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A p-Ideal in BCI-Algebras Based on Multipolar Intuitionistic Fuzzy Sets

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Abstract: In 2020, Kang, Song and Jun introduced the notion of multipolar intuitionistic fuzzy set with finite degree, which is a generalization of intuitionistic fuzzy set, and they applied it to BCK/BCI-algebras. In this paper, we used this notion to study p-ideals of BCI-algebras. The notion of k-polar intuitionistic fuzzy p-ideals in BCI-algebras is introduced, and several properties were investigated. An example to illustrate the k-polar intuitionistic fuzzy p-ideal is given. The relationship between k-polar intuitionistic fuzzy ideal and k-polar intuitionistic fuzzy p-ideal is displayed. A k-polar intuitionistic fuzzy p-ideal is found to be k-polar intuitionistic fuzzy ideal, and an example to show that the converse is not true is provided. The notions of p-ideals and k-polar $(\in, \in)$-fuzzy p-ideal in BCI-algebras are used to study the characterization of k-polar intuitionistic p-ideal. The concept of normal k-polar intuitionistic fuzzy p-ideal is introduced, and its characterization is discussed. The process of eliciting normal k-polar intuitionistic fuzzy p-ideal using k-polar intuitionistic fuzzy p-ideal is provided.

Keywords: multipolar intuitionistic fuzzy set with finite degree k; k-polar $(\in, \in)$-fuzzy ideal; k-polar intuitionistic fuzzy ideal; k-polar intuitionistic fuzzy p-ideal

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1. Introduction

BCI-algebras were introduced by Iséki [1] as the algebraic counterpart of the BCI-logic. BCI-algebras are a generalization of BCK-algebras, and they originated from two sources: set theory and propositional calculi. See the books [2,3] for more information on BCK/BCI-algebras. Fuzzy sets were first introduced by Zadeh [4], in which the membership degree is represented by only one function—the truth function. Intuitionistic fuzzy sets, which were introduced by Atanassov (see [5,6]), are a generalization of fuzzy sets. As an extension of the bipolar fuzzy set, Chen et al. [7] introduced an m-polar fuzzy set in 2014, and then this concept was applied to certain algebraic structures as BCK/BCI algebras, graph theory and decision making problem. For BCK/BCI-algebras, see [8–10], for graph theory, see [11–14] and see [15–18] for decision making problems. Al-Masarwah and Ahmad discussed the notion of m-polar fuzzy sets with applications in BCK/BCI-algebras. They introduced the notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals and gave characterizations of m-polar fuzzy subalgebras and m-polar fuzzy (commutative) ideals. They considered relations between m-polar fuzzy subalgebras, m-polar fuzzy ideals and m-polar fuzzy commutative ideals (see [8]). Using the notion of multipolar fuzzy point, Mohseni Takallo et al. [9] studied p-ideals of BCI-algebras. In [19], Kang et al. introduced the notion of multipolar intuitionistic fuzzy set with finite degree as a generalization of intuitionistic fuzzy set, and applied it to BCK/BCI-algebras. They introduced the concepts of a k-polar intuitionistic fuzzy subalgebra and a (closed) k-polar intuitionistic fuzzy ideal in a BCK/BCI-algebra, and investigated their relations and characterizations. In a BCI-algebra, they considered the relationship between a k-polar intuitionistic fuzzy ideal and a closed k-polar intuitionistic fuzzy ideal, and discussed the characterization of a closed k-polar intuitionistic fuzzy ideal. They consulted conditions for a k-polar intuitionistic fuzzy ideal to be a closed k-polar intuitionistic fuzzy ideal in a BCI-algebra. The aim of this manuscript was to use Kang et al.'s notion so called multipolar intuitionistic fuzzy set for studying p-ideal in BCI-algebras. This is a generalization of multipolar fuzzy p-ideals of BCI-algebras which is studied in [9]. We introduce the concept of k-polar intuitionistic fuzzy p-ideals in BCI-algebras, and then we study several properties. We first give an example to illustrate the k-polar intuitionistic fuzzy p-ideal. We consider the relationship between k-polar intuitionistic fuzzy ideal and k-polar intuitionistic fuzzy p-ideal. We first prove that every k-polar intuitionistic fuzzy p-ideal is a k-polar intuitionistic fuzzy ideal, and then give an example to show that the converse is not true in general. We use the notion of p-ideals in BCI-algebras to study the characterization of k-polar intuitionistic fuzzy p-ideal. We also use the notion of k-polar $(\in, \in)$-fuzzy p-ideal in BCI-algebras to study the characterization of k-polar intuitionistic fuzzy p-ideal. We define the concept of normal k-polar intuitionistic fuzzy p-ideal, and discuss its characterization. We look at the process of eliciting normal k-polar intuitionistic fuzzy p-ideal from a given k-polar intuitionistic fuzzy p-ideal.

2. Preliminaries

If a set U has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall \omega, \upsilon, \tau \in U)\ (((\omega * \upsilon) * (\omega * \tau)) * (\tau * \upsilon) = 0)$,
(II) $(\forall \omega, \upsilon \in U)\ ((\omega * (\omega * \upsilon)) * \upsilon = 0)$,
(III) $(\forall \omega \in U)\ (\omega * \omega = 0)$,
(IV) $(\forall \omega, \upsilon \in U)\ (\omega * \upsilon = 0,\ \upsilon * \omega = 0\ \Rightarrow\ \omega = \upsilon)$,

then it is said that U is a *BCI-algebra*. If a BCI-algebra U satisfies the following identity:

(V) $(\forall \omega \in U)\ (0 * \omega = 0)$,

then U is called a *BCK-algebra.*

Any BCK/BCI-algebra U satisfies the following conditions:

$$(\forall \omega \in U)\ (\omega * 0 = \omega)\,, \tag{1}$$
$$(\forall \omega, \upsilon, \tau \in U)\ ((\omega * \upsilon) * \tau = (\omega * \tau) * \upsilon)\,. \tag{2}$$

A subset I of a BCI-algebra U is called

- a *subalgebra* of U if $\omega * \upsilon \in I$ for all $\omega, \upsilon \in I$.
- an *ideal* of U if it satisfies:

$$0 \in I, \tag{3}$$
$$(\forall \omega \in U)\ (\forall \upsilon \in I)\ (\omega * \upsilon \in I\ \Rightarrow\ \omega \in I)\,. \tag{4}$$

- a *p-ideal* of U (see [20]) if it satisfies Equation (3) and

$$(\forall \omega, \upsilon, \tau \in U)\ ((\omega * \tau) * (\upsilon * \tau) \in I,\ \upsilon \in I\ \Rightarrow\ \omega \in I)\,. \tag{5}$$

Let $\{b_i \mid i \in \Gamma\}$ be a family of real numbers where Γ is any index set and we define

$$\bigvee\{b_i \mid i \in \Gamma\} := \begin{cases} \max\{b_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ \sup\{b_i \mid i \in \Gamma\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{b_i \mid i \in \Gamma\} := \begin{cases} \min\{b_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ \inf\{b_i \mid i \in \Gamma\} & \text{otherwise.} \end{cases}$$

If $\Gamma = \{1,2\}$, we will also use $b_1 \vee b_2$ and $b_1 \wedge b_2$ instead of $\bigvee\{b_i \mid i \in \Gamma\}$ and $\bigwedge\{b_i \mid i \in \Gamma\}$, respectively.

Let k be a natural number and $[0,1]^k$ denote the k-Cartesian product of $[0,1]$, that is,

$$[0,1]^k = [0,1] \times [0,1] \times \cdots \times [0,1]$$

in which $[0,1]$ is repeated k times. The order "$\leq$" in $[0,1]^k$ is given by the pointwise order.

By a *k-polar fuzzy set* on a set U (see [7]), we mean a function $\widehat{\xi} : U \to [0,1]^k$ where k is a natural number. The membership value of every element $z \in U$ is denoted by

$$\widehat{\xi}(z) = \left((\mathrm{proj}_1 \circ \widehat{\xi})(z), (\mathrm{proj}_2 \circ \widehat{\xi})(z), \cdots, (\mathrm{proj}_k \circ \widehat{\xi})(z)\right),$$

where $\mathrm{proj}_i : [0,1]^k \to [0,1]$ is the i-th projection for all $i = 1,2,\cdots,k$ and $\circ$ is the composition of functions.

A k-polar fuzzy set $\widehat{\xi}$ on a BCK/BCI-algebra U is called a *k-polar fuzzy ideal* of U (see [8]) if the following conditions are valid.

$$(\forall z \in U)\left(\widehat{\xi}(0) \geq \widehat{\xi}(z)\right), \tag{6}$$

$$(\forall z, x \in U)\left(\widehat{\xi}(z) \geq \widehat{\xi}(z * x) \wedge \widehat{\xi}(x)\right). \tag{7}$$

By a *k-polar fuzzy point* on a set U, we mean a k-polar fuzzy set $\widehat{\xi}$ on U of the form

$$\widehat{\xi}(x) = \begin{cases} \hat{r} = (r_1, r_2, \cdots, r_k) \in (0,1]^k & \text{if } x = z, \\ \hat{0} = (0,0,\cdots,0) & \text{if } x \neq z, \end{cases} \tag{8}$$

and it is denoted by $z_{\hat{r}}$ where z is a given element of U. We say that z is the *support* of $z_{\hat{r}}$ and $\hat{r}$ is the *value* of $z_{\hat{r}}$.

We say that a k-polar fuzzy point $z_{\hat{r}}$ is contained in a k-polar fuzzy set $\widehat{\xi}$, denoted by $z_{\hat{r}} \in \widehat{\xi}$, if $\widehat{\xi}(z) \geq \hat{r}$, that is, $(\mathrm{proj}_i \circ \widehat{\xi})(z) \geq r_i$ for all $i = 1,2,\cdots,k$.

A k-polar fuzzy set $\widehat{\xi}$ on a BCI-algebra U is called a *k-polar* $(\in,\in)$*-fuzzy p-ideal* of U (see [9]) if it satisfies

$$(\forall z \in U)(\forall \hat{r} \in [0,1]^k)\left(z_{\hat{r}} \in \widehat{\xi} \;\Rightarrow\; 0_{\hat{r}} \in \widehat{\xi}\right), \tag{9}$$

$$(\forall z, x, y \in U)(\forall \hat{r}, \hat{t} \in [0,1]^k)\left(((z * y) * (x * y))_{\hat{r}} \in \widehat{\xi},\, x_{\hat{t}} \in \widehat{\xi} \;\Rightarrow\; z_{\inf\{\hat{r},\hat{t}\}} \in \widehat{\xi}\right). \tag{10}$$

It is easy to show that Condition (10) is equivalent to the following condition.

$$(\forall z, x, y \in U)\left(\widehat{\xi}(z) \geq \widehat{\xi}((z * y) * (x * y)) \wedge \widehat{\xi}(x)\right). \tag{11}$$

A *multipolar intuitionistic fuzzy set with finite degree k* (briefly, *k-pIF set*) over a set U (see [19]) is a mapping

$$(\widehat{\xi}, \widehat{\varrho}) : U \to [0,1]^k \times [0,1]^k,\ z \mapsto (\widehat{\xi}(z), \widehat{\varrho}(z)) \tag{12}$$

where $\widehat{\xi} : U \to [0,1]^k$ and $\widehat{\varrho} : U \to [0,1]^k$ are k-polar fuzzy sets over a set U such that $\widehat{\xi}(z) + \widehat{\varrho}(z) \leq \hat{1}$ for all $z \in U$, that is, $(\mathrm{proj}_i \circ \widehat{\xi})(z) + (\mathrm{proj}_i \circ \widehat{\varrho})(z) \leq 1$ for all $z \in U$ and $i = 1, 2, \cdots, k$. We know that if the multipolar intuitionistic fuzzy set has degree 1, then it is an intuitionistic fuzzy set. So, the intuitionistic fuzzy set is a special case of the multipolar intuitionistic fuzzy set. From this point of view, multipolar intuitionistic fuzzy set is a generalization of intuitionistic fuzzy set.

Given a k-pIF set $(\widehat{\xi}, \widehat{\varrho})$ over a set U, we consider the sets

$$U(\widehat{\xi}, \hat{t}) := \{z \in U \mid \widehat{\xi}(z) \geq \hat{t}\} \text{ and } L(\widehat{\varrho}, \hat{s}) := \{z \in U \mid \widehat{\varrho}(z) \leq \hat{s}\}, \tag{13}$$

where $\hat{t} = (t_1, t_2, \cdots, t_k) \in [0,1]^k$ and $\hat{s} = (s_1, s_2, \cdots, s_k) \in [0,1]^k$ with $\hat{t} + \hat{s} \leq \hat{1}$, which is called a *k-polar upper* (resp., *lower*) *level set* of $(\widehat{\xi}, \widehat{\varrho})$ where "+" is the componentwise operation in $[0,1]^k$, that is, $t_i + s_i \leq 1$ for all $i = 1, 2, \cdots, k$. It is clear that $U(\widehat{\xi}, \hat{t}) = \bigcap_{i=1}^k U(\widehat{\xi}, \hat{t})^i$ and $L(\widehat{\varrho}, \hat{s}) = \bigcap_{i=1}^k L(\widehat{\varrho}, \hat{s})^i$ where

$$U(\widehat{\xi}, \hat{t})^i = \{z \in U \mid (\mathrm{proj}_i \circ \widehat{\xi})(z) \geq t_i\} \text{ and } L(\widehat{\varrho}, \hat{s})^i = \{z \in U \mid (\mathrm{proj}_i \circ \widehat{\varrho})(z) \leq s_i\}.$$

A k-pIF set $(\widehat{\xi}, \widehat{\varrho})$ over U is called a *k-polar intuitionistic fuzzy ideal* (briefly, k-pIF ideal) of U (see [19]) if it satisfies the conditions

$$(\forall z \in U)(\widehat{\xi}(0) \geq \widehat{\xi}(z),\ \widehat{\varrho}(0) \leq \widehat{\varrho}(z)), \tag{14}$$

that is, $(\mathrm{proj}_i \circ \widehat{\xi})(0) \geq (\mathrm{proj}_i \circ \widehat{\xi})(z)$ and $(\mathrm{proj}_i \circ \widehat{\varrho})(0) \leq (\mathrm{proj}_i \circ \widehat{\varrho})(z)$ for $i = 1, 2, \cdots, k$. and

$$(\forall z, x \in U) \begin{pmatrix} \widehat{\xi}(z) \geq \widehat{\xi}(z * x) \wedge \widehat{\xi}(x) \\ \widehat{\varrho}(z) \leq \widehat{\varrho}(z * x) \vee \widehat{\varrho}(x) \end{pmatrix}. \tag{15}$$

3. *k*-Polar Intuitionistic Fuzzy *p*-Ideals

In this section, let U be a BCI-algebra unless otherwise stated.

Definition 1. *A k-pIF set $(\widehat{\xi}, \widehat{\varrho})$ over U is called a k-polar intuitionistic fuzzy p-ideal (briefly, k-pIF p-ideal) of U if it satisfies Condition* (14) *and*

$$(\forall z, x, y \in U) \begin{pmatrix} \widehat{\xi}(z) \geq \widehat{\xi}((z * x) * (y * x)) \wedge \widehat{\xi}(y) \\ \widehat{\varrho}(z) \leq \widehat{\varrho}((z * x) * (y * x)) \vee \widehat{\varrho}(y) \end{pmatrix}. \tag{16}$$

Example 1. *Let $U = \{0, x, a, b\}$ be a set with a binary operation $*$ which is given in Table 1.*

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	$\boldsymbol{x}$	$\boldsymbol{a}$	$\boldsymbol{b}$
0	0	x	a	b
x	x	0	b	a
a	a	b	0	x
b	b	a	x	0

Then, U is a BCI-algebra (see [2]). Let $(\widehat{\xi}, \widehat{\varrho})$ be a 4-polar intuitionistic fuzzy set over U given by

$$(\widehat{\xi}, \widehat{\varrho}) : U \to [0,1]^4 \times [0,1]^4,$$
$$z \mapsto \begin{cases} ((0.8, 0.67, 0.9, 0.56), (0.19, 0.15, 0.07, 0.28)) & \text{if } z = 0, \\ ((0.7, 0.57, 0.7, 0.56), (0.19, 0.24, 0.07, 0.35)) & \text{if } z = x, \\ ((0.5, 0.37, 0.4, 0.32), (0.37, 0.44, 0.39, 0.58)) & \text{if } z = a, \\ ((0.5, 0.37, 0.4, 0.32), (0.37, 0.44, 0.39, 0.58)) & \text{if } z = b. \end{cases}$$

It is routine to check that $(\widehat{\xi}, \widehat{\varrho})$ is a 4-polar intuitionistic fuzzy p-ideal of U.

Theorem 1. *Let I be a subset of U and let $(\widehat{\xi}_I, \widehat{\varrho}_I)$ be a k-pIF set on U defined by*

$$\widehat{\xi}_I : U \to [0,1]^k,\ z \mapsto \begin{cases} \hat{1} & \text{if } z \in I, \\ \hat{0} & \text{otherwise} \end{cases}$$

$$\widehat{\varrho}_I : U \to [0,1]^k,\ z \mapsto \begin{cases} \hat{0} & \text{if } z \in I, \\ \hat{1} & \text{otherwise} \end{cases}$$

Then, $(\widehat{\xi}_I, \widehat{\varrho}_I)$ is a k-pIF ideal p-ideal of U if and only if I is a p-ideal of U.

Proof. Straightforward. □

In the following theorem, we look at the relationship between k-pIF ideal and k-pIF p-ideal.

Theorem 2. *Every k-pIF p-ideal is a k-pIF ideal.*

Proof. Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF p-ideal of U. If we put $x = 0$ in (16) and use (1), then

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\xi})(z) &\geq \min\{(\mathrm{proj}_i \circ \widehat{\xi})((z * 0) * (x * 0)), (\mathrm{proj}_i \circ \widehat{\xi})(x)\} \\ &= \min\{(\mathrm{proj}_i \circ \widehat{\xi})(z * x), (\mathrm{proj}_i \circ \widehat{\xi})(x)\}\end{aligned}$$

and

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\varrho})(z) &\leq \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((z * 0) * (x * 0)), (\mathrm{proj}_i \circ \widehat{\varrho})(x)\} \\ &= \max\{(\mathrm{proj}_i \circ \widehat{\varrho})(z * x), (\mathrm{proj}_i \circ \widehat{\varrho})(x)\}\end{aligned}$$

for all $z, x \in U$. Therefore $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF ideal of U. □

In the following example, we find that the converse of Theorem 2 is not true.

Example 2. *Let $U = \{0, x, b, c, d\}$ be a set with a binary operation $*$, which is given in Table 2.*

Table 2. Cayley table for the binary operation "$*$".

$*$	**0**	$\boldsymbol{x}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
0	0	0	d	c	b
x	x	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

Then, U is a BCI-algebra (see [2]). Define a 3-polar intuitionistic fuzzy set $(\widehat{\xi}, \widehat{\varrho})$ on U as follows:

$$\begin{aligned}&(\widehat{\xi}, \widehat{\varrho}) : U \to [0,1]^3 \times [0,1]^3,\\ &z \mapsto \begin{cases} ((0.6, 0.7, 0.9), (0.2, 0.25, 0.07)) & \text{if } z = 0,\\ ((0.6, 0.5, 0.7), (0.3, 0.25, 0.17)) & \text{if } z = x,\\ ((0.2, 0.3, 0.4), (0.6, 0.45, 0.27)) & \text{if } z = b,\\ ((0.5, 0.4, 0.6), (0.4, 0.35, 0.37)) & \text{if } z = c,\\ ((0.2, 0.3, 0.4), (0.6, 0.45, 0.27)) & \text{if } z = d.\end{cases}\end{aligned}$$

It is easy to confirm that $(\widehat{\xi}, \widehat{\varrho})$ is a 3-polar intuitionistic fuzzy ideal of U. But it is not a 3-polar intuitionistic fuzzy p-ideal of U since

$$(\text{proj}_2 \circ \widehat{\xi})(x) = 0.5 < 0.7 = \min\{(\text{proj}_2 \circ \widehat{\xi})((x * b) * (0 * b)), (\text{proj}_2 \circ \widehat{\xi})(0)\}$$

and/or

$$(\text{proj}_3 \circ \widehat{\varrho})(x) = 0.17 > 0.07 = \max\{(\text{proj}_3 \circ \widehat{\varrho})((x * b) * (0 * b)), (\text{proj}_3 \circ \widehat{\varrho})(0)\}.$$

Proposition 1. *Every k-pIF p-ideal $(\widehat{\xi}, \widehat{\varrho})$ of U satisfies the following inequalities.*

$$(\forall z \in U)(\widehat{\xi}(z) \geq \widehat{\xi}(0 * (0 * z)),\ \widehat{\varrho}(z) \leq \widehat{\varrho}(0 * (0 * z))). \tag{17}$$

Proof. If we change y to z and x to 0 in Equation (16), then

$$\begin{aligned}(\text{proj}_i \circ \widehat{\xi})(z) &\geq \min\{(\text{proj}_i \circ \widehat{\xi})((z * z) * (0 * z)), (\text{proj}_i \circ \widehat{\xi})(0)\}\\ &= \min\{(\text{proj}_i \circ \widehat{\xi})(0 * (0 * z)), (\text{proj}_i \circ \widehat{\xi})(0)\}\\ &= (\text{proj}_i \circ \widehat{\xi})(0 * (0 * z))\end{aligned}$$

and

$$\begin{aligned}(\text{proj}_i \circ \widehat{\varrho})(z) &\leq \max\{(\text{proj}_i \circ \widehat{\varrho})((z * z) * (0 * z)), (\text{proj}_i \circ \widehat{\varrho})(0)\}\\ &= \max\{(\text{proj}_i \circ \widehat{\varrho})(0 * (0 * z)), (\text{proj}_i \circ \widehat{\varrho})(0)\}\\ &= (\text{proj}_i \circ \widehat{\varrho})(0 * (0 * z))\end{aligned}$$

for all $z \in U$. $\square$

Proposition 2. *Every k-pIF p-ideal $(\widehat{\xi}, \widehat{\varrho})$ of U satisfies the following inequalities.*

$$(\forall z, x, y \in U)\left(\begin{array}{l}\widehat{\xi}(z * x) \leq \widehat{\xi}((z * y) * (x * y))\\ \widehat{\varrho}(z * x) \geq \widehat{\varrho}((z * y) * (x * y))\end{array}\right). \tag{18}$$

Proof. Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF p-ideal of U. Then, it is a k-pIF ideal of U by Theorem 2. For any $z, x, y \in U$, we have $((z * y) * (x * y)) * (z * x) = 0$. Hence

$$\begin{aligned}&(\text{proj}_i \circ \widehat{\xi})((z * y) * (x * y))\\ &\geq \min\{(\text{proj}_i \circ \widehat{\xi})(((z * y) * (x * y)) * (z * x)), (\text{proj}_i \circ \widehat{\xi})(z * x)\}\\ &= \min\{(\text{proj}_i \circ \widehat{\xi})(0), (\text{proj}_i \circ \widehat{\xi})(z * x)\} = (\text{proj}_i \circ \widehat{\xi})(z * x)\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y))\\&\leq \max\{(\mathrm{proj}_i \circ \widehat{\varrho})(((z*y)*(x*y))*(z*x)), (\mathrm{proj}_i \circ \widehat{\varrho})(z*x)\}\\&= \max\{(\mathrm{proj}_i \circ \widehat{\varrho})(0), (\mathrm{proj}_i \circ \widehat{\varrho})(z*x)\} = (\mathrm{proj}_i \circ \widehat{\varrho})(z*x)\end{aligned}$$

for all $z, x, y \in U$. □

We provide conditions for a k-pIF ideal to be a k-pIF p-ideal.

Theorem 3. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF ideal of U satisfying the condition*

$$(\forall z, x, y \in U)\left(\begin{array}{l}\widehat{\xi}(z*x) \geq \widehat{\xi}((z*y)*(x*y))\\ \widehat{\varrho}(z*x) \leq \widehat{\varrho}((z*y)*(x*y))\end{array}\right). \tag{19}$$

Then, it is a k-pIF p-ideal of U.

Proof. Using Equations (15) and (19), we have that

$$\widehat{\xi}(z) \geq \widehat{\xi}(z*x) \wedge \widehat{\xi}(x) \geq \widehat{\xi}((z*y)*(x*y)) \wedge \widehat{\xi}(x)$$

and

$$\widehat{\varrho}(z) \leq \widehat{\varrho}(z*x) \vee \widehat{\varrho}(x) \leq \widehat{\varrho}((z*y)*(x*y)) \vee \widehat{\varrho}(x)$$

for all $z, x, y \in U$. Therefore $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U. □

Lemma 1. *Every k-pIF ideal $(\widehat{\xi}, \widehat{\varrho})$ of U satisfies the following inequalities.*

$$(\forall z \in U)(\widehat{\xi}(z) \leq \widehat{\xi}(0*(0*z)),\ \widehat{\varrho}(z) \geq \widehat{\varrho}(0*(0*z))). \tag{20}$$

Proof. For any $z, x \in U$, we obtain

$$\widehat{\xi}(0*(0*z)) \geq \widehat{\xi}((0*(0*z))*z) \wedge \widehat{\xi}(z) = \widehat{\xi}((0*z)*(0*z)) \wedge \widehat{\xi}(z) = \widehat{\xi}(0) \wedge \widehat{\xi}(z) = \widehat{\xi}(z)$$

and

$$\widehat{\varrho}(0*(0*z)) \leq \widehat{\varrho}((0*(0*z))*z) \vee \widehat{\varrho}(z) = \widehat{\varrho}((0*z)*(0*z)) \vee \widehat{\varrho}(z) = \widehat{\varrho}(0) \vee \widehat{\varrho}(z) = \widehat{\varrho}(z)$$

by Equations (2), (3), (14) and (15). □

Theorem 4. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF set over U. If $(\widehat{\xi}, \widehat{\varrho})$ satisfies the following inequalities*

$$(\forall z \in U)(\widehat{\xi}(z) \geq \widehat{\xi}(0*(0*z)),\ \widehat{\varrho}(z) \leq \widehat{\varrho}(0*(0*z))). \tag{21}$$

Proof. For any $z, x, y \in U$ and $i = 1, 2, \cdots, k$, we have

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)) &\leq (\mathrm{proj}_i \circ \widehat{\xi})(0*(0*(z*y)*(x*y)))\\&= (\mathrm{proj}_i \circ \widehat{\xi})((0*x)*(0*y))\\&= (\mathrm{proj}_i \circ \widehat{\xi})(0*(0*(z*y)))\\&\leq (\mathrm{proj}_i \circ \widehat{\xi})(z*x),\end{aligned}$$

and

$$
\begin{aligned}
(\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)) &\geq (\mathrm{proj}_i \circ \widehat{\varrho})(0*(0*(z*y)*(x*y))) \\
&= (\mathrm{proj}_i \circ \widehat{\varrho})((0*x)*(0*y)) \\
&= (\mathrm{proj}_i \circ \widehat{\varrho})(0*(0*(z*y))) \\
&\geq (\mathrm{proj}_i \circ \widehat{\varrho})(z*x),
\end{aligned}
$$

which imply that $\widehat{\xi}((z*y)*(x*y)) \leq \widehat{\xi}(z*x)$ and $\widehat{\varrho}((z*y)*(x*y)) \geq \widehat{\varrho}(z*x)$ for all $z, x, y \in U$. Therefore $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U by Theorem 3. $\square$

We consider characterizations of a k-pIF p-ideal.

Theorem 5. *Given a k-pIF set $(\widehat{\xi}, \widehat{\varrho})$ over U, the following assertions are equivalent.*

(i) *$(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U.*
(ii) *The k-polar upper and lower level sets $U(\widehat{\xi}, \hat{r})$ and $L(\widehat{\varrho}, \hat{q})$ are p-ideals of U for all $(\hat{r}, \hat{q}) \in [0,1]^k \times [0,1]^k$ with $U(\widehat{\xi}, \hat{r}) \neq \emptyset \neq L(\widehat{\varrho}, \hat{q})$.*

Proof. Assume that $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U. It is clear that $0 \in U(\widehat{\xi}; \hat{r})$ and $0 \in L(\widehat{\varrho}; \hat{q})$ for any $\hat{r} = (r_1, r_2, \cdots, r_k) \in (0,1]^k$ and $\hat{q} = (q_1, q_2, \cdots, q_k) \in (0,1]^k$. Let $z, x, y, b, c, d \in U$ be such that $(z*y)*(x*y) \in U(\widehat{\xi}; \hat{r})$, $x \in U(\widehat{\xi}; \hat{r})$, $(b*d)*(c*d) \in L(\widehat{\varrho}; \hat{q})$ and $c \in L(\widehat{\varrho}; \hat{q})$. Then, $(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)) \geq r_i$, $(\mathrm{proj}_i \circ \widehat{\xi})(x) \geq r_i$, $(\mathrm{proj}_i \circ \widehat{\varrho})((b*d)*(c*d)) \leq q_i$ and $(\mathrm{proj}_i \circ \widehat{\varrho})(c) \leq q_i$. It follows from Equations (16) that

$$(\mathrm{proj}_i \circ \widehat{\xi})(z) \geq \min\{(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\xi})(x)\} \geq r_i$$

and

$$(\mathrm{proj}_i \circ \widehat{\varrho})(b) \leq \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((b*d)*(c*d)), (\mathrm{proj}_i \circ \widehat{\varrho})(c)\} \leq q_i$$

for $i = 1, 2, \cdots, k$. Hence $z \in U(\widehat{\xi}; \hat{r})$ and $b \in L(\widehat{\varrho}; \hat{q})$ and therefore $U(\widehat{\xi}; \hat{r})$ and $L(\widehat{\varrho}; \hat{q})$ are p-ideals of U.

Conversely, suppose that the k-polar upper and lower level sets $U(\widehat{\xi}, \hat{r})$ and $L(\widehat{\varrho}, \hat{q})$ are p-ideals of U for all $(\hat{r}, \hat{q}) \in [0,1]^k \times [0,1]^k$ with $U(\widehat{\xi}, \hat{r}) \neq \emptyset \neq L(\widehat{\varrho}, \hat{q})$. If $\widehat{\xi}(0) < \widehat{\xi}(b)$ for some $b \in U$, then $b \in U(\widehat{\xi}; \hat{r})$ and $0 \notin U(\widehat{\xi}; \hat{r})$ where $\hat{r} := \widehat{\xi}(b)$. This is a contradiction, and so $\widehat{\xi}(0) \geq \widehat{\xi}(z)$ for all $z \in U$. If $\widehat{\varrho}(0) > \widehat{\varrho}(c)$ for some $c \in U$, then $(\mathrm{proj}_i \circ \widehat{\varrho})(0) > (\mathrm{proj}_i \circ \widehat{\varrho})(c)$ for $i = 1, 2, \cdots, k$. If we take $q_i := (\mathrm{proj}_i \circ \widehat{\varrho})(c)$ for $i = 1, 2, \cdots, k$, then $c \in L(\widehat{\varrho}, \hat{q})^i$ and $0 \notin L(\widehat{\varrho}, \hat{q})^i$ for $i = 1, 2, \cdots, k$. Thus $c \in \bigcap_{i=1}^{k} L(\widehat{\varrho}, \hat{q})^i = L(\widehat{\varrho}, \hat{q})$ and $0 \notin L(\widehat{\varrho}, \hat{q})$, which is a contradiction; hence $\widehat{\varrho}(0) \leq \widehat{\varrho}(z)$ for all $z \in U$. Now, suppose that there exist $b, c, d \in U$ such that $\widehat{\xi}(b) < \widehat{\xi}((b*d)*(c*d)) \wedge \widehat{\xi}(c)$ or $\widehat{\varrho}(b) > \widehat{\varrho}((b*d)*(c*d)) \vee \widehat{\varrho}(c)$. If we take

$$\hat{r} := \widehat{\xi}((b*d)*(c*d)) \wedge \widehat{\xi}(c)$$

and

$$\hat{q} := \widehat{\varrho}((b*d)*(c*d)) \vee \widehat{\varrho}(c),$$

then

$$(b*d)*(c*d) \in U(\widehat{\xi}; \hat{r}) \text{ and } c \in U(\widehat{\xi}; \hat{r})$$

or

$$(b*d)*(c*d) \in L(\widehat{\varrho}, \hat{q}) \text{ and } c \in L(\widehat{\varrho}, \hat{q}).$$

Since $U(\widehat{\xi};\hat{r})$ and $L(\widehat{\varrho},\hat{q})$ are p-ideals of U by assumption, it follows that $b \in U(\widehat{\xi};\hat{r})$ or $b \in L(\widehat{\varrho};\hat{q})$. Hence $\widehat{\xi}(b) \geq \hat{r} = \widehat{\xi}((b*d)*(c*d)) \wedge \widehat{\xi}(c)$ or $\widehat{\varrho}(b) \leq \hat{q} = \widehat{\varrho}((b*d)*(c*d)) \vee \widehat{\varrho}(c)$, which is a contradiction. Thus $\widehat{\xi}(z) \geq \widehat{\xi}((z*y)*(x*y)) \wedge \widehat{\xi}(x)$ and $\widehat{\varrho}(z) \leq \widehat{\varrho}((z*y)*(x*y)) \vee \widehat{\varrho}(x)$ for all $z,x,y \in U$; therefore $(\widehat{\xi},\widehat{\varrho})$ is a k-pIF p-ideal of U. □

Given a k-pIF set $(\widehat{\xi},\widehat{\varrho})$ over U and $(\hat{t},\hat{s}) \in (0,1]^k \times [0,1)^k$, we consider the sets:

$$R_{(\widehat{\xi},\hat{t})}(U) := \{z \in U \mid \widehat{\xi}(z) + \hat{t} > \hat{1}\}$$

and

$$R_{(\widehat{\varrho},\hat{s})}(U) := \{z \in U \mid \widehat{\varrho}(z) + \hat{s} < \hat{1}\}.$$

Then, $R_{(\widehat{\xi},\hat{t})}(U) = \bigcap_{i=1}^{k} R_{(\widehat{\xi},\hat{t})}(U)^i$ and $R_{(\widehat{\varrho},\hat{s})}(U) = \bigcap_{i=1}^{k} R_{(\widehat{\varrho},\hat{s})}(U)^i$ where

$$R_{(\widehat{\xi},\hat{t})}(U)^i := \{z \in U \mid (\mathrm{proj}_i \circ \widehat{\xi})(z) + t_i > 1\}$$

and

$$R_{(\widehat{\varrho},\hat{s})}(U)^i := \{z \in U \mid (\mathrm{proj}_i \circ \widehat{\varrho})(z) + s_i < 1\}$$

for $i = 1,2,\cdots,k$.

Theorem 6. *Given a k-pIF set $(\widehat{\xi},\widehat{\varrho})$ over U, the following assertions are equivalent.*

(i) *$(\widehat{\xi},\widehat{\varrho})$ is a k-pIF p-ideal of U.*
(ii) *The sets $R_{(\widehat{\xi},\hat{t})}(U)$ and $R_{(\widehat{\varrho},\hat{s})}(U)$ are p-ideals of U for all $(\hat{t},\hat{s}) \in (0,1]^k \times [0,1)^k$ with $R_{(\widehat{\xi},\hat{t})}(U) \neq \emptyset \neq R_{(\widehat{\varrho},\hat{s})}(U)$.*

Proof. Assume that $(\widehat{\xi},\widehat{\varrho})$ is a k-pIF p-ideal of U. It is clear that $0 \in R_{(\widehat{\xi},\hat{t})}(U)$ and $0 \in R_{(\widehat{\varrho},\hat{s})}(U)$. Let $z,x,y,b,c,d \in U$ be such that $(z*y)*(x*y) \in R_{(\widehat{\xi},\hat{t})}(U)$, $x \in R_{(\widehat{\xi},\hat{t})}(U)$, $(b*d)*(c*d) \in R_{(\widehat{\varrho},\hat{s})}(U)$ and $c \in R_{(\widehat{\varrho},\hat{s})}(U)$. Then, $\widehat{\xi}((z*y)*(x*y)) + \hat{t} > \hat{1}$, $\widehat{\xi}(x) + \hat{t} > \hat{1}$, $\widehat{\varrho}((b*d)*(c*d)) + \hat{s} < \hat{1}$ and $\widehat{\varrho}(c) + \hat{s} < \hat{1}$. It follows that

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\xi})(z) + t_i &\geq \min\{(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\xi})(x)\} + t_i \\ &= \min\{(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)) + t_i, (\mathrm{proj}_i \circ \widehat{\xi})(x) + t_i\} > 1\end{aligned}$$

and

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\varrho})(b) + s_i &\leq \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((b*d)*(c*d)), (\mathrm{proj}_i \circ \widehat{\varrho})(c)\} + s_i \\ &= \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((b*d)*(c*d)) + s_i, (\mathrm{proj}_i \circ \widehat{\varrho})(c) + s_i\} < 1\end{aligned}$$

for all $i = 1,2,\cdots,k$. Hence $z \in \bigcap_{i=1}^{k} R_{(\widehat{\xi},\hat{t})}(U)^i = R_{(\widehat{\xi},\hat{t})}(U)$ and $b \in \bigcap_{i=1}^{k} R_{(\widehat{\varrho},\hat{s})}(U)^i = R_{(\widehat{\varrho},\hat{s})}(U)$; therefore $R_{(\widehat{\xi},\hat{t})}(U)$ and $R_{(\widehat{\varrho},\hat{s})}(U)$ are p-ideals of U for all $(\hat{t},\hat{s}) \in (0,1]^k \times [0,1)^k$.

Conversely suppose that (ii) is valid. If $\widehat{\xi}(0) < \widehat{\xi}(z)$ or $\widehat{\varrho}(0) > \widehat{\varrho}(b)$ for some $z,b \in U$, then $\widehat{\xi}(0) + \hat{t} \leq \hat{1} < \widehat{\xi}(z) + \hat{t}$ or $\widehat{\varrho}(0) + \hat{s} \geq \hat{1} > \widehat{\varrho}(b) + \hat{s}$ for some $(\hat{t},\hat{s}) \in (0,1]^k \times [0,1)^k$. Thus $0 \notin R_{(\widehat{\xi},\hat{t})}(U)$ or $0 \notin R_{(\widehat{\varrho},\hat{s})}(U)$ which is a contradiction. Hence $(\widehat{\xi},\widehat{\varrho})$ satisfies Condition (14). Suppose that $\widehat{\xi}(b) < \widehat{\xi}((b*d)*(c*d)) \wedge \widehat{\xi}(c)$ for some $b,c \in U$. Then, $\widehat{\xi}(b) + \hat{t} \leq \hat{1} < (\widehat{\xi}((b*d)*(c*d)) \wedge \widehat{\xi}(c)) + \hat{t} = (\widehat{\xi}((b*d)*(c*d)) + \hat{t}) \wedge (\widehat{\xi}(c) + \hat{t})$ for some $\hat{t} \in (0,1]^k$. It follows that $(b*d)*(c*d) \in R_{(\widehat{\xi},\hat{t})}(U)$ and

$c \in R_{(\widehat{\xi},\hat{t})}(U)$, which implies that $b \in R_{(\widehat{\xi},\hat{t})}(U)$ since $R_{(\widehat{\xi},\hat{t})}(U)$ is a p-ideal of U; hence $\widehat{\xi}(b) + \hat{t} > \hat{1}$, which is a contradiction. If $\widehat{\varrho}(z) > \widehat{\varrho}((z*y)*(x*y)) \vee \widehat{\varrho}(x)$ for some $z, x \in U$, then

$$\widehat{\varrho}(z) + \hat{s} \geq \hat{1} > (\widehat{\xi}((z*y)*(x*y)) \vee \widehat{\xi}(x)) + \hat{s} = (\widehat{\xi}((z*y)*(x*y)) + \hat{s}) \vee (\widehat{\xi}(x) + \hat{s})$$

for some $\hat{s} \in [0,1)^k$. Thus $(z*y)*(x*y) \in R_{(\widehat{\varrho},\hat{s})}(U)$ and $x \in R_{(\widehat{\varrho},\hat{s})}(U)$. Since $R_{(\widehat{\varrho},\hat{s})}(U)$ is a p-ideal of U, it follows that $z \in R_{(\widehat{\varrho},\hat{s})}(U)$, that is, $\widehat{\varrho}(z) + \hat{s} < \hat{1}$. This is a contradiction. This shows that $(\widehat{\xi}, \widehat{\varrho})$ satisfies Condition (16); therefore $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U. □

The following theorem shows the characterization of k-pIF p-ideal using k-polar $(\in, \in)$-fuzzy p-ideal.

Theorem 7. *A k-pIF set $(\widehat{\xi}, \widehat{\varrho})$ over U is a k-pIF p-ideal of U if and only if $\widehat{\xi}$ and $\widehat{\varrho}^c$ are k-polar $(\in, \in)$-fuzzy p-ideals of U where $\widehat{\varrho}^c = 1 - \widehat{\varrho}$, i.e., $(\mathrm{proj}_i \circ \widehat{\varrho})^c = 1 - (\mathrm{proj}_i \circ \widehat{\varrho})$ for $i = 1, 2, \cdots, k$.*

Proof. Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF p-ideal of U. It is clear that $\widehat{\xi}$ is a k-polar $(\in, \in)$-fuzzy p-ideal of U. Let $z, x, y \in U$. Then,

$$(\mathrm{proj}_i \circ \widehat{\varrho})^c(0) = 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(0) \geq 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(z) = (\mathrm{proj}_i \circ \widehat{\varrho})^c(z)$$

and

$$\begin{aligned}(\mathrm{proj}_i \circ \widehat{\varrho})^c(z) = 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(z) &\geq 1 - \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\varrho})(x)\} \\ &= \min\{1 - (\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)), 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(x)\} \\ &= \min\{(\mathrm{proj}_i \circ \widehat{\varrho})^c((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\varrho})^c(x)\}.\end{aligned}$$

Thus $\widehat{\varrho}^c$ is a k-polar $(\in, \in)$-fuzzy p-ideal of U.

Conversely, suppose that $\widehat{\xi}$ and $\widehat{\varrho}^c$ are k-polar $(\in, \in)$-fuzzy p-ideals of U. For any $z, x \in U$, we have $(\mathrm{proj}_i \circ \widehat{\xi})(0) \geq (\mathrm{proj}_i \circ \widehat{\xi})(z)$, $(\mathrm{proj}_i \circ \widehat{\xi})(z) \geq \min\{(\mathrm{proj}_i \circ \widehat{\xi})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\xi})(x)\}$, $1 - (\mathrm{proj}_i \circ \widehat{\varrho})(0) = (\mathrm{proj}_i \circ \widehat{\varrho})^c(0) \geq (\mathrm{proj}_i \circ \widehat{\varrho})^c(z) = 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(z)$, i.e., $(\mathrm{proj}_i \circ \widehat{\varrho})(0) \leq (\mathrm{proj}_i \circ \widehat{\varrho})(z)$ and

$$\begin{aligned}1 - (\mathrm{proj}_i \circ \widehat{\varrho})(z) = (\mathrm{proj}_i \circ \widehat{\varrho})^c(z) &\geq \min\{(\mathrm{proj}_i \circ \widehat{\varrho})^c((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\varrho})^c(x)\} \\ &= \min\{1 - (\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)), 1 - (\mathrm{proj}_i \circ \widehat{\varrho})(x)\} \\ &= 1 - \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\varrho})(x)\},\end{aligned}$$

that is, $(\mathrm{proj}_i \circ \widehat{\varrho})(z) \leq \max\{(\mathrm{proj}_i \circ \widehat{\varrho})((z*y)*(x*y)), (\mathrm{proj}_i \circ \widehat{\varrho})(x)\}$; therefore $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U. □

The following corollary is an immediate consequence of Theorem 7.

Corollary 1. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF set over U. Then, $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U if and only if the necessary operator $\Box(\widehat{\xi}, \widehat{\varrho}) = (\widehat{\xi}, \widehat{\xi}^c)$ and the possibility operator $\Diamond(\widehat{\xi}, \widehat{\varrho}) = (\widehat{\varrho}^c, \widehat{\varrho})$ of $(\widehat{\xi}, \widehat{\varrho})$ are k-pIF p-ideals of U.*

Definition 2. *A k-pIF p-ideal $(\widehat{\xi}, \widehat{\varrho})$ of U is said to be normal if there exists $z, x \in U$ such that $\widehat{\xi}(z) = \hat{1}$ and $\widehat{\varrho}(x) = \hat{0}$.*

Example 3. *Consider the BCI-algebra $U = \{0, x, a, b\}$, which is given in Example 1. Let $(\widehat{\xi}, \widehat{\varrho})$ be a 3-polar intuitionistic fuzzy set over U given by*

$$(\widehat{\xi}, \widehat{\varrho}) : U \to [0,1]^3 \times [0,1]^3,$$
$$z \mapsto \begin{cases} ((1.00, 1.00, 1.00), (0.00, 0.00, 0.00)) & \text{if } z = 0, \\ ((0.72, 0.57, 1.00), (0.00, 0.24, 0.35)) & \text{if } z = x, \\ ((0.52, 0.37, 0.32), (0.37, 0.44, 0.58)) & \text{if } z = a, \\ ((0.52, 0.37, 0.32), (0.37, 0.44, 0.58)) & \text{if } z = b. \end{cases}$$

It is routine to check that $(\widehat{\xi}, \widehat{\varrho})$ is a normal 3-polar intuitionistic fuzzy p-ideal of U.

It is clear that if a k-pIF p-ideal $(\widehat{\xi}, \widehat{\varrho})$ of U is normal, then $\widehat{\xi}(0) = \hat{1}$ and $\widehat{\varrho}(0) = \hat{0}$, that is, $(\text{proj}_i \circ \widehat{\xi})(0) = 1$ and $(\text{proj}_i \circ \widehat{\varrho})(0) = 0$ for all $i = 1, 2, \cdots, k$.

Lemma 2. *A k-pIF p-ideal $(\widehat{\xi}, \widehat{\varrho})$ of U is normal if and only if $\widehat{\xi}(0) = \hat{1}$ and $\widehat{\varrho}(0) = \hat{0}$.*

Proof. Straightforward. □

In the following theorem we look at the process of eliciting normal k-pIF p-ideal from a given k-pIF p-ideal.

Theorem 8. *If $(\widehat{\xi}, \widehat{\varrho})$ is k-pIF p-ideal of U, then the k-pIF set $(\widehat{\xi}, \widehat{\varrho})^+ = (\widehat{\xi}^+, \widehat{\varrho}^+)$ on U defined by*

$$\begin{aligned} &\widehat{\xi}^+ : U \to [0,1]^k,\ z \mapsto \hat{1} + \widehat{\xi}(z) - \widehat{\xi}(0), \\ &\widehat{\varrho}^+ : U \to [0,1]^k,\ z \mapsto \widehat{\varrho}(z) - \widehat{\varrho}(0) \end{aligned} \tag{22}$$

is a normal k-pIF p-ideal of U containing $(\widehat{\xi}, \widehat{\varrho})$.

Proof. Assume that $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF p-ideal of U. Then, $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF ideal of U by Theorem 2. For any $z, x \in U$, we have

$$(\text{proj}_i \circ \widehat{\xi})(0) = 1 + (\text{proj}_i \circ \widehat{\xi})(0) - (\text{proj}_i \circ \widehat{\xi})(0) = 1 \geq (\text{proj}_i \circ \widehat{\xi})(z),$$

$$(\text{proj}_i \circ \widehat{\varrho})(0) = (\text{proj}_i \circ \widehat{\varrho})(0) - (\text{proj}_i \circ \widehat{\varrho})(0) = 0 \leq (\text{proj}_i \circ \widehat{\varrho})(z),$$

$$\begin{aligned} (\text{proj}_i \circ \widehat{\xi})^+(z) &= 1 + (\text{proj}_i \circ \widehat{\xi})(z) - (\text{proj}_i \circ \widehat{\xi})(0) \\ &\geq 1 + \min\{(\text{proj}_i \circ \widehat{\xi})((z * y) * (x * y)), (\text{proj}_i \circ \widehat{\xi})(x)\} - (\text{proj}_i \circ \widehat{\xi})(0) \\ &= \min\{1 + (\text{proj}_i \circ \widehat{\xi})((z * y) * (x * y)) - (\text{proj}_i \circ \widehat{\xi})(0), 1 + (\text{proj}_i \circ \widehat{\xi})(x) - (\text{proj}_i \circ \widehat{\xi})(0)\} \\ &= \min\{(\text{proj}_i \circ \widehat{\xi})^+((z * y) * (x * y)), (\text{proj}_i \circ \widehat{\xi})^+(x)\} \end{aligned}$$

and

$$\begin{aligned} (\text{proj}_i \circ \widehat{\varrho})^+(z) &= (\text{proj}_i \circ \widehat{\varrho})(z) - (\text{proj}_i \circ \widehat{\varrho})(0) \\ &\leq \max\{(\text{proj}_i \circ \widehat{\varrho})((z * y) * (x * y)), (\text{proj}_i \circ \widehat{\varrho})(x)\} - (\text{proj}_i \circ \widehat{\varrho})(0) \\ &= \max\{(\text{proj}_i \circ \widehat{\varrho})((z * y) * (x * y)) - (\text{proj}_i \circ \widehat{\varrho})(0), (\text{proj}_i \circ \widehat{\varrho})(x) - (\text{proj}_i \circ \widehat{\varrho})(0)\} \\ &= \max\{(\text{proj}_i \circ \widehat{\varrho})^+((z * y) * (x * y)), (\text{proj}_i \circ \widehat{\varrho})^+(x)\} \end{aligned}$$

for all for $i = 1, 2, \cdots, k$. Hence $(\widehat{\xi}, \widehat{\varrho})^+$ is a k-pIF p-ideal of U and it is normal by Lemma 2. It is clear that $(\widehat{\xi}, \widehat{\varrho})$ is contained in $(\widehat{\xi}, \widehat{\varrho})^+$. □

Theorem 9. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF p-ideal of U. Then, $(\widehat{\xi}, \widehat{\varrho})$ is normal if and only if $(\widehat{\xi}, \widehat{\varrho})^+ = (\widehat{\xi}, \widehat{\varrho})$, that is, $\widehat{\xi}^+ = \widehat{\xi}$ and $\widehat{\varrho}^+ = \widehat{\varrho}$.*

Proof. The sufficiency is clear. Assume that $(\widehat{\xi}, \widehat{\varrho})$ is normal. Then,

$$(\mathrm{proj}_i \circ \widehat{\xi})^+(z) = 1 + (\mathrm{proj}_i \circ \widehat{\xi})(z) - (\mathrm{proj}_i \circ \widehat{\xi})(0) = (\mathrm{proj}_i \circ \widehat{\xi})(z)$$
$$(\mathrm{proj}_i \circ \widehat{\varrho})^+(z) = (\mathrm{proj}_i \circ \widehat{\varrho})(z) - (\mathrm{proj}_i \circ \widehat{\xi})(0) = (\mathrm{proj}_i \circ \widehat{\xi})(z)$$

for all $z \in U$ by Lemma 2. This completes the proof. □

Corollary 2. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a k-pIF p-ideal of U. If $(\widehat{\xi}, \widehat{\varrho})$ is normal, then $((\widehat{\xi}, \widehat{\varrho})^+)^+ = (\widehat{\xi}, \widehat{\varrho})$.*

Theorem 10. *Let $(\widehat{\xi}, \widehat{\varrho})$ be a non-constant normal k-pIF p-ideal of U, which is maximal in the poset of normal k-pIF p-ideals under set inclusion. Then, $\widehat{\xi}$ and $\widehat{\varrho}$ have the values $\hat{0}$ and $\hat{1}$ only.*

Proof. Since $(\widehat{\xi}, \widehat{\varrho})$ is normal, we have $\widehat{\xi}(0) = \hat{1}$ and $\widehat{\varrho}(0) = \hat{0}$ by Lemma 2. Let $z, x \in U$ be such that $\widehat{\xi}(z) \neq \hat{1}$ and $\widehat{\varrho}(x) \neq \hat{0}$. It is sufficient to show that $\widehat{\xi}(z) = \hat{0}$ and $\widehat{\varrho}(x) = \hat{1}$. If $\widehat{\xi}(z) \neq \hat{0}$ and $\widehat{\varrho}(x) \neq \hat{1}$, then there exists $b, c \in U$ such that $\hat{0} < \widehat{\xi}(b) < \hat{1}$ and $\hat{0} < \widehat{\varrho}(c) < \hat{1}$. Let $(\widehat{\xi}, \widehat{\varrho})_* = (\widehat{\xi}_*, \widehat{\varrho}_*)$ be a k-pIF set on U given by

$$\widehat{\xi}_* : U \to [0,1]^k,\ z \mapsto \tfrac{1}{2}\left(\widehat{\xi}(z) + \widehat{\xi}(b)\right).$$

and

$$\widehat{\varrho}_* : U \to [0,1]^k,\ z \mapsto \tfrac{1}{2}\left(\widehat{\varrho}(z) + \widehat{\varrho}(c)\right).$$

It is clear that $(\widehat{\xi}, \widehat{\varrho})_*$ is well-defined. For any $z, x \in U$, we have

$$\widehat{\xi}_*(0) = \tfrac{1}{2}\left(\widehat{\xi}(0) + \widehat{\xi}(b)\right) = \tfrac{1}{2}\left(\hat{1} + \widehat{\xi}(b)\right) \geq \tfrac{1}{2}\left(\widehat{\xi}(z) + \widehat{\xi}(b)\right) = \widehat{\xi}_*(z),$$

$$\widehat{\varrho}_*(0) = \tfrac{1}{2}\left(\widehat{\varrho}(0) + \widehat{\varrho}(c)\right) = \tfrac{1}{2}\left(\hat{0} + \widehat{\varrho}(c)\right) \leq \tfrac{1}{2}\left(\widehat{\varrho}(z) + \widehat{\varrho}(c)\right) = \widehat{\varrho}_*(z),$$

$$\begin{aligned}\widehat{\xi}_*(z) &= \tfrac{1}{2}\left(\widehat{\xi}(z) + \widehat{\xi}(b)\right) \geq \tfrac{1}{2}\left((\widehat{\xi}(z*x) \wedge \widehat{\xi}(x)) + \widehat{\xi}(b)\right)\\ &= \tfrac{1}{2}((\widehat{\xi}(z*x) + \widehat{\xi}(b)) \wedge (\widehat{\xi}(x) + \widehat{\xi}(b)))\\ &= \tfrac{1}{2}(\widehat{\xi}(z*x) + \widehat{\xi}(b)) \wedge \tfrac{1}{2}(\widehat{\xi}(x) + \widehat{\xi}(b))\\ &= \widehat{\xi}_*(z*x) \wedge \widehat{\xi}_*(x)\end{aligned}$$

and

$$\begin{aligned}\widehat{\varrho}_*(z) &= \tfrac{1}{2}\left(\widehat{\varrho}(z) + \widehat{\varrho}(c)\right) \leq \tfrac{1}{2}\left((\widehat{\varrho}(z*x) \vee \widehat{\varrho}(x)) + \widehat{\varrho}(c)\right)\\ &= \tfrac{1}{2}((\widehat{\varrho}(z*x) + \widehat{\varrho}(c)) \vee (\widehat{\varrho}(x) + \widehat{\varrho}(c)))\\ &= \tfrac{1}{2}(\widehat{\varrho}(z*x) + \widehat{\varrho}(c)) \vee \tfrac{1}{2}(\widehat{\varrho}(x) + \widehat{\varrho}(c))\\ &= \widehat{\varrho}_*(z*x) \vee \widehat{\varrho}_*(x).\end{aligned}$$

Hence $(\widehat{\xi}, \widehat{\varrho})$ is a k-pIF ideal of U. We have

$$\widehat{\xi}_*(z) = \tfrac{1}{2}\left(\widehat{\xi}(z) + \widehat{\xi}(b)\right) \geq \tfrac{1}{2}\left(\widehat{\xi}(0*(0*z)) + \widehat{\xi}(b)\right) = \widehat{\xi}_*(0*(0*z))$$

and

$$\widehat{\varrho}_*(z) = \tfrac{1}{2}\left(\widehat{\varrho}(z) + \widehat{\varrho}(c)\right) \leq \tfrac{1}{2}\left(\widehat{\varrho}(0 * (0 * z)) + \widehat{\varrho}(c)\right) = \widehat{\varrho}_*(0 * (0 * z))$$

for all $z \in U$. Hence $(\widehat{\xi}, \widehat{\varrho})_*$ is a k-pIF p-ideal of U by Theorem 4. Now, we get

$$\widehat{\xi}_*^+(z) = \hat{1} + \widehat{\xi}_*(z) - \widehat{\xi}_*(0) = \hat{1} + \tfrac{1}{2}\left(\widehat{\xi}(z) + \widehat{\xi}(b)\right) - \tfrac{1}{2}\left(\widehat{\xi}(0) + \widehat{\xi}(b)\right) = \tfrac{1}{2}\left(\hat{1} + \widehat{\xi}(z)\right),$$

and

$$\widehat{\varrho}_*^+(z) = \widehat{\varrho}_*(z) - \widehat{\varrho}_*(0) = \tfrac{1}{2}\left(\widehat{\varrho}(z) + \widehat{\varrho}(c)\right) - \tfrac{1}{2}\left(\widehat{\varrho}(0) + \widehat{\varrho}(c)\right) = \tfrac{1}{2}\widehat{\varrho}(z),$$

and so $\widehat{\xi}_*^+(0) = \frac{1}{2}\left(\hat{1} + \widehat{\xi}(0)\right) = \hat{1}$ and $\widehat{\varrho}_*^+(z) = \frac{1}{2}\widehat{\varrho}(0) = \hat{0}$. Hence $(\widehat{\xi}, \widehat{\varrho})_*$ is normal. Note that

$$\widehat{\xi}_*^+(0) = \hat{1} > \widehat{\xi}_*^+(b) = \tfrac{1}{2}\left(\hat{1} + \widehat{\xi}(b)\right) > \widehat{\xi}(b)$$

and

$$\widehat{\varrho}_*^+(0) = \hat{0} < \widehat{\varrho}_*^+(c) = \tfrac{1}{2}\left(\hat{0} + \widehat{\varrho}(c)\right) < \widehat{\varrho}(c).$$

Hence $(\widehat{\xi}, \widehat{\varrho})_*^+$ is non-constant and $(\widehat{\xi}, \widehat{\varrho})$ is not maximal, which is a contradiction; therefore $\widehat{\xi}$ and $\widehat{\varrho}$ have the values $\hat{0}$ and $\hat{1}$ only. □

4. Conclusions and Future Works

As a generalization of intuitionistic fuzzy set, Kang et al. [19] introduced the notion of multipolar intuitionistic fuzzy set with finite degree, and then they applied the notion to BCK/BCI-algebras. In this manuscript, we used Kang et al.'s multipolar intuitionistic fuzzy set to study p-ideal in BCI-algebras. We introduced the notion of k-polar intuitionistic fuzzy p-ideals (see Definition 1) in BCI-algebras, and then we studied several properties (See Proposition 1, Proposition 2). We gave an example to illustrate the k-polar intuitionistic fuzzy p-ideal (see Example 1), and considered the relationship between k-polar intuitionistic fuzzy ideal and k-polar intuitionistic fuzzy p-ideal. We have shown that every k-polar intuitionistic fuzzy p-ideal is a k-polar intuitionistic fuzzy ideal (see Theorem 2), and then provided an example to show that the converse is not true in general (see Example 2). We used the notion of p-ideals in BCI-algebras to study the characterization of k-polar intuitionistic fuzzy p-ideal (see Theorem 1, Theorem 5 and Theorem 6), and also used the notion of k-polar $(\in, \in)$-fuzzy p-ideal in BCI-algebras to study the characterization of k-polar intuitionistic fuzzy p-ideal (see Theorem 7). We defined the concept of normal k-polar intuitionistic fuzzy p-ideal (see Definition 2), and discussed its characterization (see Lemma 2 and Theorem 9). We looked at the process of eliciting normal k-polar intuitionistic fuzzy p-ideal from a given k-polar intuitionistic fuzzy p-ideal (see Theorem 8). Our goal in the future is to apply the ideas and results of this paper to other forms of ideals, filters, etc. in BCK/BCI-algebras. We will also apply the ideas and results of this paper to other algebraic structures, for example, MV-algebras, EQ-algebras, equality algebras, hoops, etc.

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