mathematics

MDPI

Article

More Effective Criteria for Oscillation of Second-Order Differential Equations with Neutral Arguments

Osama Moaaz [1,*], Mona Anis [1], Dumitru Baleanu [2,3,4] and Ali Muhib [1,5]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; mona_anis1985@yahoo.com (M.A.); muhib39@students.mans.edu.eg or muhib39@yahoo.com (A.M.)
2 Department of Mathematics and Computer Science, Faculty of Arts and Sciences, Çankaya University Ankara, 06790 Etimesgut, Turkey; dumitru@cankaya.edu.tr or Baleanu@mail.cmuh.org.tw
3 Institue of Space Sciences, Magurele-Bucharest, 077125 Magurele, Romania
4 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
5 Department of Mathematics, Faculty of Education (Al-Nadirah), Ibb University, Ibb, Yemen
* Correspondence: o_moaaz@mans.edu.eg

Received: 22 April 2020; Accepted: 3 June 2020; Published: 16 June 2020

Abstract: The motivation for this paper is to create new criteria for oscillation of solutions of second-order nonlinear neutral differential equations. In more than one respect, our results improve several related ones in the literature. As proof of the effectiveness of the new criteria, we offer more than one practical example.

Keywords: second order differential equation; neutral delayed argument; oscillation

1. Introduction

In recent decades, an increasing interest in establishing sufficient criteria for oscillatory and non-oscillatory properties of different classes of differential equations has been observed; see, for instance, the monographs [1–3] and the references cited therein. Many authors were concerned with the oscillation and nonoscillation of delay differential equations of second-order [4–17] and higher-order [18–23]. The growing interest of delay differential equation is due to the many applications of these equations in different fields of science, see [24,25].

In this work, by using different techniques, we create new criteria for oscillation of 2nd-order neutral differential equation

$$\left(r\left(\zeta\right)\left(z'\left(\zeta\right)\right)^{\alpha}\right)' + q\left(\zeta\right)x^{\beta}\left(\sigma\left(\zeta\right)\right) = 0, \tag{1}$$

where $\zeta \geq \zeta_0$ and $z\left(\zeta\right) := x\left(\zeta\right) + p\left(\zeta\right)x\left(\tau\left(\zeta\right)\right)$. Moreover, we assume that:

[N$_1$] $\alpha, \beta \in \mathbb{Q}^{+}_{odd}$, where $\mathbb{Q}^{+}_{odd} := \{a/b : a, b \in \mathbb{Z}^{+}$ are odd$\}$;
[N$_2$] $r \in C\left([\zeta_0,\infty)\right), r\left(\zeta\right) > 0$ and

$$\theta_s\left(\zeta\right) := \int_s^{\zeta} r^{-1/\alpha}\left(s\right) \mathrm{d}s;$$

[N$_3$] $p, q \in C\left([\zeta_0,\infty)\right), q\left(\zeta\right) \geq 0, 0 \leq p\left(\zeta\right) \leq p_0 < \infty$ and $q\left(\zeta\right)$ is not identically zero for large ζ;
[N$_4$] $\tau, \sigma \in C\left([\zeta_0,\infty)\right), \tau\left(\zeta\right) \leq \zeta, \sigma\left(\zeta\right) < \zeta$, and $\lim_{\zeta\to\infty}\tau\left(\zeta\right) = \lim_{\zeta\to\infty}\sigma\left(\zeta\right) = \infty$.

If $x \in C^1[\zeta_x,\infty)$, $\zeta_x \geq \zeta_0$, $r\left(\zeta\right)\left(z'\left(\zeta\right)\right)^{\alpha} \in C^1[\zeta_x,\infty)$ for all $\zeta_x \geq \zeta_0$, and x satisfies (1) on $[\zeta_x,\infty)$, then x is called a solution of (1). Our attention will be solely to the solutions which satisfy

$\sup\{|x(\zeta)| : \zeta \geq \zeta\} > 0$, for all $\zeta \geq \zeta_x$. If there exists a $\zeta_1 \geq \zeta_0$ such that either $x(\zeta) > 0$ or $x(\zeta) < 0$ for all $\zeta \geq \zeta_1$, then x is said to be a non-oscillatory solution; otherwise, it is said to be an oscillatory solution.

First, we will shed light on the studies of canonical equations that require $\theta_{\zeta_0}(\infty) = \infty$. By employing the Riccati transformation, Sun and Meng [16] studied the oscillation of delay Equation (1), with $p(\zeta) = 0$ and $\alpha = \beta$. They proved that (1) is oscillatory if

$$\int_{\zeta_1}^{\infty} \left(\theta_{\zeta_1}^{\alpha}(\sigma(s))\, q(s) - \frac{\alpha^{\alpha+1}\sigma'(s)}{(\alpha+1)^{\alpha+1}\theta_{\zeta_1}(\sigma(s))\, r^{1/\alpha}(\sigma(s))} \right) \mathrm{d}s = \infty.$$

Likewise, Xu and Meng [17] extended the results of [16] to the neutral case, and proved that if

$$\int_{\zeta_1}^{\infty} \left(\theta_{\zeta_1}^{\alpha}(\sigma(s))\, G(s) - \frac{\alpha^{\alpha+1}\sigma'(s)}{(\alpha+1)^{\alpha+1}\theta_{\zeta_1}(\sigma(s))\, r^{1/\alpha}(\sigma(s))} \right) \mathrm{d}s = \infty,\ \alpha = \beta,$$

then (1) is oscillatory, where $G(\zeta) := q(\zeta)(1 - p(\sigma(\zeta)))^{\alpha}$.

By a different approach, by using comparison theorems that compare the second-order equation with first-order equations, Baculikova and Dzurina [5] proved that (1) is oscillatory if $\alpha \geq \beta$, $\sigma(\zeta) \leq \tau(\zeta)$ and

$$\liminf_{\zeta\to\infty} \int_{\tau^{-1}(\sigma(\zeta))}^{\zeta} \widehat{G}(s) \left(\int_{\zeta_1}^{\sigma(\zeta)} r^{-1/\alpha}(u)\,\mathrm{d}u \right)^{\beta} \mathrm{d}s > \left(1 + \frac{p_0^{\beta}}{\tau_0} \right)^{\beta/\alpha} \frac{1}{\kappa \mathrm{e}}, \tag{2}$$

where

$$\widehat{G}(\zeta) := \min\{q(\zeta), q(\tau(\zeta))\} \tag{3}$$

and

$$\kappa := \begin{cases} 1 & \text{if } 0 < \beta \leq 1; \\ 2^{1-\beta} & \text{if } \beta > 1. \end{cases} \tag{4}$$

Very recently, complementing the approach taken in [7], Grace et al. [11] improved the results in [5,13,17]. They established the following criteria for oscillation of (1) with $\alpha = \beta$:

(a) By using comparison theory:

$$\liminf_{\zeta\to\infty} \int_{\sigma(\zeta)}^{\zeta} G(s) \left(\theta_{\zeta_1}^{*}(\sigma(s)) \right)^{\alpha} \mathrm{d}s > \frac{1}{\mathrm{e}},$$

where

$$\theta_{\zeta_1}^{*}(\zeta) := \theta_{\zeta_1}(\zeta) + \frac{1}{\alpha} \int_{\zeta_1}^{\zeta} \theta_{\zeta_1}(u)\, \theta_{\zeta_1}^{\alpha}(u)\,\mathrm{d}u;$$

(b) By employing the Riccati transformation

$$\limsup_{\zeta\to\infty} \int_{\zeta_1}^{\infty} \left(\phi(s) \exp\left(-\int_{\sigma(s)}^{s} \frac{\mathrm{d}u}{r^{1/\alpha}(u)\, \theta_{\zeta_1}^{*}(u)} \right) - \frac{r(s)\left(\phi_{+}'(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\phi^{\alpha}(s)} \right) \mathrm{d}s = \infty, \tag{5}$$

where $\phi \in C([\zeta_0, \infty), (0, \infty))$ and $\phi_{+}'(\zeta) := \max\{\phi'(\zeta), 0\}$.

Moreover, Moaaz [14] extended the results of [11] to (1) when $\alpha > \beta$ and $\alpha < \beta$.

On the other hand, there are many studies to improve the criteria for oscillation of solutions of non-canonical equations $\theta_{\zeta_0}(\infty) < \infty$, some of which we will refer to below.

Recently, Agarwal et al. [4] established conditions for oscillation of (1) which are an improvement for its predecessors. They proved that (1), with $\beta = \alpha$, is oscillatory if

$$\limsup_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} \left(\rho(s) q(s) (1 - p(\sigma(s)))^{\alpha} - \frac{((\rho'(s))_+)^{\alpha+1} r(\tau(s))}{(\alpha+1)^{\alpha+1} \rho^{\alpha}(s) (\tau'(s))^{\alpha}} \right) \mathrm{d}s = \infty \tag{6}$$

and

$$\limsup_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} \left(\psi(s) - \frac{\delta(s) r(s) ((\varphi(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right) \mathrm{d}s = \infty \tag{7}$$

hold, where

$$\psi(\zeta) = \delta(\zeta) \left(q(\zeta) \left(1 - p(\sigma(\zeta)) \frac{\pi(\tau(\sigma(\zeta)))}{\pi(\sigma(\zeta))} \right)^{\alpha} + \frac{1-\alpha}{r^{1/\alpha}(\zeta) \pi^{\alpha+1}(\zeta)} \right),$$

and $\varphi(\zeta) = (\delta'(\zeta)/\delta(\zeta)) + (1+\alpha) / \left(r^{1/\alpha}(\zeta) \pi(\zeta) \right)$ and $(\varphi(\zeta))_+ = \max\{\varphi(\zeta), 0\}$.

Bohner et al. in [7], Theorem 2.4 simplified the criteria for oscillatory by setting one sufficient condition

$$\liminf_{\zeta\to\infty} \int_{\sigma(\zeta)}^{\zeta} \left(\frac{1}{r(\zeta)} \int_{\zeta_1}^{\zeta} q(\zeta) \left(1 - p(\sigma(\zeta)) \frac{\pi(\tau(\sigma(\zeta)))}{\pi(\sigma(\zeta))} \right)^{\alpha} \right)^{1/\alpha} \mathrm{d}s > \frac{1}{\mathrm{e}}.$$

The main purpose of this work is to improve conditions (2) and (5) by establishing a new criterion for oscillation of (1), which also takes into account the influence of delay argument $\tau(\zeta)$. Our approach is essentially based on presenting sharper criteria for oscillation solutions of (1) than criteria in [5,11]. Moreover, for non-canonical case, we improve and complete some results in [4,7]. As proof of the effectiveness of the new criteria, we offer more than one practical example.

2. Oscillation Theorems in Canonical Case

In this section, we establish new criteria for oscillation of solution of (1) in canonical case $\theta_{\zeta_0}(\infty) = \infty$. For convenience, we denote that: $Q(\zeta) := q(\zeta)(1 - p(\sigma(\zeta)))^{\beta}$,

$$\begin{aligned}
\widetilde{\theta}_{\zeta_0}(\zeta) &: = \theta_{\zeta_0}(\zeta) + \frac{1}{\alpha} \int_{\zeta_0}^{\zeta} \theta_{\zeta_0}(u) \theta_{\zeta_0}^{\alpha}(\sigma(\zeta)) \eta(\sigma(u)) Q(u) \mathrm{d}u, \\
\widehat{\theta}_{\zeta_0}(\zeta) &: = \exp\left(-\alpha \int_{\sigma(\zeta)}^{\zeta} \frac{1}{\widetilde{\theta}_{\zeta_0}(s) r^{1/\alpha}(s)} \mathrm{d}s \right), \\
\psi_k(\zeta) &: = \int_{\zeta}^{\infty} \widehat{\theta}^{k}(u) \eta(\sigma(u)) Q(u) \mathrm{d}u, \ k = 0, 1,
\end{aligned}$$

and

$$\eta(\zeta) := \begin{cases} c_1^{\beta-\alpha} & \text{if } \alpha \le \beta; \\ c_2 \theta_{\zeta_2}^{\beta-\alpha}(\zeta) & \text{if } \alpha > \beta, \end{cases}$$

where c_1 and c_2 are positive constants.

Lemma 1. *Assume that x is an eventually positive solution of (1). Then,*

$$z(\zeta) > 0, \ z'(\zeta) > 0 \ \text{and} \ \left(r(\zeta) (z'(\zeta))^{\alpha} \right) \le 0, \tag{8}$$

for $\zeta \ge \zeta_1$, where ζ_1 is sufficiently large. Moreover, $z^{\beta-\alpha}(\zeta) \ge \eta(\zeta)$, eventually.

Proof. First, we postulate that x is a positive solution of (1). From [5], Lemma 3, we have that (8) holds.

Next, let $\alpha \le \beta$. From the monotonicity of z, we get that $z(\zeta) \ge z(\zeta_2) := c_1 > 0$ for $\zeta \ge \zeta_2$, where ζ_2 is sufficiently large.

In the case where $\alpha > \beta$, since $r(\zeta)(z'(\zeta))^{\alpha}$ is non-increasing, we have that

$$z(\zeta) \leq z(\zeta_2) + r^{1/\alpha}(\zeta_2) z'(\zeta_2) \theta_{\zeta_2}(\zeta),$$

for $\zeta \geq \zeta_2$. In view of the fact that $\theta_{\zeta_2}(\infty) = \infty$, there exists a $N > 0$ and $\zeta_N > \zeta_2$ such that $\theta_{\zeta_2}(\zeta) > N$ for all $\zeta \geq \zeta_N$. Thus, $z(\zeta) \leq c_2 \theta_{\zeta_2}(\zeta)$ where $c_3 := \left(z(\zeta_2)/N + r^{1/\alpha}(\zeta_2) z'(\zeta_2)\right)$. Consequently, $z^{\beta-\alpha}(\zeta) \geq \eta(\zeta)$. The proof is complete. □

Theorem 1. *Assume that*

$$\sigma(\zeta) \leq \tau(\zeta), \ \tau'(\zeta) \geq \tau_0 > 0 \text{ and } \tau \circ \sigma = \sigma \circ \tau.$$

If

$$\liminf_{\zeta \to \infty} \int_{\tau^{-1}(\sigma(\zeta))}^{\zeta} \widehat{G}(s) \widetilde{\theta}_{\zeta_1}^{\beta}(\sigma(\zeta)) \, \mathrm{d}s > \left(1 + \frac{p_0^{\beta}}{\tau_0}\right)^{\beta/\alpha} \frac{1}{\kappa \mathrm{e}}, \tag{9}$$

where $\widehat{G}(s)$ and κ defined as in (3) and (4), respectively, then all solutions of (1) are oscillatory.

Proof. Suppose the contrary; that (1) has an eventually non-oscillatory solution. Without loss of generality, we assume that $x(\zeta) > 0$, $x(\tau(\zeta)) > 0$ and $x(\sigma(\zeta)) > 0$ for $\zeta \geq \zeta_1$, where ζ_1 is sufficiently large. By Lemma 1, we have that (8) holds. Taking (8) into account, we obtain $x(\zeta) \geq z(\zeta)(1 - p(\sigma(\zeta)))$, which with (1) gives

$$\left(r(\zeta)(z'(\zeta))^{\alpha}\right)' \leq -Q(\zeta) z^{\beta}(\sigma(\zeta)). \tag{10}$$

Using the chain rule and simple computation, we see that

$$\alpha \left(r^{1/\alpha} z'\right)^{\alpha-1} \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(z - \theta_{\zeta_1} r^{1/\alpha} z'\right) = -\alpha \left(r^{1/\alpha} z'\right)^{\alpha-1} \theta_{\zeta_1} \left(r^{1/\alpha} z'\right)' = -\theta_{\zeta_1}\left(r (z')^{\alpha}\right)'. \tag{11}$$

From (10) and (11), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(z(\zeta) - \theta_{\zeta_1}(\zeta) r^{1/\alpha}(\zeta) z'(\zeta)\right) &\geq \frac{1}{\alpha}\left(r^{1/\alpha}(\zeta) z'(\zeta)\right)^{1-\alpha} \theta_{\zeta_1}(\zeta) Q(\zeta) z^{\beta}(\sigma(\zeta)) \\
&\geq \frac{1}{\alpha}\left(r^{1/\alpha}(\zeta) z'(\zeta)\right)^{1-\alpha} \theta_{\zeta_1}(\zeta) Q(\zeta) \eta(\sigma(\zeta)) z^{\alpha}(\sigma(\zeta)).
\end{aligned}$$

Integrating this inequality from ζ_1 to ζ, we arrive at

$$z(\zeta) \geq \theta_{\zeta_1}(\zeta) r^{1/\alpha}(\zeta) z'(\zeta) + \frac{1}{\alpha}\int_{\zeta_1}^{\zeta}\left(r^{1/\alpha}(u) z'(u)\right)^{1-\alpha} \theta_{\zeta_1}(u) Q(u) \eta(\sigma(u)) z^{\alpha}(\sigma(u)) \, \mathrm{d}u. \tag{12}$$

From the monotonicity of $r^{1/\alpha}(\zeta) z'(\zeta)$, we have

$$z(\sigma(\zeta)) \geq \theta_{\zeta_1}(\sigma(\zeta)) r^{1/\alpha}(\sigma(\zeta)) z'(\sigma(\zeta)) \geq \theta_{\zeta_1}(\sigma(\zeta)) r^{1/\alpha}(\zeta) z'(\zeta).$$

Thus, (12) becomes

$$z(\zeta) \geq \widetilde{\theta}_{\zeta_1}(\zeta) r^{1/\alpha}(\zeta) z'(\zeta). \tag{13}$$

Proceeding as in the proof of Theorem 1 in [5] and using (13) instead of ((2.10) in [5]), we arrive at

$$\left(\omega(\zeta) + \frac{p_0^{\beta}}{\tau_0}\omega(\tau(\zeta))\right)' + \kappa \widehat{G}(\zeta) \widetilde{\theta}_{\zeta_1}^{\beta}(\sigma(\zeta)) \omega^{\beta/\alpha}(\sigma(\zeta)) \leq 0,$$

where $\omega := r^{1/\alpha} z'$. Next, set $\psi(\zeta) := \omega(\zeta) + \left(p_0^{\beta}/\tau_0\right)\omega(\tau(\zeta)) > 0$. Since $\tau(\zeta) \le \zeta$ and $\omega'(t) \le 0$, we obtain

$$\begin{aligned}\psi(\zeta) &\le \omega(\tau(\zeta)) + \left(p_0^{\beta}/\tau_0\right)\omega(\tau(\zeta)) \\ &\le \omega(\tau(\zeta))\left(1 + \frac{p_0^{\beta}}{\tau_0}\right).\end{aligned}$$

Thus, ψ is a positive solution of

$$\psi'(\zeta) + \kappa\left(\frac{\tau_0}{\tau_0 + p_0^{\beta}}\right)^{\beta/\alpha} \widehat{G}(\zeta)\,\widetilde{\theta}_{\zeta_1}^{\beta}(\sigma(\zeta))\,\psi^{\beta/\alpha}\left(\tau^{-1}(\sigma(\zeta))\right) \le 0.$$

From Theorem 1 in [26], the equation

$$\psi'(\zeta) + \kappa\left(\frac{\tau_0}{\tau_0 + p_0^{\beta}}\right)^{\beta/\alpha} \widehat{G}(\zeta)\,\widetilde{\theta}_{\zeta_1}^{\beta}(\sigma(\zeta))\,\psi^{\beta/\alpha}\left(\tau^{-1}(\sigma(\zeta))\right) = 0, \tag{14}$$

also has a positive solution. It is well-known (see, e.g., [27], Theorem 2) that condition (9) implies oscillation of (14). This contradiction completes the proof. □

Theorem 2. *If*

$$\liminf_{\zeta\to\infty} \frac{\alpha}{\psi_1(\zeta)} \int_{\zeta}^{\infty} r^{-1/\alpha}(u)\,\psi_1^{(\alpha+1)/\alpha}(u)\,\mathrm{d}u > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{15}$$

then all solutions of (1) are oscillatory.

Proof. Proceeding as in the proof of Theorem 1, we arrive at (13). From (13), we see that

$$\frac{z(\sigma(\zeta))}{z(\zeta)} \ge \exp\left(-\int_{\sigma(\zeta)}^{\zeta} \frac{1}{\widetilde{\theta}_{\zeta_1}(s)\,r^{1/\alpha}(s)}\mathrm{d}s\right). \tag{16}$$

Define the function

$$w(\zeta) := \frac{r(\zeta)\left(z'(\zeta)\right)^{\alpha}}{z^{\alpha}(\zeta)}. \tag{17}$$

Then $\omega(\zeta) > 0$ for $\zeta \ge \zeta_1$. From (1) and (17), we get

$$w'(\zeta) \le -Q(\zeta)\frac{z^{\alpha}(\sigma(\zeta))}{z^{\alpha}(\zeta)} z^{\beta-\alpha}(\sigma(\zeta)) - \frac{\alpha}{r^{1/\alpha}(\zeta)} w^{(\alpha+1)/\alpha}(\zeta).$$

Using Lemma 1 and (16), we obtain

$$w'(\zeta) \le -\widehat{\theta}(\zeta)\,\eta(\sigma(\zeta))\,Q(\zeta) - \frac{\alpha}{r^{1/\alpha}(\zeta)} w^{(\alpha+1)/\alpha}(\zeta) < 0. \tag{18}$$

By integrating (18) from ζ to s, we conclude that

$$\int_{\zeta}^{s} \widehat{\theta}(u)\,\eta(\sigma(u))\,Q(u)\,\mathrm{d}u + \alpha\int_{\zeta}^{s} r^{-1/\alpha}(u)\,w^{(\alpha+1)/\alpha}(u)\,\mathrm{d}u \le w(\zeta) - w(s).$$

Since w is positive decreasing function, we see that

$$\psi_1(\zeta) + \alpha\int_{\zeta}^{\infty} r^{-1/\alpha}(u)\,w^{(\alpha+1)/\alpha}(u)\,\mathrm{d}u \le w(\zeta).$$

Hence,

$$1+\frac{\alpha}{\psi_1(\zeta)}\int_{\zeta}^{\infty} r^{-1/\alpha}(u)\,\psi^{(\alpha+1)/\alpha}(u)\left(\frac{w(u)}{\psi(u)}\right)^{(\alpha+1)/\alpha}\mathrm{d}u \le \frac{w(\zeta)}{\psi_1(\zeta)}. \tag{19}$$

Set

$$\delta := \inf_{\zeta\ge\zeta_1}\frac{w(\zeta)}{\psi_1(\zeta)}.$$

From (19), $\delta \ge 1$. Taking (15) and (19) into account, we find

$$1+\alpha\left(\frac{\delta}{\alpha+1}\right)^{1+1/\alpha} \le \delta$$

or

$$\left(\frac{\delta}{\alpha+1}\right)^{\alpha+1} \le \left(\frac{\delta-1}{\alpha}\right)^{\alpha},$$

which is not possible with the permissible value $\alpha > 0$ and $\delta \ge 1$. This contradiction completes the proof. □

In the following, we give an example to illustrate our main results.

Example 1. *Consider the differential equation*

$$\left(\left(\left(x(\zeta)+p_0x(\mu\zeta)\right)'\right)^{\alpha}\right)'+\frac{q_0}{\zeta^{\alpha+1}}x^{\alpha}(\lambda\zeta)=0, \tag{20}$$

where $q_0 > 0$ *and* $\mu,\lambda \in (0,1)$. *We note that* $\theta_{\zeta_0}(\zeta)=\zeta$,

$$Q(\zeta)=\frac{q_0}{\zeta^{\alpha+1}}(1-p_0)^{\alpha},\ \widehat{G}(\zeta):=\frac{q_0}{\zeta^{\alpha+1}},\ \widetilde{\theta}_{\zeta_0}(\zeta)=A\zeta,\ \widehat{\theta}(\zeta)=\lambda^{\alpha/A}$$

and

$$\psi_1(\zeta)=\frac{1}{\alpha}\lambda^{\alpha/A}q_0(1-p_0)^{\alpha}\frac{1}{\zeta^{\alpha}},$$

where $A := 1+\frac{1}{\alpha}q_0\lambda^{\alpha}(1-p_0)^{\alpha}$.

From Theorem 1, Equation (20) is oscillatory if $\lambda < \mu$ *and*

$$q_0(A\lambda)^{\alpha}\ln\frac{\mu}{\lambda} > \left(1+\frac{p_0^{\beta}}{\mu}\right)^{\beta/\alpha}\frac{1}{\kappa \mathrm{e}}. \tag{21}$$

Using Theorem 2, we have that if

$$\left(\frac{1}{\alpha}\lambda^{\alpha/A}q_0(1-p_0)^{\alpha}\right)^{1/\alpha} > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}$$

or

$$q_0\lambda^{\alpha/A}(1-p_0)^{\alpha} > \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}, \tag{22}$$

then (20) is oscillatory

As a particular case, the known criteria for oscillation of equation

$$\left(\left((x(\zeta))'\right)^{1/3}\right)'+\frac{q_0}{\zeta^{4/3}}x^{1/3}\left(\frac{9}{10}\zeta\right)=0, \tag{23}$$

where $q_0 > 0$, are:

Using Comparison Theory (As in [5])		
$\mathbf{C}_1$	Corollary 2 in [5]	$q_0 > 3.61643$
$\mathbf{C}_2$	Condition (21)	$q_0 > 1.92916$
Using comparison theory (as in [11])		
$\mathbf{C}_3$	Corollary 2.1 in [11]	$q_0 > 1.92916$
By employing the Riccati transformation		
$\mathbf{C}_4$	Corollary 2.1 in [16]	$q_0 > 0.16312$
$\mathbf{C}_5$	Theorem 6 in [11]	$q_0 > 0.16243$
$\mathbf{C}_6$	Condition (22)	$q_0 > 0.16131$

Remark 1. *Note that,Theorem 1 improves Theorem 2 in [5]. The criterion of oscillation in Theorem 1 (as (21)) essentially takes into account the influence of delay argument $\tau(\zeta)$, which has been neglected in results of [11]. Moreover, Theorem 2 improves Theorem 6 in [11], and supports the most efficient condition for oscillation of Equation (20).*

3. Oscillation Theorems in Non-Canonical Case

In the following, we derive new criteria for oscillation of solution of (1) in non-canonical case $\theta_{\zeta_0}(\infty) < \infty$. For convenience, we denote that: $\theta_\zeta(\infty) := \overline{\theta}(\zeta)$ and

$$\Theta(\zeta) := q(\zeta)\left(1 - p(\sigma(\zeta))\frac{\overline{\theta}(\tau(\sigma(\zeta)))}{\overline{\theta}(\sigma(\zeta))}\right)^{\beta}.$$

Lemma 2. *Let $\Phi(\varrho) = L\varrho - M(\varrho - N)^{(\alpha+1)/\alpha}$, where $M > 0$, L and N are constants. Then, the maximum value of Φ on $\mathbb{R}$ is at $\varrho^* = N + (\alpha L/((\alpha+1)M))^{\alpha}$ and is given by*

$$\max_{\varrho\in R} \Phi(\varrho) = \Phi(\varrho^*) = LN + \frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)}}\frac{L^{\alpha+1}}{M^{\alpha}}.$$

Lemma 3. *Let x be an eventually positive solution of (1), $z'(\zeta) < 0$ and*

$$\int_{\zeta_0}^{\infty}\left(\frac{1}{r(\zeta)}\int_{\zeta_1}^{\zeta}\Theta(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\zeta = \infty \tag{24}$$

holds. If there exists a constant $\delta \in [0,1)$ such that

$$\overline{\theta}(\zeta)\left(\eta(\zeta)\int_{\zeta_0}^{\zeta}\Theta(s)\,\mathrm{d}s\right)^{1/\alpha} \geq \delta, \tag{25}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{z(\zeta)}{\overline{\theta}^{\delta}(\zeta)}\right) \leq 0. \tag{26}$$

Proof. Suppose that x positive solution of (1) and $z'(\zeta) < 0$. We assume that $x(\zeta) > 0$, $x(\tau(\zeta)) > 0$ and $x(\sigma(\zeta)) > 0$ for $\zeta \geq \zeta_1$, where ζ_1 is sufficiently large. Then, from (1), we obtain $z(\zeta) > 0$ and

$$\left(r(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' = -q(\zeta)\,x^{\beta}(\sigma(\zeta)) < 0. \tag{27}$$

Thus, we have

$$z(\zeta) \geq -\int_{\zeta}^{\infty} \frac{1}{r^{1/\alpha}(\varrho)} r^{1/\alpha}(\varrho) z'(\varrho)\, \mathrm{d}\varrho \geq -\overline{\theta}(\zeta) r^{1/\alpha}(\zeta) z'(\zeta) \tag{28}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{z(\zeta)}{\overline{\theta}(\zeta)}\right) = \frac{\overline{\theta}(\zeta) z'(\zeta) + (r(\zeta))^{-1/\alpha} z(\zeta)}{\overline{\theta}^2(\zeta)} \geq 0.$$

Hence, we find

$$x(\zeta) \geq \left(1 - p(\zeta)\frac{\overline{\theta}(\tau(\zeta))}{\overline{\theta}(\zeta)}\right) z(\zeta),$$

which with (27) gives

$$\left(r(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' \leq -\Theta(\zeta) z^{\beta}(\sigma(\zeta)). \tag{29}$$

Next, since z is a positive decreasing function, we have that $\lim_{\zeta\to\infty} z(\zeta) = \epsilon \geq 0$. Let $\epsilon > 0$. Then, from (29), there exists a $\zeta_1 \geq \zeta_0$ such that

$$\left(r(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' \leq -\Theta(\zeta) z^{\beta}(\sigma(\zeta)) \leq -\epsilon^{\beta}\Theta(\zeta).$$

Integrating this inequality from ζ_1 to ζ, we have

$$-z'(\zeta) \geq \epsilon^{\beta/\alpha}\left(\frac{1}{r(\zeta)}\int_{\zeta_1}^{\zeta}\Theta(s)\,\mathrm{d}s\right)^{1/\alpha}. \tag{30}$$

Integrating (30) from ζ_1 to ζ, we get

$$z(\zeta) \leq z(\zeta_1) - \epsilon^{\beta/\alpha}\int_{\zeta_1}^{\zeta}\left(\frac{1}{r(\varrho)}\int_{\zeta_1}^{\varrho}\Theta(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\varrho.$$

Taking $\lim_{\zeta\to\infty}$ of this inequality and using (24), we arrive at a contradiction with positivity of z. Therefore, we get that $\epsilon = 0$ that is

$$\lim_{\zeta\to\infty} x(\zeta) = \lim_{\zeta\to\infty} z(\zeta) = 0. \tag{31}$$

Integrating (29) from ζ_1 to ζ, we get

$$\begin{aligned} r(\zeta)\left(z'(\zeta)\right)^{\alpha} &\leq r(\zeta_1)\left(z'(\zeta_1)\right)^{\alpha} - \int_{\zeta_1}^{\zeta}\Theta(s) z^{\beta}(\sigma(s))\,\mathrm{d}s \\ &\leq r(\zeta_1)\left(z'(\zeta_1)\right)^{\alpha} - z^{\beta}(\sigma(\zeta))\int_{\zeta_1}^{\zeta}\Theta(s)\,\mathrm{d}s. \end{aligned} \tag{32}$$

In view of (31), we see that

$$r(\zeta_1)\left(z'(\zeta_1)\right)^{\alpha} + z^{\beta}(\sigma(\zeta))\int_{\zeta_0}^{\zeta_1}\Theta(s)\,\mathrm{d}s > 0, \tag{33}$$

for $\zeta \geq \zeta_2$, where ζ_2 large enough. Combining (32) and (33) and using the fact that $z^{\beta-\alpha}(\zeta) \geq \eta(\zeta)$, we find

$$r(\zeta)\left(z'(\zeta)\right)^{\alpha} \leq -z^{\beta}(\sigma(\zeta))\int_{\zeta_0}^{\zeta}\Theta(s)\,\mathrm{d}s \leq -\eta(\zeta) z^{\alpha}(\zeta)\int_{\zeta_0}^{\zeta}\Theta(s)\,\mathrm{d}s, \tag{34}$$

which with (25) gives

$$\begin{aligned} z'(\zeta) &\le -r(\zeta)^{-1/\alpha} z(\zeta)\left(\eta(\zeta)\int_{\zeta_0}^{\zeta}\Theta(s)\,\mathrm{d}s\right)^{1/\alpha} \\ &\le -\frac{\delta}{\overline{\theta}(\zeta)} r(\zeta)^{-1/\alpha} z(\zeta). \end{aligned}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{z(\zeta)}{\overline{\theta}^{\delta}(\zeta)}\right) = \frac{\overline{\theta}^{\delta-1}(\zeta)\left(\overline{\theta}(\zeta)z'(\zeta)+\delta(\mu(\zeta)r(\zeta))^{-1/\alpha}z(\zeta)\right)}{\overline{\theta}^{2\delta}(\zeta)} \le 0.$$

This completes the proof. □

Theorem 3. *Assume that (24) holds, $\overline{\theta}(\tau(\zeta)) \ge \overline{\theta}(\zeta)$ and there exists a $\delta \in [0,1)$ such that (25) holds. If there exist a function $\rho \in C^1([\zeta_0,\infty),(0,\infty))$ and a $\zeta_1 \in [\zeta_0,\infty)$ such that*

$$\limsup_{\zeta\to\infty}\left\{\frac{\overline{\theta}^{\alpha}(\zeta)}{\rho(\zeta)}\int_{\zeta_1}^{\zeta}\left(\rho(\varrho)\Theta(\varrho)\eta(\varrho)\left(\frac{\overline{\theta}(\sigma(\varrho))}{\overline{\theta}(\varrho)}\right)^{\delta\beta} - \frac{r(\varrho)(\rho'(\varrho))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(\varrho)}\right)\mathrm{d}\varrho\right\} > 1, \tag{35}$$

then all solutions of (1) are oscillatory.

Proof. Assuming that the required result is not fulfilled, we suppose, without loss of generality, that x is a positive solution of (1) on $[\zeta_0,\infty)$. Then, there exists $\zeta_1 \ge \zeta_0$ such that $x(\tau(\zeta)) > 0$ and $x(\sigma(\zeta)) > 0$ for all $\zeta \ge \zeta_1$. Obviously, for all $\zeta \ge \zeta_1$, $z(\zeta) \ge x(\zeta) > 0$ and $\left(r(z')^{\alpha}\right)' \le 0$. Therefore, $z'(\zeta)$ is either eventually negative or eventually positive.

Assume first that $z'(\zeta) < 0$. From Lemma 3, we get that (28) and (29) hold. We define the function

$$w := \rho\left(\frac{r(z')^{\alpha}}{z^{\alpha}} + \frac{1}{\overline{\theta}^{\alpha}}\right). \tag{36}$$

Using (28), we see that $w(\zeta) \ge 0$ for all $\zeta \ge \zeta_2 \ge \zeta_1$. Differentiating (36), we get

$$w' = \frac{\rho'}{\rho}w + \rho\frac{\left(r(z')^{\alpha}\right)'}{z^{\alpha}} - \alpha\rho\frac{r(z')^{\alpha+1}}{z^{\alpha+1}} + \alpha\rho\frac{1}{r^{1/\alpha}\overline{\theta}^{\alpha+1}}.$$

From Lemma 1, (29) and (36), we obtain

$$w' \le -\frac{\alpha}{(\rho r)^{1/\alpha}}\left(w - \frac{\rho}{\overline{\theta}^{\alpha}}\right)^{1+1/\alpha} - \rho\Theta\eta\frac{z^{\beta}(\sigma)}{z^{\beta}} + \alpha\rho\frac{1}{r^{1/\alpha}\overline{\theta}^{\alpha+1}} + \frac{\rho'}{\rho}w. \tag{37}$$

Using Lemma 2 with $L := \rho'/\rho$, $M := \alpha/(\rho r)^{1/\alpha}$, $N = \rho/\overline{\theta}^{\alpha}$ and $\varrho := w$, we get

$$\frac{\rho'}{\rho}w - \frac{\alpha}{(\rho r)^{1/\alpha}}\left(w - \frac{\rho}{\overline{\theta}^{\alpha}}\right)^{1+1/\alpha} \le \frac{\rho'}{\overline{\theta}^{\alpha}} + \frac{r}{(\alpha+1)^{\alpha+1}}\frac{(\rho')^{\alpha+1}}{\rho^{\alpha}},$$

which with (37) gives

$$w' \le \frac{\rho'}{\overline{\theta}^{\alpha}} + \frac{r}{(\alpha+1)^{\alpha+1}}\frac{(\rho')^{\alpha+1}}{\rho^{\alpha}} - \rho\Theta\eta\frac{z^{\beta}(\sigma)}{z^{\beta}} + \alpha\rho\frac{1}{r^{1/\alpha}\overline{\theta}^{\alpha+1}}. \tag{38}$$

From Lemma 3 we arrive at (26). From (26), we conclude that

$$\frac{z(\sigma(\zeta))}{z(\zeta)} \geq \frac{\overline{\theta}^{\delta}(\sigma(\zeta))}{\overline{\theta}^{\delta}(\zeta)}. \tag{39}$$

Thus, (38) becomes

$$w' \leq -\rho\Theta\eta\left(\frac{\overline{\theta}(\sigma(\zeta))}{\overline{\theta}(\zeta)}\right)^{\delta\beta} + \frac{r}{(\alpha+1)^{\alpha+1}}\frac{(\rho')^{\alpha+1}}{\rho^{\alpha}} + \left(\frac{\rho}{\overline{\theta}^{\alpha}}\right)'.$$

Integrating this inequality from ζ_2 to ζ, we have

$$\begin{aligned}\int_{\zeta_2}^{\zeta}\left(\rho(\varrho)\Theta(\varrho)\eta(\varrho)\left(\frac{\overline{\theta}(\sigma(\varrho))}{\overline{\theta}(\varrho)}\right)^{\delta\beta} - \frac{r(\varrho)(\rho'(\varrho))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(\varrho)}\right)\mathrm{d}\varrho \\ \leq \left(\frac{\rho(\zeta)}{\overline{\theta}^{\alpha}(\zeta)} - w(\zeta)\right)\Bigg|_{\zeta_2}^{\zeta} \\ \leq -\left(\rho(\zeta)\frac{r(\zeta)(z'(\zeta))^{\alpha}}{z^{\alpha}(\zeta)}\right)\Bigg|_{\zeta_2}^{\zeta}.\end{aligned} \tag{40}$$

From (28), we have

$$-\frac{r(\zeta)^{1/\alpha}z'(\zeta)}{z(\zeta)} \leq \frac{1}{\overline{\theta}(\zeta)},$$

which in view of (40) implies

$$\frac{\overline{\theta}^{\alpha}(\zeta)}{\rho(\zeta)}\int_{\zeta_2}^{\zeta}\left(\rho(\varrho)\Theta(\varrho)\eta(\varrho)\left(\frac{\overline{\theta}(\sigma(\varrho))}{\overline{\theta}(\varrho)}\right)^{\delta\beta} - \frac{r(\varrho)(\rho'(\varrho))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(\varrho)}\right)\mathrm{d}\varrho \leq 1.$$

Taking the lim sup on both sides of this inequality, we arrive at a contradiction with (35).
Now assume that $z'(\zeta) > 0$. Since $z' > 0$ and $\overline{\theta}(\tau(\zeta)) \geq \overline{\theta}(\zeta)$. Then

$$x(\zeta) \geq (1-p(\zeta))z(\zeta) \geq \left(1 - p(\zeta)\frac{\overline{\theta}(\tau(\zeta))}{\overline{\theta}(\zeta)}\right)z(\zeta). \tag{41}$$

From (1) and (41), we have

$$\begin{aligned}\left(r(\zeta)(z'(\zeta))^{\alpha}\right)' &\leq -q(\zeta)\left(1 - p(\sigma(\zeta))\frac{\overline{\theta}(\tau(\sigma(\zeta)))}{\overline{\theta}(\sigma(\zeta))}\right)^{\beta} z^{\beta}(\sigma(\zeta)) \\ &\leq -\Theta(\zeta)z^{\beta}(\sigma(\zeta)).\end{aligned}$$

Integrating the above inequality from ζ_2 to ζ, we have

$$\begin{aligned}r(\zeta)(z'(\zeta))^{\alpha} &\leq r(\zeta_2)(z'(\zeta_2))^{\alpha} - \int_{\zeta_2}^{\zeta}\Theta(s)z^{\beta}(\sigma(s))\,\mathrm{d}s \\ &\leq r(\zeta_2)(z'(\zeta_2))^{\alpha} - z^{\beta}(\sigma(\zeta_2))\int_{\zeta_2}^{\zeta}\Theta(s)\,\mathrm{d}s.\end{aligned} \tag{42}$$

Now, from (24) and $\theta_{\zeta_0}(\infty) < \infty$, we get that $\int_{\zeta_1}^{\zeta}\Theta(s)\,\mathrm{d}s$ must be unbounded, that is

$$\int_{\zeta_1}^{\infty}\Theta(s)\,\mathrm{d}s = \infty. \tag{43}$$

From (42) and (43), we get contradictions to $z' > 0$ as $\zeta \to \infty$. The proof is complete. □

Corollary 1. *Assume that (24) holds, $\overline{\theta}(\tau(\zeta)) \geq \overline{\theta}(\zeta)$ and there exists a $\delta \in [0,1)$ such that (25) holds. If there exists a $\zeta_1 \in [\zeta_0, \infty)$ such that*

$$\limsup_{\zeta\to\infty} \int_{\zeta_1}^{\zeta} \left(\overline{\theta}^{\alpha}(\varrho)\,\Theta(\varrho)\,\eta(\varrho) \left(\frac{\overline{\theta}(\sigma(\varrho))}{\overline{\theta}(\varrho)} \right)^{\delta\beta} - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{r^{1/\alpha}(\varrho)\,\overline{\theta}(\varrho)} \right) \mathrm{d}\varrho > 1, \tag{44}$$

then all solutions of (1) are oscillatory.

Example 2. *Consider the second-order differential equation*

$$\left(\zeta^{\alpha+1} \left((x(\zeta) + p_0 x(\kappa\zeta))' \right)^{\alpha} \right)' + q_0 x^{\alpha}(\lambda\zeta) = 0,\ \zeta \geq 1, \tag{45}$$

where $q_0 > 0$ and $\kappa, \lambda \in (0,1]$. Note that, $r(\zeta) = \zeta^{\alpha+1}$, $p(\zeta) = p_0$, $\tau(\zeta) = \kappa\zeta$, $q(\zeta) = q_0$, $\sigma(\zeta) = \lambda\zeta$ and $\beta = \alpha$. It is easy to conclude that $\eta(\zeta) = 1$, $\overline{\theta}(\zeta) = \alpha/\zeta^{1/\alpha}$ and $\Theta(\zeta) = q_0 \left(1 - 2^{1/\alpha} p_0\right)^{\alpha}$. We note that the conditions (24) and (25) hold. Furthermore, condition (44) becomes

$$\limsup_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} \left(\frac{\alpha^{\alpha}}{s} q_0 \left(1 - 2^{1/\alpha} p_0\right)^{\alpha} \frac{1}{\lambda^{\delta}} - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{\alpha s} \right) \mathrm{d}s > 1.$$

Using Corollary 1, we get that every solution of Equation (45) is oscillatory if

$$q_0 \left(1 - 2^{1/\alpha} p_0\right)^{\alpha} \frac{1}{\lambda^{\delta}} > \frac{1}{(\alpha+1)^{\alpha+1}}. \tag{46}$$

From Theorem 2.2 in [4], (45) is oscillatory if $\alpha \geq 1$ and

$$q_0 \left(1 - 2^{1/\alpha} p_0\right)^{\alpha} > \frac{1 - (1-\alpha)(\alpha+1)^{\alpha+1}}{\alpha^{\alpha+1}(\alpha+1)^{\alpha+1}}. \tag{47}$$

By using Theorem 2.4 in [7], (45) is oscillatory if

$$q_0^{1/\alpha} \left(1 - 2^{1/\alpha} p_0\right) \ln \frac{1}{\lambda} > \frac{1}{\mathrm{e}}. \tag{48}$$

Consider the special case in which $\alpha = 1$, $\kappa = \lambda = 1/2$, $p_0 = 1/4$ and $q_0 = \sqrt{2}/3$, that is,

$$\left(\zeta^2 \left(\left(x(\zeta) + \frac{1}{4} x\left(\frac{1}{2}\zeta\right) \right)' \right)^{\alpha} \right)' + \frac{\sqrt{2}}{3} x\left(\frac{1}{2}\zeta\right) = 0.$$

We note that the criteria arrive at $\sqrt{2}/3 > 0.4307$, $\sqrt{2}/3 > 0.5$ and $\sqrt{2}/3 > 1.0615$. Hence, it is obvious that (47) and (48) fail to apply.

4. Conclusions

During this work, we highlighted the oscillatory properties of solutions of differential Equation (1). By using many techniques, we have created new criteria that are more effective than the relevant criteria in the literature. Moreover, we discussed oscillatory behavior in both canonical and non-canonical cases. Through the examples, it turns out that our results improve and complete some of the results in [4,5,7,11,16]. Finally, we can try to extend our results to differential equations with a damping term, in the future.

Author Contributions: Formal analysis, O.M., M.A., D.B., A.M.; investigation, O.M., D.B.; writing—original draft preparation, O.M., D.B.; writing—review and editing, M.A., A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Springer Science & Business, Media: Berlin/Heidelberg, Germany, 2013.
2. Bainov, D.D.; Mishev, D.P. *Oscillation Theory for Neutral Differential Equations with Delay*; Adam Hilger: New York, NY, USA, 1991.
3. Saker, S.H. *Oscillation Theory of Delay Differential and Difference Equations*; VDM Verlag Dr. Muller Saarbrucken: Saarbrucken, Germany, 2010.
4. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [CrossRef]
5. Baculikova, B.; Dzurina, J. Oscillation theorems for second-order nonlinear neutral differential equations. *Comput. Math. Appl.* **2011**, *62*, 4472–4478. [CrossRef]
6. Bazighifan, O.; Cesarano, C. Some New Oscillation Criteria for Second Order Neutral Differential Equations with Delayed Arguments. *Mathematics* **2019**, *7*, 619. [CrossRef]
7. Bohner, M.; Grace, S.R.; Jadlovska, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *2017*, 1–12. [CrossRef]
8. Chatzarakis, G.E.; Dzurina, J.; Jadlovska, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [CrossRef]
9. Chatzarakis, G.E.; Jadlovska, I.; Li, T. Oscillations of differential equations with non-monotone deviating arguments. *Adv. Differ. Equ.* **2019**, *2019*, 1–20. [CrossRef]
10. Chatzarakis, G.E.; Moaaz, O.; Li, T.; Qaraad, B. Some oscillation theorems for nonlinear second-order differential equations with an advanced argument. *Adv. Differ. Equ.* **2020**, *2020*, 1–17. [CrossRef]
11. Grace, S.R.; Dzurina, J.; Jadlovska, I.; Li, T. An improved approach for studying oscillation of second-order neutral delay differential equations. *J. Inequal. Appl.* **2018**, *2018*, 1–13. [CrossRef]
12. Elabbasy, E.M.; Hassan, T.S.; Moaaz, O. Oscillation behavior of second order nonlinear neutral differential equations with deviating arguments. *Opuscula Math.* **2012**, *32*, 719–730. [CrossRef]
13. Koplatadze, R.; Kvinikadze, G.; Stavroulakis, I.P. Oscillation of second order linear delay differential equations. *Funct. Differ. Equ.* **2000**, *7*, 121–145.
14. Moaaz, O. New criteria for oscillation of nonlinear neutral differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 1–11. [CrossRef]
15. Moaaz, O.; Elabbasy, E.M.; Qaraad, B. An improved approach for studying oscillation of generalized Emden–Fowler neutral differential equation. *J. Inequal. Appl.* **2020**, *2020*, 1–18 [CrossRef]
16. Sun, Y.G.; Meng, F.W. Note on the paper of Dzurina and Stavroulakis: "Oscillation criteria for second-order delay differential equations" [Appl. Math. Comput. 140 (2003) 445–453]. *Appl. Math. Comput.* **2006**, *174*, 1634–1641.
17. Xu, R.; Meng, F. Some new oscillation criteria for second order quasi-linear neutral delay differential equations. *Appl. Math. Comput.* **2006**, *182*, 797–803. [CrossRef]
18. Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequal. Appl.* **2019**, *2019*, 1–9. [CrossRef]
19. El-Nabulsi R.A.; Moaaz O.; Bazighifan O. New results for oscillatory behavior of fourth-order differential equations. *Symmetry* **2020**, *12*, 136. [CrossRef]
20. Moaaz, O.; Awrejcewicz, J.; Bazighifan, O. A new approach in the study of oscillation criteria of even-Order neutral differential equations. *Mathematics* **2020**, *8*, 197. [CrossRef]
21. Moaaz, O.; Furuichi, S.; Muhib, A. New comparison theorems for the nth order neutral differential equations with delay inequalities. *Mathematics* **2020**, *8*, 454. [CrossRef]
22. Moaaz, O.; Muhib, A. New oscillation criteria for nonlinear delay differential equations of fourth-order. *Appl. Math. Comput.* **2020**, *377*, 125192. [CrossRef]
23. Moaaz, O.; Elabbasy, E.M.; Muhib, A. Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Equ.* **2019**, *2019*, 297. [CrossRef]
24. Elsgolts, L.E.; Norkin, S.B. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*; Academic Press: New York, NY, USA, 1973.

25. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
26. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
27. Kitamura, Y.; Kusano, T. Oscillation of first-order nonlinear differential equations with deviating arguments. *Proc. Amer. Math. Soc.* **1980**, *78*, 64–68. [CrossRef]