2^r v_l)$ in (11), we have

$$N_b\left(\frac{1}{2^r} D\Psi\left(2^r v_1, 2^r v_2, \cdots, 2^r v_l\right), \alpha\right) \geq N_n\left(\phi\left(2^r v_1, 2^r v_2, \cdots, 2^r v_l\right), 2^r \alpha\right)$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$. Since

$$\lim_{r \to +\infty} N_n\left(\phi(2^r v_1, 2^r v_2, \cdots, 2^r v_l), 2^r \alpha\right) = 1.$$

Hence A_1 fulfils (2). Therefore, $A_1 : E \to F$ is an additive mapping. Now, to claim that the uniqueness of A_1, consider A_2 is another additive function which maps from E to F and fulfilling (2) and (13). Hence,

$$\begin{aligned}
N_b\left(A_1(v) - A_2(v), \alpha\right) &= N_b\left(\frac{A_1(2^r v)}{2^r} - \frac{A_2(2^r v)}{2^r}, \alpha\right) \\
&\geq \min\left\{N_b\left(\frac{A_1(2^r v)}{2^r} - \frac{\Psi(2^r v)}{2^r}, \frac{\alpha}{2}\right), N_b\left(\frac{\Psi(2^r v)}{2^r} - \frac{A_2(2^r v)}{2^r}, \frac{\alpha}{2}\right)\right\} \\
&\geq N_n\left(\phi(2^r v, 2^r v, 0, \cdots, 0), \frac{2^r (l-2)\alpha(2-\varrho)}{2}\right) \\
&\geq N_n\left(\phi(v, v, 0, \cdots, 0), \frac{2^r (l-2)\alpha(2-\varrho)}{2\varrho^r}\right)
\end{aligned}$$

for all $v \in E, \alpha > 0$. We know that, $\lim_{r \to +\infty} \frac{2^r (l-2)\alpha(2-\varrho)}{2\varrho^r} = +\infty$, we get

$$\lim_{r \to +\infty} N_n\left(\phi(v, v, 0, \cdots, 0), \frac{2^r (l-2)\alpha(2-\varrho)}{2\varrho^r}\right) = 1.$$

Thus, $N_b\left(A_1(v) - A_2(v), \alpha\right) = 1$ for all $v \in E$ and all $\alpha > 0$. Hence, $A_1(v) = A_2(v)$. Therefore, $A_1(v)$ is unique. This completes the proof. □

Theorem 4. *Let $\phi : E^l \to Z$ be a mapping with $\varrho > 0$ and $\varrho > 2$*

$$N_n\left(\phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), \alpha\right) \geq N_n\left(\frac{1}{\varrho}\phi\left(v, v, 0, \cdots, 0\right), \alpha\right) \tag{23}$$

for all $v \in E$and all $\alpha > 0$, and

$$\lim_{r\to+\infty} N_n\left(\phi\left(\frac{v_1}{2^r}, \frac{v_2}{2^r}, \cdots, \frac{v_l}{2^r}\right), \frac{\alpha}{2^r}\right) = 1 \tag{24}$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$. Suppose an odd mapping $\Psi : E \to F$ fulfils

$$N_b\left(D\Psi\left(v_1, v_2, \cdots, v_l\right), \alpha\right) \geq N_n\left(\phi\left(v_1, v_2, \cdots, v_l\right), \alpha\right) \tag{25}$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$. Then the limit

$$A_1(v) = N_b - \lim_{r\to+\infty} 2^r \Psi\left(\frac{v}{2^r}\right) \tag{26}$$

exists for all $v \in E$ and the mapping $A_1 : E \to F$ is the unique additive mapping such that

$$N_b\left(\Psi(v) - A_1(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), (l-2)\alpha(\varrho-2)\right) \tag{27}$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Interchanging $(v_1, v_2, \cdots, v_l)$ by $(v, v, 0, \cdots, 0)$ in (25), we obtain

$$N_b\left((l-2)\Psi(2v) - 2(l-2)\Psi(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{28}$$

for all $v \in E$ and all $\alpha > 0$. From (28), we reach

$$N_b\left(\Psi(2v) - 2\Psi(v), \frac{\alpha}{(l-2)}\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{29}$$

for all $v \in E$ and all $\alpha > 0$. Switching v by $\frac{v}{2}$ in (29), we get

$$N_b\left(\Psi(v) - 2\Psi\left(\frac{v}{2}\right), \frac{\alpha}{(l-2)}\right) \geq N_n\left(\phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), \alpha\right) \tag{30}$$

for all $v \in E$ and all $\alpha > 0$. Again, replacing v by $\frac{v}{2^r}$ in (30), we obtain

$$N_b\left(\Psi\left(\frac{v}{2^r}\right) - 2\Psi\left(\frac{v}{2^{(r+1)}}\right), \frac{\alpha}{(l-2)}\right) \geq N_n\left(\phi\left(\frac{v}{2^{r+1}}, \frac{v}{2^{r+1}}, 0, \cdots, 0\right), \alpha\right) \tag{31}$$

for all $v \in E$ and all $\alpha > 0$. From (31) and (23) that

$$N_b\left(2^r\Psi\left(\frac{v}{2^r}\right) - 2^{(r+1)}\Psi\left(\frac{v}{2^{(r+1)}}\right), \frac{2^r\alpha}{(l-2)}\right) \geq N_n\left(\phi\left(v, v, 0, \cdots, 0\right), \varrho^{r+1}\alpha\right) \tag{32}$$

for all $v \in E$ and all $\alpha > 0$. The remaining part of the proof is similar to the proof of Theorem 3. □

Corollary 1. *Suppose an odd function $\Psi : E \to F$ fulfils the inequality*

$$N_b(D\Psi(v_1, v_2, \cdots, v_l), \alpha) \geq N_n\left(\vartheta, \alpha\right),$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$ where ϑ is a real constant with $\vartheta > 0$, then there exists a unique additive mapping $A_1 : E \to F$ such that

$$N_b(\Psi(v) - A_1(v), \alpha) \geq N_n(\vartheta, |2-1|(l-2)\alpha)$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Let us define $\phi(v_1, v_2, \cdots, v_l) = \vartheta$, then the proof is raised from Theorems 3 and 4 by taking $\varrho = 2^0$. □

Corollary 2. *Suppose an odd function $\Psi : E \to F$ fulfils the inequality*

$$N_b(D\Psi(v_1, v_2, \cdots, v_l), \alpha) \geq N_n\Big(\epsilon \sum_{j=1}^{l} \|v_j\|^{\beta}, \alpha\Big),$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$, where ϵ and β are real constants with $\beta \in (0,1) \cup (1, +\infty)$, then there exists a unique additive mapping $A_1 : E \to F$ such that

$$N_b(\Psi(v) - A_1(v), \alpha) \geq N_n(2\epsilon\|v\|^{\beta}, |2-2^{\beta}|(l-2)\alpha)$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Let us define $\phi(v_1, v_2, \cdots, v_l) = \epsilon \sum_{j=1}^{l} \|v_j\|^{\beta}$, then the proof is raised from Theorems 3 and 4 by taking $\varrho = 2^{\beta}$. □

Corollary 3. *Suppose an odd function $\Psi : E \to F$ fulfils the inequality*

$$N_b(D\Psi(v_1, v_2, \cdots, v_l), \alpha) \geq N_n\Big(\epsilon \sum_{j=1}^{l} \|v_j\|^{l\beta} + \vartheta \prod_{j=1}^{l} \|v_j\|^{\gamma}, \alpha\Big),$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$ where $\epsilon, \vartheta, \beta$ and γ are real constants with $l\beta, l\gamma \in (0,1) \cup (1, +\infty)$, then there exists a unique additive mapping $A_1 : E \to F$ such that

$$N_b(\Psi(v) - A_1(v), \alpha) \geq N_n(2\epsilon\|v\|^{l\beta}, |2-2^{l\beta}|(l-2)\alpha)$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Let us define $\phi(v_1, v_2, \cdots, v_l) = \epsilon \sum_{j=1}^{l} \|v_j\|^{l\beta} + \vartheta \prod_{j=1}^{l} \|v_j\|^{\gamma}$, then the proof is raised from Theorems 3 and 4 by taking $\varrho = 2^{l\beta}$. □

Corollary 4. *Suppose an odd function $\Psi : E \to F$ fulfils the inequality*

$$N_b(D\Psi(v_1, v_2, \cdots, v_l), \alpha) \geq N_n\Big(\vartheta \prod_{j=1}^{l} \|v_j\|^{\gamma}, \alpha\Big),$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$, where ϑ and γ are real constants with $0 < l\gamma \neq 1$, then Ψ is additive.

Proof. Let us define $\phi(v_1, v_2, \cdots, v_l) = \vartheta \prod_{j=1}^{l} \|v_j\|^{\gamma}$, then the proof is raised from Theorems 3 and 4. □

4. Result and Discussion: Fixed Point Method

In this section, we investigate the stability results for (2) in fuzzy normed spaces by means of fixed point method.

First, we define ς_a as a constant such that

$$\varsigma_a = \begin{cases} 2 & if \quad a = 0 \\ \frac{1}{2} & if \quad a = 1 \end{cases}$$

and we consider $\Lambda = \{\varphi_1 : E \to F : \ \varphi_1(0) = 0\}$.

Theorem 5. *Let $\Psi : E \to F$ be an odd mapping for which there exists a function $\phi : E^l \to Z$ with condition*

$$\lim_{r\to+\infty} N_n\left(\phi(\varsigma_a^r v_1, \varsigma_a^r v_2, \cdots, \varsigma_a^r v_l), \varsigma_a^r \alpha\right) = 1 \tag{33}$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$, and fulfilling

$$N_b(D\Psi(v_1, v_2, \cdots, v_l), \alpha) \geq N_n(\phi(v_1, v_2, \cdots, v_l), \alpha) \tag{34}$$

for all $v_1, v_2, \cdots, v_l \in E$ and all $\alpha > 0$. Let $\rho(v) = \frac{1}{(l-2)}\phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$ for all $v \in E$. If there exist $L = L_a \in (0,1)$ such that

$$N_n\left(\frac{1}{\varsigma_a}\rho(\varsigma_a v), \alpha\right) \geq N_n\left(L\rho(v), \alpha\right) \tag{35}$$

for all $v \in E$ and all $\alpha > 0$, then there exist a unique additive function $A_1 : E \to F$ fulfilling

$$N_b(\Psi(v) - A_1(v), \alpha) \geq N_n\left(\frac{L^{1-a}}{1-L}\rho(v), \alpha\right) \tag{36}$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Let μ be the generalized metric on Λ:

$$\mu(\varphi_1, \varphi_2) = \inf\left\{w \in (0, +\infty) : \ N_b(f\varphi_1(v) - \varphi_2(v), \alpha) \geq N_n(w\rho(v), \alpha), \right\},$$

for all $v \in E$ and all $\alpha > 0$ and we take, as usual, $\inf \emptyset = +\infty$. A similar argument provided in ([46], Lemma 2.1) shows that (Λ, μ) is a complete generalized metric space. Define $\Phi_a : \Lambda \longrightarrow \Lambda$ by $\Phi_a\varphi_1(v) = \frac{1}{\varsigma_a}\varphi_1(\varsigma_a v)$ for all $v \in E$. Let φ_1, φ_2 in Λ be given such that $\mu(\varphi_1, \varphi_2) \leq \epsilon$. Then

$$N_b\left(\varphi_1(v) - \varphi_2(v), \alpha\right) \geq N_n\left(\epsilon\rho(v), \alpha\right)$$

for all $v \in E$ and all $\alpha > 0$, whence

$$N_b(\Phi_a\varphi_1(v) - \Phi_a\varphi_2(v), \alpha) \geq N_n\left(\frac{\epsilon}{\varsigma_a}\rho(\varsigma_a v), \alpha\right)$$

for all $v \in E$ and all $\alpha > 0$. It follows from (4) that

$$N_b(\Phi_a\varphi_1(v) - \Phi_a\varphi_2(v), \alpha) \geq N_n(\epsilon L\rho(v), \alpha)$$

for all $v \in E$ and all $\alpha > 0$. Hence, we have $\mu(\Phi_a\varphi_1, \Phi_a\varphi_2) \leq \epsilon L$. This shows $\mu(\Phi_a\varphi_1, \Phi_a\varphi_2) \leq L\mu(\varphi_1, \varphi_2)$, i.e., Φ_a is strictly contractive mapping on Λ with the Lipschitz constant L. Interchanging $(v_1, v_2, \cdots, v_l)$ by $(v, v, 0, \cdots, 0)$ in (34) , we get

$$N_b\left((l-2)\Psi(2v) - 2(l-2)\Psi(v), \alpha\right) \geq N_n\left(\phi(v, v, 0, \cdots, 0), \alpha\right) \tag{37}$$

for all $v \in E$ and all $\alpha > 0$. Utilizing (4) and (N_3) when $a = 0$, it follows from (37) that

$$\begin{aligned} N_b\left(\frac{\Psi(2v)}{2} - \Psi(v), \alpha\right) &\geq N_n\left(\frac{\phi(v, v, 0, \cdots, 0)}{2(l-2)}, \alpha\right) \\ &\geq N_n(L\rho(v), \alpha) \end{aligned}$$

for all $v \in E$ and all $\alpha > 0$. Therefore,

$$\mu(\Phi_0\Psi, \Psi) \leq L = L^{1-a}. \tag{38}$$

Switching v through $\frac{v}{2}$ in (37) (i.e., when $a = 1$) and using N_3, we have

$$\begin{aligned} N_b\left(\Psi(v) - 2\Psi\left(\frac{v}{2}\right), \alpha\right) &\geq N_n\left(\frac{1}{(l-2)}\phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), \alpha\right) \\ &= N_n(L\rho(v), \alpha) \end{aligned}$$

for all $v \in E$ and all $\alpha > 0$. Therefore,

$$\mu(\Phi_1\Psi, \Psi) \leq 1 = L^{1-a}. \tag{39}$$

Then from (38) and (39), we conclude $\mu(\Phi_a\Psi, \Psi) \leq L^{1-a} < +\infty$. Now from Theorem 1, implies that there exists a fixed point A_1 of Φ_a in Λ such that

(i) $\Phi_a A_1 = A_1$ and $\lim_{r\to+\infty} \mu(\Phi_a^r\Psi, A_1) = 0$;
(ii) A_1 is the unique fixed point of Φ in the set $E = \{\varphi_1 \in \Lambda : \ d(\Psi, \varphi_1) < +\infty\}$;
(iii) $\mu(\Psi, A_1) \leq \frac{1}{1-L}\mu(\Psi, \Phi_a\Psi)$.

Letting $\mu(\Phi_a^r\Psi, A_1) = \epsilon_n$, we get $N_b(\Phi_a^r\Psi(v) - A_1(v), \alpha) \geq N_n(\epsilon_r\rho(v), \alpha)$ for all $v \in E$ and all $\alpha > 0$. Since $\lim_{r\to+\infty} \epsilon_r = 0$, we infer

$$A_1(v) = N_b - \lim_{r\to+\infty} \frac{\Psi(\varsigma_a^r v)}{\varsigma_a^r}$$

for all $v \in E$. Switching $(v_1, v_2, \cdots, v_l)$ by $(\varsigma_a^r v_1, \varsigma_a^r v_2, \cdots, \varsigma_a^r v_l)$ in (34), we obtain

$$N_b\left(\frac{1}{\varsigma_a^r} D\Psi(\varsigma_a^r v_1, \varsigma_a^r v_2, \cdots, \varsigma_a^r v_l), \alpha\right) \geq N_n(\phi(\varsigma_a^r v_1, \varsigma_a^r v_2, \cdots, \varsigma_a^r v_l), \varsigma_a^r \alpha),$$

for all $\alpha > 0$ and all $v_1, v_2, \cdots, v_l \in E$. Utilizing the similar argument as in the proof of Theorem 3, we can prove the function $A_1 : E \to F$ is additive. Since $\mu(\Phi_a\Psi, \Psi) \leq L^{1-a}$, it follows from (iii) that $\mu(\Psi, A_1) \leq \frac{L^{1-a}}{1-L}$ which means (36). Next, we show the uniqueness of A_1; consider another additive function A_2 which fulfils (36). Since $A_1(2^r v) = 2^r A_1(v)$ and $A_2(2^r v) = 2^r A_2(v)$ for all $v \in E$ and all $r \in \mathbb{N}$, we have

$$\begin{aligned} N_b(A_1(v) - A_2(v), \alpha) &= N_b\left(\frac{A_1(2^r v)}{2^r} - \frac{A_2(2^r v)}{2^r}, \alpha\right) \\ &\geq \min\left\{N_b\left(\frac{A_1(2^r v)}{2^r} - \frac{\Psi(2^r v)}{2^r}, \frac{\alpha}{2}\right), N_b\left(\frac{\Psi(2^r v)}{2^r} - \frac{A_2(2^r v)}{2^r}, \frac{\alpha}{2}\right)\right\} \\ &\geq N_n\left(\frac{L^{1-a}}{1-L}\rho(2^r v), \frac{2^r\alpha}{2}\right). \end{aligned}$$

By (33), we have

$$\lim_{r\to+\infty} N_n\left(\frac{L^{1-a}}{1-L}\rho(2^r v), \frac{2^r\alpha}{2}\right) = 1.$$

Consequently, $N_b(A_1(v) - A_2(v), \alpha) = 1$ for all $v \in E$ and all $\alpha > 0$. So $A_1(v) = A_2(v)$ for all $v \in E$, which ends the proof. $\square$

Corollary 5. *Suppose that an odd function $\Psi : E \to F$ fulfils the inequality*

$$N_b\left(D\Psi\left(v_1, v_2, \cdots, v_l\right), \alpha\right) \geq \begin{cases} N_n\left(\vartheta, \alpha\right), \\ N_n\left(\vartheta \sum_{j=1}^{l} \left\|v_j\right\|^{\beta}, \alpha\right), \\ N_n\left(\vartheta\left(\prod_{j=1}^{l} \|v_i\|^{\beta} + \sum_{j=1}^{l} \|v_i\|^{l\beta}\right), \alpha\right), \end{cases} \tag{40}$$

for all $v_1, v_2, \cdots, v_l \in E$ and $\alpha > 0$, where ϑ and β are constants along with $\vartheta > 0$, then there exists a mapping $A_1 : E \to F$ is the unique additive such that

$$N_b\left(\Psi(v) - A_1(v), \alpha\right) \geq \begin{cases} N_n\left(\vartheta, |2-1|(l-2)\alpha\right), \\ N_n\left(2\vartheta\|v\|^{\beta}, |2-2^{\beta}|(l-2)\alpha\right); & \beta \neq 1, \\ N_n\left(\vartheta\|v\|^{l\beta}, |2-2^{l\beta}|(l-2)\alpha\right); & \beta \neq \frac{1}{l}, \end{cases}$$

for all $v \in E$ and all $\alpha > 0$.

Proof. Considering

$$\phi\left(v_1, v_2, \cdots, v_l\right) \geq \begin{cases} \vartheta, \\ \vartheta \sum_{j=1}^{l} \left\|v_j\right\|^{\beta}, \\ \vartheta\left(\prod_{j=1}^{l} \left\|v_j\right\|^{\beta} + \sum_{j=1}^{l} \left\|v_j\right\|^{l\beta}\right), \end{cases}$$

for all $v_1, v_2, \cdots, v_l \in E$. Then

$$\begin{aligned} N_n\left(\phi\left(\varsigma_a^r v_1, \varsigma_a^r v_2, \cdots, \varsigma_a^r v_l\right), \varsigma_a^r \alpha\right) &= \begin{cases} N_n\left(\vartheta, \varsigma_a^r \alpha\right), \\ N_n\left(\vartheta \sum_{j=1}^{l} \left\|v_j\right\|^{\beta}, \varsigma_a^{(1-\beta)r} \alpha\right), \\ N_n\left(\vartheta\left(\prod_{j=1}^{l} \left\|v_j\right\|^{\beta} + \sum_{j=1}^{l} \left\|v_j\right\|^{l\beta}\right), \varsigma_a^{(1-l\beta)r} \alpha\right), \end{cases} \\ &= \begin{cases} \to 1 \quad as \quad r \to +\infty, \\ \to 1 \quad as \quad r \to +\infty, \\ \to 1 \quad as \quad r \to +\infty. \end{cases} \end{aligned}$$

Thus (33) holds but we have

$$\rho(v) = \frac{1}{(l-2)} \phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$$

has the property

$$N_n\left(\frac{1}{\varsigma_a} \rho(\varsigma_a v), \alpha\right) \geq N_n\left(L\rho(v), \alpha\right), \quad v \in E, \ \alpha > 0.$$

for all $v \in E$ and all $\alpha > 0$. Hence

$$\begin{aligned} N_n\left(\rho(v), \alpha\right) &= N_n\left(\phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right), (l-2)\alpha\right) \\ &= \begin{cases} N_n\left(\vartheta, (l-2)\alpha\right), \\ N_n\left(\frac{2\vartheta}{2^{\beta}}\|v\|^{\beta}, (l-2)\alpha\right), \\ N_n\left(\frac{2\vartheta}{2^{l\beta}}\|v\|^{l\beta}, (l-2)\alpha\right). \end{cases} \end{aligned}$$

So,

$$N_n\left(\frac{1}{\varsigma_a}\rho(\varsigma_a v),\alpha\right)=\begin{cases}N_n\left(\varsigma_a^{-1}\rho(v),\alpha\right);\\ N_n\left(\varsigma_a^{\beta-1}\rho(v),\alpha\right);\\ N_n\left(\varsigma_a^{l\beta-1}\rho(v),\alpha\right);\end{cases}$$

From the following cases for the conditions of ς_a.
Case (i) $L=\frac{1}{2}$ for $\beta=0$ if $a=0$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{2^{-1}}{1-2^{-1}}\rho(v),\alpha\right)=N_n\left(\vartheta,(l-2)\alpha\right).$$

Case (ii) $L=\left(\frac{1}{2}\right)^{-1}$ for $\beta=0$ if $a=1$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{1}{1-2}\rho(v),\alpha\right)=N_n\left(\vartheta,(2-l)\alpha\right).$$

Case (iii) $L=2^{\beta-1}$ for $\beta<1$ if $a=0$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{2^{\beta-1}}{1-2^{\beta-1}}\rho(v),\alpha\right)=N_n\left(2\vartheta\|v\|^{\beta},(2-2^{\beta})(l-2)\alpha\right).$$

Case (iv) $L=2^{1-\beta}$ for $\beta>1$ if $a=1$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{1}{1-2^{1-\beta}}\rho(v),\alpha\right)=N_n\left(2\vartheta\|v\|^{\beta},(2^{\beta}-2)(l-2)\alpha\right).$$

Case (v) $L=2^{l\beta-1}$ for $\beta<\frac{1}{l}$ if $a=0$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{2^{l\beta-1}}{1-2^{l\beta-1}}\rho(v),\alpha\right)=N_n\left(2\vartheta\|v\|^{l\beta},(2-2^{l\beta})(l-2)\alpha\right).$$

Case (vi) $L=2^{1-l\beta}$ for $\beta>\frac{1}{l}$ if $a=1$

$$N_b\left(\Psi(v)-A_1(v),\alpha\right)\geq N_n\left(\frac{1}{1-2^{1-l\beta}}\rho(v),\alpha\right)=N_n\left(2\vartheta\|v\|^{l\beta},(2^{l\beta}-2)(l-2)\alpha\right).$$

Hence the proof is completed. □

5. Counter Example

Next, we show the upcoming counter example changed by the well-known counter example of Gajda [47] to (2):

Example 1. *Let $\tau: E\to F$ defined by:*

$$\tau(v)=\sum_{l=0}^{+\infty}\frac{\sigma\left(2^l v\right)}{2^l}$$

where

$$\sigma(v)=\begin{cases}\zeta v, & -1<v<1\\ \zeta, & otherwise,\end{cases}\tag{41}$$

where ζ is a constant, then $\tau : E \to F$ fulfils the inequality

$$|\tau(v_1, v_2, \cdots, v_l)| \leq (l-1)\, 8\zeta \left(\sum_{j=1}^{l} |v_j| \right), \tag{42}$$

for all $v_1, v_2, \cdots, v_l \in E$, but there does not arise an additive function $A_1 : E \to F$ along with a constant δ such that

$$|\Psi(v) - A_1(v)| \leq \delta |v| \tag{43}$$

for all $v \in E$.

Proof. It is easy to notice that Ψ is bounded by 2ζ on E. If $\sum_{j=1}^{l} |v_j| \geq \frac{1}{2}$ or 0, then the left side of (42) is less than $(l-1)\, 2\zeta$, and thus (42) is true. At once, assume that $0 < \sum_{j=1}^{l} |v_j| < \frac{1}{2}$. Then there exists an integer m such that

$$\frac{1}{2^{(m+2)}} \leq \sum_{j=1}^{l} |v_j| < \frac{1}{2^{(m+1)}}. \tag{44}$$

So that $2^m|v_1| < \frac{1}{2}, 2^m|v_2| < \frac{1}{2}, \cdots, 2^m|v_l| < \frac{1}{2}$ and $2^l v_1, 2^l v_2, \cdots, 2^l v_l \in (-1,1)$ for all $l = 0, 1, 2, \cdots, m-1$. So, for $l = 0, 1, \cdots, m-1$

$$\sum_{a=1}^{l} \sigma \left(2^l \left(-v_a + \sum_{b=1; b \neq a}^{l} v_b \right) \right) - (l-2) \sum_{a=1}^{l} \sigma \left(2^l \left(v_a \right) \right) = 0.$$

By the definition of Ψ, we obtain

$$\begin{aligned}
|\tau(v_1, v_2, \cdots, v_l)| &\leq \sum_{j=m}^{+\infty} \frac{1}{2^j} |\sigma(2^j v_1, 2^j v_2, \cdots, 2^j v_l)| \\
&\leq \sum_{j=m}^{+\infty} \frac{1}{2^j} \left| \sum_{a=1}^{l} \sigma \left(2^j \left(-v_a + \sum_{b=1; b \neq a}^{l} v_b \right) \right) - (l-2) \sum_{a=1}^{l} \sigma \left(2^j \left(v_a \right) \right) \right| \\
&\leq \sum_{j=m}^{+\infty} \frac{1}{2^j} \left| \sum_{a=1}^{l} \sigma \left(2^j \left(-v_a + \sum_{b=1; b \neq a}^{l} v_b \right) \right) \right| + (l-2) \left| \sum_{a=1}^{l} \sigma \left(2^j \left(v_a \right) \right) \right| \\
&\leq \sum_{j=m}^{+\infty} \frac{1}{2^j} (l-1)\, \zeta \\
&\leq (l-1)\, 2^{(1-m)} \zeta.
\end{aligned}$$

It follows from (44) that

$$|\tau(v_1, v_2, \cdots, v_l)| \leq (l-1)\, 8\zeta \left(\sum_{j=1}^{l} |v_j| \right), \tag{45}$$

for all $v_1, v_2, \cdots, v_l \in E$. Thus Ψ satisfies (42) for all $v_1, v_2, \cdots, v_l \in E$. We propose that there arises an additive mapping $A_1 : E \to F$ along with a constant $\delta > 0$ fulfilling (43). As Ψ is bounded and continuous for every v in E, A_1 is bounded on any open interval containing the origin and continuous at the origin. By Remark 1, A_1 must have the form $A_1(v) = av$ for all $v \in E$. Thus we have

$$|\Psi(v)| \leq (\delta + |a|)\, |v|$$

for all $v \in E$. However, we can select a non-negative integer m and $m\zeta > \delta + |a|$. If $v \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^l v \in (0,1)$ for all $l = 0, 1, \cdots, m-1$ and for this v, we obtain

$$\Psi(v) = \sum_{l=0}^{+\infty} \frac{\sigma(2^l v)}{2^l} \geq \sum_{l=0}^{m-1} \frac{\zeta(2^l v)}{2^l} = m\zeta v > (\delta + |a|)v,$$

which is contradictory. □

6. Conclusions

We have introduced the finite variable additive functional Equation (2) and have obtained the general solution of the finite variable additive functional Equation (2) in fuzzy normed spaces by means of direct method and fixed point method. Furthermore, we discussed the counter example for the non-stability to the finite variable additive functional Equation (2).

Author Contributions: Conceptualization, K.T and K.L.; Formal analysis, A.M.A., K.T., A.E. and K.L.; Investigation, E.N.A.; Methodology, G.M. and A.M.A; Writing-original draft, G.M.; Writing – review and editing, K.T. and K.L.; All authors contributed equally to this work. And all the authors have read and approved the final version manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for funding this work through Research Group no. RGP-0207-1440.

Conflicts of Interest: The authors declare that they have no conflict of interest.

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