



Article On the Effect of the Absorption Coefficient in a Differential Game of Pollution Control

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Abstract: We consider various approaches for a characteristic function construction on the example of an *n* players differential game of pollution control with a prescribed duration. We explore the effect of the presence of an absorption coefficient in the game on characteristic functions. As an illustration, we consider a game in which the parameters are calculated based on the real ecological situation of the Irkutsk region. For this game, we compute a number of characteristic functions and compare their properties.

Keywords: differential games; prescribed duration; characteristic function; environmental resource management; pollution control

1. Introduction

Differential games are used for describing continuous processes of decision making in conflict situations that happen in industry, ecology, biology, political science, and so on. Models of differential games are often utilized for solving problems in the field of environmental protection policy and the optimal exploitation of natural resources [1–7].

To solve the problems of environmental management, it is effective to use games with negative externalities [4–7]. In that class of games, the increasing of the controls of some players, which are the volumes of environmental pollution over time, leads to the decreasing value of the payoff functions for others.

In this paper, we consider a cooperative differential game of pollution control with negative externalities modeling the behavior of several enterprises. They have an agreement to limit environmental pollution. This limitation has a negative effect on the total profit of each enterprise. Moreover, the high level of environmental pollution leads to profit loss associated with the higher environmental costs and taxes. We focus our attention on two cases when there is an absorption coefficient and when there is not.

In the theory of cooperative differential games, the concept of a characteristic function is one of the basic ones. The characteristic function shows the worth of a coalition and affects the formation of coalitions [8,9]. Therefore, players or coalitions have the motivation to cooperate with each other if the characteristic function of the joint coalition is greater than the sum of the original characteristic functions. Additionally, the significance of each player in the coalition can be determined by its marginal contribution to the characteristic function. For example, this approach is used in constructing the Shapley value and Banzhaf power index [10–12].

We present different techniques for characteristic function construction [13–15]. For the above-mentioned cooperative differential game, we compute a number of characteristic functions and compare their properties. Further, we analyze the effect of the absorption coefficient representing the natural environment purification on player's payoffs.

As an illustration, we explore a game of pollution control based on the acute ecological situation of the Irkutsk region. We use a cooperative differential game of three players, which are the largest enterprises of Bratsk [16]. Furthermore, we add the absorption coefficient to this model and focus on the effect this coefficient has on the payoffs.

2. Cooperative Differential Game in the form of the Characteristic Function

2.1. Differential Game in Normal Form

Let $N = \{1, 2, ..., n\}$ be a set of players participating in a classical cooperative differential game $\Gamma(x_0, t_0, T)$ with a prescribed duration [17]. The game starts from the initial state x_0 at time t_0 and evolves over the interval $t \in [t_0, T]$.

The dynamics of the game are described by the system of differential equations:

$$\dot{x}(t) = f(x, u_1, ..., u_n), \quad x(t_0) = x_0,$$
(1)

where $x \in R^n$, $u_i \in U_i \subset \text{comp} R^k$.

We assume that all standard restrictions [18] on the parameters, controls, and trajectory function are satisfied.

The payoff function of the i^{th} player is:

$$K_i(t_0, x_0, u) = \int_{t_0}^T h_i(x(\tau), u(\tau)) d\tau, \quad i = \overline{1, n},$$

where $h_i(x, u)$ are continuous functions and x(t) is a solution of the Cauchy problem for System (1) under controls $u(t) = (u_1(t), ..., u_n(t))$.

2.2. Different Methods of Characteristic Function Construction

To define the cooperative game, we have to construct a characteristic function V(S) for every coalition $S \in N$ in the game.

A characteristic function is a mapping from the set of all possible coalitions:

$$V(S): 2^N \to R, \quad V(\emptyset) = 0.$$

The value V(S) is typically interpreted as the worth or the power of the coalition *S*. One of the most important properties of the characteristic function is superadditivity:

$$V(S_1 \cup S_2) \ge V(S_1) + V(S_2), \qquad \forall S_1, S_2 \subseteq N, \ S_1 \cap S_2 = \emptyset.$$

$$(2)$$

Superadditive characteristic functions provide some useful advantages in solving various problems in the field of cooperative game theory in static and dynamic settings. More information about this can be found in [19].

Currently, there are different approaches to the calculation of the characteristic function (see [7,19–22]). A systematic overview of different characteristic functions and their properties was presented in [23]. This paper provides an analysis of α -, ζ -, and η -characteristic functions.

2.2.1. α -Characteristic Function

A classical approach to the construction the characteristic function is called the α -characteristic function. It was introduced in [20] and was the only way to construct a cooperative game for a long time. The main idea of this method is using the lower value of the zero-sum game $\Gamma_{S,N\setminus S}$ between the coalition *S* as the first player and coalition $N \setminus S$ as the second player.

$$V^{\alpha}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max_{u_{i, i}, u_{i, j}} \sum_{i \in S} K_i(t_0, x_0, u), & S \subseteq N. \end{cases}$$
(3)

We assume that the maximum and minimum is achieved on (3). The value $V^{\alpha}(S)$ is interpreted as the maximum value that coalition *S* can get when $N \setminus S$ acts against *S*.

It was proved in [24] that the α -characteristic function is superadditive.

This approach of defining the characteristic function has some issues. It is necessary to solve $2^n - 1$ complicated optimization problems. It is hard to find (3) in analytical form in differential games due to computational problems. Finally, from an economic standpoint, it is unlikely that players of $N \setminus S$ form an anti-coalition [7].

2.2.2. δ -Characteristic Function

The technique of the construction the δ -characteristic function was proposed in [7]. The process of the calculation of this function consists of two steps. Firstly, one has to calculate the Nash equilibrium strategies for all players. Secondly, players from *S* maximize their total payoff $\sum_{i \in S} K_i$ while players from

 $N \setminus S$ use strategies from the Nash equilibrium.

$$V^{\delta}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \max_{\substack{u_i, i \in S \\ u_j = u_i^{NE}, j \in N \setminus S}} \sum_{i \in S} K_i(t_0, x_0, u_S, u_{N \setminus S}^{NE}), & S \subseteq N. \end{cases}$$
(4)

This form of the characteristic function requires fewer computational operations compared the with α -characteristic function. Additionally, the previously constructed Nash equilibrium simplifies the computation of $V^{\delta}(S)$. Moreover, (4) has a practical economic interpretation. Players not from the coalition *S* do not tend to form anti-coalition $N \setminus S$ in real models (see [25–27]).

Nevertheless there are some problems of this approach. In general, the δ -characteristic function is a non-superadditive function (see examples in [28]). Besides, one has to consider the problem of the existence and uniqueness of the Nash equilibrium solution.

2.2.3. ζ -Characteristic Function

The ζ -characteristic function was introduced in [19]. The first step of calculation of this characteristic function for coalition *S* is finding optimal controls maximizing the total payoff of the players. In the second step, players from coalition *S* use the cooperative optimal strategies, while the left-out players from $N \setminus S$ use the strategies minimizing the total payoff of the players from *S*.

$$V^{\zeta}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \min_{\substack{u_j \in U_j, \ j \in N \setminus S, \\ u_i = u_i^*, \ i \in S}} \sum_{i \in S} K_i(t_0, x_0, u_S^*, u_{N \setminus S}), & S \subseteq N. \end{cases}$$
(5)

We assume that the maximum and minimum are attained in (5).

The constructed $V^{\zeta}(S)$ is superadditive in general [19]. Additionally, already computed optimal controls are used for the ζ -characteristic function, which simplifies the computation process compared with the α -characteristic function. Besides, these controls exist and could be found for a wide class of games under rather weak constraints. Lastly, the ζ -characteristic function is applicable to games with fixed coalition structures [29].

2.2.4. η -Characteristic Function

The idea of the η -characteristic function was presented in [22]. This characteristic function is based on strategies from the optimal profile u^* and strategies from the Nash equilibrium u^{NE} . We use u_S^* for players from *S* (as in the ζ -characteristic function) and $u_{N\setminus S}^{NE}$ for players from $N \setminus S$ (as in the δ -characteristic function).

$$V^{\eta}(S) = \begin{cases} 0, & S = \{\emptyset\}, \\ \sum_{i \in S} K_i(t_0, x_0, u_S^*, u_{N \setminus S}^{NE}), & S \subseteq N. \end{cases}$$
(6)

This function models the case when players from $N \setminus S$ decide instead of optimal strategies to use strategies from Nash equilibrium u^{NE} .

The construction of the η -characteristic function has some technical advantages. It is much simpler in terms of calculation compared with the α -characteristic function. As mentioned above, optimal controls exist and could be found for a wide class of games. The drawback of this function is the problem of the existence and uniqueness of the Nash equilibrium solution. Furthermore, $V^{\eta}(S)$ is not superadditive in the general case [23].

3. Problem of Optimal Pollution Control

3.1. Problem Statement

We assume there is an enterprise having a production site in its territory. The volume of production is directly proportional to harmful emissions to the atmosphere $u(t) \in [0, b]$, which the enterprise controls.

The dynamics of the total amount of pollution x(t) is described by the following differential equation:

$$\dot{x}(t) = u(t) - \delta x(t), \quad x(t_0) = x_0,$$

where $\delta \ge 0$ is the absorption coefficient, $[\delta] = 1/[t]$ (we use square brackets to denote the dimension of the respective variable). Note that δ does not need to belong to the interval [0, 1] as the value of δ is determined by the dimension of the time unit. For instance, 1 [1/day]=30 [1/month].

The instantaneous profit of the enterprise is defined as:

$$R(u(t)) = \left(b - \frac{1}{2}u(t)\right)u(t), \quad t \in [t_0, T].$$

There is also an ecotax that is proportional to the amount of pollution. Hence, the net instantaneous payoff is obtained as the difference between the profit and the tax:

$$R(u(t)) - dx(t),$$

where d > 0 is the tax coefficient. Thus, the total payoff is:

$$K(t_0, x_0, u) = \int_{t_0}^T \left(\left(b - \frac{1}{2}u(t) \right) u(t) - dx(t) \right) dt.$$

It is straightforward to show that the optimal control maximizing the payoff $K(t_0, x_0, u)$ is:

$$u^{*}(t) = b + d \frac{e^{-\delta(T-t)} - 1}{\delta}.$$
(7)

We start by analyzing what values the optimal control can achieve. Before proceeding to the main result, we define:

$$\bar{\delta} = \frac{d}{b} + \frac{1}{T - t_0} W_0 \left(-\frac{d(T - t_0)}{b} e^{-\frac{d(T - t_0)}{b}} \right),$$

where $W_0(z)$ is the principal branch of the Lambert function, defined as the solution to the equation $we^w = z$ [30].

Proposition 1. The optimal control Function (7) is bounded by b. The optimal control $u^*(t) \ge 0$ for all $t \in [t_0, T]$ if $(T - t_0) \le \frac{b}{d}$ or $(T - t_0) > \frac{b}{d}$ and $\delta \ge \overline{\delta}$. If $(T - t_0) > \frac{b}{d}$ and $\delta < \overline{\delta}$, then the optimal control function changes sign from minus to plus at the point:

$$\bar{t} = T + \frac{1}{\delta} \ln \left(1 - \frac{b}{d} \delta \right) \in (t_0, T).$$

The proof is given in Appendix A.

Proposition 1 gives the conditions guaranteeing that the optimal control is defined by (7). Then, the corresponding trajectory is given by:

$$x^*(t) = \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(de^{\delta t} + 2(b\delta - d)e^{\delta T} \right) - e^{-\delta(T-t_0)} \left(de^{\delta t_0} + 2(b\delta - d - \delta^2 x_0)e^{\delta T} \right) \right)$$

and the payoff is:

$$K(t_0, x_0, u^*) = x_0 \frac{d(e^{-\delta\Delta} - 1)}{\delta} - \frac{d^2}{4\delta^3} e^{-2\delta\Delta} + \left(\frac{d^2}{\delta^3} - \frac{bd}{\delta^2}\right) e^{-\delta\Delta} + \left(\frac{d^2}{2\delta^2} - \frac{bd}{\delta} + \frac{b^2}{2}\right) \Delta - \frac{3d^2}{4\delta^3} + \frac{bd}{\delta^2}, \quad (8)$$

where $\Delta = T - t_0$.

If $(T - t_0) > \frac{b}{d}$ and $\delta < \overline{\delta}$, the function $u^*(t)$ is defined as:

$$u^{*}(t) = \begin{cases} 0, & t \in [t_{0}, \bar{t}], \\ b + d \frac{e^{-\delta(T-t)} - 1}{\delta}, & t \in (\bar{t}, T]. \end{cases}$$

and the corresponding trajectory is:

$$x^*(t) = \begin{cases} e^{-\delta(t-t_0)}x_0, & t \in [t_0, \overline{t}], \\ \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(de^{\delta t} + 2(b\delta - d)e^{\delta T} \right) + 2\delta^2 x_0 e^{\delta t_0} + \frac{(b\delta - d)^2}{d}e^{\delta T} \right), & t \in (\overline{t}, T]. \end{cases}$$

The corresponding value of the payoff is:

$$K(t_0, x_0, u^*) = x_0 \frac{d(e^{-\delta\Delta} - 1)}{\delta} - \frac{d^2}{4\delta^3} e^{-2\delta\bar{\Delta}} + \left(\frac{d^2}{\delta^3} - \frac{bd}{\delta^2}\right) e^{-\delta\bar{\Delta}} + \left(\frac{d^2}{2\delta^2} - \frac{bd}{\delta} + \frac{b^2}{2}\right) \bar{\Delta} - \frac{3d^2}{4\delta^3} + \frac{bd}{\delta^2},$$

where $\overline{\Delta} = T - \overline{t}$. Note that the latter case differs from the former one in that the optimal control is equal to zero on the interval $[t_0, \overline{t}]$.

3.2. Influence of the Absorption Coefficient on the Payoff

Proposition 2.

$$\lim_{\delta \to +\infty} K(t_0, x_0, u^*) = \frac{b^2}{2} \Delta.$$

Proposition 2 is proven by applying L'Hospital's rule to (8) three times.

Figures 1–5 demonstrate the dependence of the payoff from the absorption coefficient in special cases when the control has a switch point. Here and later on, we follow the convention that the overall plot is shown on the left, while the right plot shows a zoomed-in part of the graph.



Figure 1. $d = 10, b = 25, t_0 = 0, T = 200, x_0 = 0.25$. (a) $\delta \in [0, 8]$ and (b) $\delta \in [0, 0.6]$.



Figure 2. d = 10, b = 25, $t_0 = 0$, T = 200, $x_0 = 10$. (a) $\delta \in [0, 20]$ and (b) $\delta \in [0, 1]$.











Figure 5. d = 1000, b = 1, $t_0 = 8$, T = 100, $x_0 = 10$. (a) $\delta \in [0, 2000]$ and (b) $\delta \in [0, 90]$.

4. Game-Theoretical Model of Pollution Control

4.1. No Absorption Coefficient Model

In this section, we consider a differential game of pollution control with a prescribed duration based on the game-theoretical models issued in [4]. The game involves *n* players (companies or countries), and each of them has an industrial production site in its territory. A three player game was considered in [31].

Let $N = \{1, 2, ..., n\}$ with $n \ge 2$ be a set of players. The strategy of player *i* is the amount of pollution emitted to the atmosphere over time $u_i \in [0; b_i]$. We will look for the solution in the class of open-loop strategies $u_i(t)$.

The dynamics of the total amount of pollution x(t) is described by the following differential equation:

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t), \quad x(t_0) = x_0 \ge 0.$$
 (9)

The total payoff of the i^{th} player is defined along the same lines as in Section 3.1. Thus, we write the payoff of the i^{th} player as:

$$K_{i}(t_{0}, x_{0}, u) = \int_{t_{0}}^{T} \left(\left(b_{i} - \frac{1}{2}u_{i} \right) u_{i} - d_{i}x \right) dt,$$
(10)

where b_i and d_i are the respective parameters that can be different for each player.

To simplify notation, we introduce the following parameters:

$$D_N = \sum_{i=1}^n d_i, \quad D_S = \sum_{i \in S} d_i, \quad D_{N \setminus S} = \sum_{i \in N \setminus S} d_i, \quad B_N = \sum_{i=1}^n b_i, \quad \tilde{B}_S = \sum_{i \in S} b_i^2, \text{ and } s = |S|.$$

We also assume that the regularity constraints $(T - t_0) \leq \frac{b_i}{D_N}$ hold for all $i \in N$. These constraints guarantee that $u_i(t) \geq 0, \forall t \in [t_0, T]$.

We assume the players have made a cooperative agreement on maximizing the total payoff of the players. To compute the cooperative solution (Shapley value, core, Harsanyi dividend, Banzhaf power index, etc.), one has to compute a characteristic function that plays a key role in cooperative game theory. In [23], we constructed the following characteristic functions for every coalition $S \subset N$.

$$V^{\alpha}(S,\Delta,x_0) = -D_S x_0 \Delta + \frac{1}{2} \tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6} s D_S^2 \Delta^3.$$
(11)

$$V^{\delta}(S,\Delta,x_0) = -D_S x_0 \Delta + \frac{1}{2} \tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6} (2D_{N\setminus S} D_S + sD_S^2) \Delta^3.$$
(12)

$$V^{\zeta}(S,\Delta,x_0) = -D_S x_0 \Delta + \frac{1}{2} \tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 - \frac{1}{6} s D_N (D_N - 2D_S) \Delta^3.$$
(13)

$$V^{\eta}(S,\Delta,x_0) = -D_S x_0 \Delta + \frac{1}{2} \tilde{B}_S \Delta - \frac{1}{2} B_N D_S \Delta^2 + \frac{1}{6} (-sD_N^2 + 2sD_N D_S + 2D_{N\setminus S} D_S) \Delta^3.$$
(14)

Functions (11)–(14) can be shown to be superadditive (see [23]). Moreover, the relations between characteristic functions were found in [23].

$$V^{\delta}(S,\Delta,x_{0}) \geq V^{\alpha}(S,\Delta,x_{0}),$$

$$V^{\eta}(S,\Delta,x_{0}) \geq V^{\zeta}(S,\Delta,x_{0}),$$

$$V^{\alpha}(S,\Delta,x_{0}) \geq V^{\zeta}(S,\Delta,x_{0}),$$

$$V^{\delta}(S,\Delta,x_{0}) \geq V^{\eta}(S,\Delta,x_{0})$$
(15)

4.2. Absorption Coefficient Model

We consider a modification of the differential game of pollution control. A special case of this game where n = 3 was considered in detail in [32]. In the paper [33], the construction of the η -characteristic function was presented for the example based on real data.

The dynamics of the total amount of pollution x(t) is described by an extended equation:

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t) - \delta x(t), \quad x(t_0) = x_0 \ge 0,$$
(16)

where $\delta > 0$ is an absorption coefficient introduced earlier. The payoff function is defined by (10).

In the following, we assume that all additional regularity constraints defined above are satisfied. Using the definitions (3)–(6), we get the following characteristic functions:

$$V^{\alpha}(S,\Delta,x_{0}) = x_{0} \frac{D_{S}\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{sD_{S}^{2}}{4\delta^{3}}e^{-2\delta\Delta} + \left(\frac{sD_{S}^{2}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta\Delta} + \left(\frac{sD_{S}^{2}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)\Delta - \frac{3sD_{S}^{2}}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}.$$
 (17)

$$V^{\delta}(S,\Delta,x_{0}) = x_{0} \frac{D_{S}\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{4\delta^{3}}e^{-2\delta\Delta} + \left(\frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta\Delta} + \left(\frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)\Delta - \frac{3(sD_{S}^{2}+2D_{S}D_{N\setminus S})}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}.$$
 (18)

$$V^{\zeta}(S,\Delta,x_{0}) = x_{0} \frac{D_{S}\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{-sD_{N}^{2}+2sD_{S}D_{N}}{4\delta^{3}}e^{-2\delta\Delta} + \left(\frac{-sD_{N}^{2}+2sD_{S}D_{N}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta\Delta} + \left(\frac{-sD_{N}^{2}+2sD_{S}D_{N}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)\Delta - \frac{3(-sD_{N}^{2}+2sD_{S}D_{N})}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}.$$
 (19)

$$V^{\eta}(S,\Delta,x_{0}) = x_{0} \frac{D_{S}\left(e^{-\delta\Delta}-1\right)}{\delta} - \frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{4\delta^{3}}e^{-2\delta\Delta} + \left(\frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta\Delta} + \left(\frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)\Delta - \frac{3\left(s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}\right)}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}.$$
 (20)

Functions (17)–(20) are superadditive. The construction of characteristic functions and the proof of their superadditivity are given in Appendix B and Appendix C. It can also be checked that the relations between characteristic Functions (15) are satisfied. In addition, we obtain the result:

$$V^{\delta}(S,\Delta,x_0) + V^{\zeta}(S,\Delta,x_0) = V^{\eta}(S,\Delta,x_0) + V^{\alpha}(S,\Delta,x_0).$$

Theorem 1. The limits of the α -, δ -, ζ -, and η -characteristic functions exist and are equal as δ tends to infinity.

$$\lim_{\delta \to +\infty} V^{\alpha}(S, \Delta, x_0) = \lim_{\delta \to +\infty} V^{\delta}(S, \Delta, x_0) = \lim_{\delta \to +\infty} V^{\zeta}(S, \Delta, x_0) = \lim_{\delta \to +\infty} V^{\eta}(S, \Delta, x_0) = \frac{B_S}{2}\Delta.$$

Theorem 1 is proven by computing limits with the use of the L'Hospital rule.

5. Optimal Control of Pollution Emissions for the Irkutsk Region

As an illustration, we considered a differential game of pollution control based on real data for enterprises of the Irkutsk region [16,33]. The ecological situation in Bratsk is one of the most acute in the region. We observed that the three largest enterprises of Bratsk pollute the environment: OJSC «RUSAL Bratsk», OJSC «ILIM Group», and units of OJSC «Irkutskenergo».

We used the coefficients of the differential game that were found in the paper [16].

We considered two cases. In the first case, the dynamics of the total amount of pollution x(t) is described by (9). This means that there is no absorption coefficient in the model.

Using the provided numerical values of the parameters, we used (11)–(14) to compute α -, δ -, ζ -, and η -characteristic functions as functions of x_0 and Δ .

Figure 6 shows the influence of the initial time t_0 on the characteristic functions of the individual players. Similar results can be shown for the case of coalitions. In the following figures, we present the respective dependence on the left and the zoomed-in fragments of the respective plots on the right side.

We now proceed to the case with non-zero pollution absorption. The dynamics of x(t) is described by (16). Using the numerical values from Table 1 and the expressions for the characteristic Functions (17)–(20), we calculated the α -, δ -, ζ -, and η -characteristic functions.

 $\times 10^{10}$

0.0

-0.2

-0.4

-0.6

-0.8

-1.0

-1.2

0.0

 $\times 10^{12}$

4

3

2

1

0.0

 $\times 10^8$

C

0.5





Figure 6. (a) $V(\{1\}), t_0 \in [0, 3.5]$, (b) $V(\{1\}), t_0 \in [0, 2 \times 10^{-5}]$, (c) $V(\{2\}), t_0 \in [0, 3.5]$, (d) $V(\{2\}), t_0 \in [0, 2 \times 10^{-5}]$, (e) $V(\{3\}), t_0 \in [0, 3.5]$, and (f) $V(\{3\}), t_0 \in [0, 2 \times 10^{-5}]$.

Table 1. Coefficient values.

Enterprise	b _i	di
OJSC «RUSAL Bratsk»	28,838.01	1254.97
OJSC «ILIM Group»	1,530,463	102.27
Units of OJSC «Irkutskenergo»	5228.4	36.65

Below, we present the plots showing the effect of the initial time t_0 and the absorption coefficient δ on the values of the respective characteristic functions. In particular, Figure 7 shows the influence of

the initial time t_0 on the characteristic functions of the individual players, while Figure 8 illustrates the dependence of characteristic functions on the parameter δ . All presented results can be demonstrated for the coalitions as well.

Finally, in Figure 9, we illustrate the difference between the characteristic functions for the cases when the absorption coefficient is equal to zero (blue line) and when the absorption coefficient is taken from Table 1. For the illustration, we chose the η -characteristic function.



Figure 7. (a) $V(\{1\}), t_0 \in [0, 3.5]$, (b) $V(\{1\}), t_0 \in [0, 2 \times 10^{-8}]$, (c) $V(\{2\}), t_0 \in [0, 3.5]$, (d) $V(\{2\}), t_0 \in [0, 2 \times 10^{-8}]$, (e) $V(\{3\}), t_0 \in [0, 3.5]$, and (f) $V(\{3\}), t_0 \in [0, 2 \times 10^{-8}]$.



Figure 8. (a) $V(\{1\}), \delta \in [0.1, 20]$, (b) $V(\{1\}), \delta \in [0.0001, 0.00018]$, (c) $V(\{2\}), \delta \in [0.1, 20]$, (d) $V(\{2\}), \delta \in [0.0001, 0.00018]$, (e) $V(\{3\}), \delta \in [0.1, 20]$, and (f) $V(\{3\}), \delta \in [0.0001, 0.00018]$.



Conclusions

In this paper, we considered a cooperative differential game of pollution control with a prescribed duration. We analyzed two cases when there was an absorption coefficient δ and when there was not. For these games, α -, δ -, ζ -, and η -characteristic functions were constructed for the general case of *n* players.

It was shown that the parameter δ had a significant impact on the characteristic functions. We also obtained analytical formulas for the limiting values of the characteristic functions with increasing absorption coefficient.

All results were illustrated with the differential game of pollution control based on real data for enterprises of the Irkutsk region.

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Appendix A. Construction of Optimal Pollution Control

Proof of Proposition 1

Proof. The proof of $u^*(t) \le b$ is trivial because the second term is not positive.

We differentiate $u^*(t)$ with respect to *t*:

$$\frac{\mathrm{d}u^*(t)}{\mathrm{d}t} = de^{-\delta(T-t)} > 0, \quad \forall t \in [t_0, T].$$

Therefore, $u^*(t)$ is an increasing function of t. This means that the function takes a minimum value at a point $t = t_0$:

$$u^*(t_0) = b + d \frac{e^{-\delta(T-t_0)} - 1}{\delta}.$$

We differentiate $u^*(t)$ with respect to δ :

$$\frac{\mathrm{d}u^*(t)}{\mathrm{d}\delta} = \frac{\mathrm{d}e^{-\delta(T-t)}(-\delta(T-t) + e^{\delta(T-t)} - 1)}{\delta^2}.$$

Obviously, $de^{-\delta(T-t)} > 0$ and $\delta^2 > 0$. To define the sign of $(-\delta(T-t) + e^{\delta(T-t)} - 1)$:

$$\frac{d(-\delta(T-t) + e^{\delta(T-t)} - 1)}{d\delta} = (T-t)(e^{\delta(T-t)} - 1) > 0.$$

Hence, $(-\delta(T-t) + e^{\delta(T-t)} - 1)$ is an increasing function of δ . It follows that the function is positive for $\delta > 0$. This completes the proof that $\frac{du^*(t)}{d\delta} > 0$.

Thus, $u^*(t)$ is an increasing function of δ , then:

$$u^*(t_0) \ge \lim_{\delta \to 0} \left(b + d \frac{e^{-\delta(T-t_0)} - 1}{\delta} \right) = b - d(T-t_0)$$

If $b - d(T - t_0) \ge 0$, which is the same as $(T - t_0) \le \frac{b}{d}$, then $u^*(t) \ge 0$, $\forall \delta$ and $t \in [t_0, T]$. Next, we consider the case $(T - t_0) > \frac{b}{d}$. We find δ that provides $u^*(t) \ge 0$.

Note that $u^*(t_0) \ge 0$ for sufficiently large δ . If δ tends to zero, then $u^*(t) < 0$ at $t = t_0$ at least. Define the delta at which $u^*(t_0) = 0$ under the condition $u^*(t)$ is an increasing function of t and δ .

$$b+d\frac{e^{-\delta(T-t_0)}-1}{\delta}=0,$$

whence, upon some algebraic manipulation, we get:

$$\delta(T-t_0) - \frac{d(T-t_0)}{b} + \frac{d(T-t_0)}{b}e^{-\delta(T-t_0)} = 0.$$

We define:

$$\rho = \frac{d(T-t_0)}{b}, \quad \varepsilon = \delta(T-t_0).$$

Since $(T - t_0) > \frac{b}{d}$, then $\rho > 1$. Hence, we obtain the equation $\varepsilon - \rho + \rho e^{-\varepsilon} = 0$. Solving this equation with respect to ε , we get:

$$\varepsilon = \rho + W(-\rho e^{-\rho}).$$

Finally, we arrive at the following expression for the threshold value of δ :

$$\begin{split} \delta &= \frac{1}{T-t_0} \left(\frac{d(T-t_0)}{b} + W\left(-\frac{d(T-t_0)}{b} \exp\left(-\frac{d(T-t_0)}{b} \right) \right) \right) \\ &= \frac{d}{b} + \frac{1}{T-t_0} W\left(-\frac{d(T-t_0)}{b} \exp\left(-\frac{d(T-t_0)}{b} \right) \right) = \bar{\delta}. \end{split}$$

If $\delta \geq \overline{\delta}$, then $u^*(t_0) \geq 0$ and $u^*(t) \geq 0$, $\forall t \in [t_0, T]$. Otherwise, if $\delta < \overline{\delta}$, then $u^*(t_0) < 0$. We determine the moment $\overline{t} > t_0$ at which the optimal control turns to zero: $u^*(\overline{t}) = 0$. Solving:

$$b+d\frac{e^{-\delta(T-t)}-1}{\delta}=0,$$

we get:

$$\bar{t} = T + \frac{1}{\delta} \ln \left(1 - \frac{b\delta}{d} \right).$$

Thus, \bar{t} is the point where the control changes sign. \Box

Appendix B. Computation of the Characteristic Functions

Appendix B.1. Nash Equilibrium

The computation of the Nash equilibrium strategies (NE) is fairly obvious, so we skip most details. Using the Pontryagin maximum principle, we find the Nash equilibrium strategies:

$$u^{NE}(t) = \begin{pmatrix} b_1 + d_1 \frac{e^{-\delta(T-t)} - 1}{\delta} \\ \dots \\ b_n + d_n \frac{e^{-\delta(T-t)} - 1}{\delta} \end{pmatrix}$$
(A1)

and the corresponding trajectory:

$$x^{NE}(t) = \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(D_N e^{\delta t} + 2(B_N \delta - D_N) e^{\delta T} \right) - e^{-\delta(T-t_0)} \left(D_N e^{\delta t_0} + 2(B_N \delta - D_N - \delta^2 x_0) e^{\delta T} \right) \right).$$

Following the same scheme as that shown in Appendix A, we obtain that if $(T - t_0) \le \frac{b_i}{d_i}$, then $u_i^{NE}(t) \ge 0, \forall \delta > 0$ and $\forall t \in [t_0, T]$.

Appendix B.2. Cooperative Agreement

The optimal cooperative strategies are:

$$u^{*}(t) = \begin{pmatrix} b_{1} + D_{N} \frac{e^{-\delta(T-t)} - 1}{\delta} \\ \dots \\ b_{n} + D_{N} \frac{e^{-\delta(T-t)} - 1}{\delta} \end{pmatrix}$$
(A2)

and the optimal trajectory is:

$$x^*(t) = \frac{e^{-\delta t}}{2\delta^2} \left(e^{-\delta(T-t)} \left(nD_N e^{\delta t} + 2(B_N \delta - nD_N) e^{\delta T} \right) - e^{-\delta(T-t_0)} \left(nD_N e^{\delta t_0} + 2(B_N \delta - nD_N - \delta^2 x_0) e^{\delta T} \right) \right)$$

For the optimal control, we have that if $b_i - D_N(T - t_0) \ge 0$, which is the same as $(T - t_0) \le \frac{b_i}{D_N}$, then $u^*(t) \ge 0, \forall \delta$ and $t \in [t_0, T]$.

Appendix B.3. Construction of the α -Characteristic Function

We consider a minimization problem:

$$\min_{\substack{u_j\\j\in N\setminus S}}\sum_{i\in S}\int_{t_0}^T \Big(\Big(b_i-\frac{1}{2}u_i\Big)u_i-d_ix\Big)dt, \quad S\in N.$$

For this problem, the optimal strategies are:

$$u_j = b_j. \tag{A3}$$

Next, we consider a maximization problem:

$$\max_{\substack{u_i, i \in S, \\ u_j = b_j, j \in N \setminus S}} \sum_{i \in S} \int_{t_0}^T \left(\left(b_i - \frac{1}{2} u_i \right) u_i - d_i x \right) dt, \quad S \in N.$$

For this problem, we obtain the controls:

$$u_i^S(t) = b_i + D_S \frac{e^{-\delta(T-t)} - 1}{\delta}, \quad i \in S.$$
 (A4)

The corresponding trajectory is:

$$x^{S}(t) = \frac{e^{-\delta t}}{2\delta^{2}} \left(e^{-\delta(T-t)} \left(sD_{S}e^{\delta t} + 2(B_{N}\delta - sD_{S})e^{\delta T} \right) - e^{-\delta(T-t_{0})} \left(sD_{S}e^{\delta t_{0}} + 2(B_{N}\delta - sD_{S} - \delta^{2}x_{0})e^{\delta T} \right) \right).$$

For the controls (A4), we have that if $(T - t_0) \leq \frac{b_i}{D_S}$, then $u_i^S(t) \geq 0, \forall \delta > 0$ and $t \in [t_0, T]$. Combining (3), (A3), and (A4), we construct the α -characteristic function:

$$\begin{split} V^{\alpha}(S,t_{0},x_{0}) &= x_{0} \frac{D_{S}\left(e^{-\delta(T-t_{0})}-1\right)}{\delta} - \frac{sD_{S}^{2}}{4\delta^{3}}e^{-2\delta(T-t_{0})} + \left(\frac{sD_{S}^{2}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta(T-t_{0})} \\ &+ \left(\frac{sD_{S}^{2}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)(T-t_{0}) - \frac{3sD_{S}^{2}}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}. \end{split}$$

Appendix B.4. Construction of the δ -Characteristic Function

We consider a maximization problem:

$$\max_{\substack{u_i, i \in S, \\ u_j = u_i^{NE}, j \in N \setminus S}} \sum_{i \in S} \int_{t_0}^T \left(\left(b_i - \frac{1}{2} u_i \right) u_i - d_i x \right) dt, \quad S \in N.$$

For this problem, the optimal controls are:

$$u_{i}^{S}(t) = b_{i} + D_{S} \frac{e^{-\delta(T-t)} - 1}{\delta}$$
(A5)

and the corresponding trajectory is:

$$\begin{aligned} x^{S}(t) &= \frac{e^{-\delta t}}{2\delta^{2}} \left(e^{-\delta(T-t)} \left((sD_{S} + D_{N\setminus S})e^{\delta t} + 2(B_{N}\delta - sD_{S} - D_{N\setminus S})e^{\delta T} \right) \\ &- e^{-\delta(T-t_{0})} \left((sD_{S} + D_{N\setminus S})e^{\delta t_{0}} + 2(B_{N}\delta - sD_{S} - D_{N\setminus S} - \delta^{2}x_{0})e^{\delta T} \right) \end{aligned}$$

Similar to the previous cases, we have that if $(T - t_0) \leq \frac{b_i}{D_S}$, then $u_i^S(t) \geq 0$, $\forall \delta > 0$ and $t \in [t_0, T]$. Combining (4), (A1), and (A5), we construct the δ -characteristic function:

$$\begin{split} V^{\delta}(S,t_{0},x_{0}) &= x_{0} \frac{D_{S}\left(e^{-\delta(T-t_{0})}-1\right)}{\delta} - \frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{4\delta^{3}}e^{-2\delta(T-t_{0})} \\ &+ \left(\frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta(T-t_{0})} \\ &+ \left(\frac{sD_{S}^{2}+2D_{S}D_{N\setminus S}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)(T-t_{0}) - \frac{3(sD_{S}^{2}+2D_{S}D_{N\setminus S})}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}. \end{split}$$

Appendix B.5. Construction of the ζ -Characteristic Function

According to (5), players from coalition $S \in N$ use optimal controls (A2). We consider the minimization problem:

$$\min_{u_j,j\in N\setminus S}\sum_{i\in S}\int_{t_0}^T \left(\left(b_i-\frac{1}{2}u_i^*\right)u_i^*-d_ix\right)dt, \quad S\in N.$$

In this case, the optimal strategies are:

$$u_j = b_j. \tag{A6}$$

Combining (5), (A2), and (A6), we construct the ζ -characteristic function:

$$\begin{split} V^{\zeta}(S,t_0,x_0) &= x_0 \frac{D_S \left(e^{-\delta(T-t_0)} - 1 \right)}{\delta} - \frac{-sD_N^2 + 2sD_S D_N}{4\delta^3} e^{-2\delta(T-t_0)} \\ &+ \left(\frac{-sD_N^2 + 2sD_S D_N}{\delta^3} - \frac{B_N D_S}{\delta^2} \right) e^{-\delta(T-t_0)} \\ &+ \left(\frac{-sD_N^2 + 2sD_S D_N}{2\delta^2} - \frac{B_N D_S}{\delta} + \frac{\tilde{B}_S}{2} \right) (T-t_0) - \frac{3(-sD_N^2 + 2sD_S D_N)}{4\delta^3} + \frac{B_N D_S}{\delta^2} \end{split}$$

Appendix B.6. Construction of the η-Characteristic Function

According to (6), players from $S \in N$ use (A2) when players from $N \setminus S$ use (A1). We construct the η -characteristic function:

$$\begin{split} V^{\eta}(S,t_{0},x_{0}) &= x_{0} \frac{D_{S}\left(e^{-\delta(T-t_{0})}-1\right)}{\delta} - \frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{4\delta^{3}}e^{-2\delta(T-t_{0})} \\ &+ \left(\frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{\delta^{3}} - \frac{B_{N}D_{S}}{\delta^{2}}\right)e^{-\delta(T-t_{0})} \\ &+ \left(\frac{s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}}{2\delta^{2}} - \frac{B_{N}D_{S}}{\delta} + \frac{\tilde{B}_{S}}{2}\right)(T-t_{0}) \\ &- \frac{3\left(s(2D_{S}-D_{N})D_{N}+2D_{S}D_{N\setminus S}\right)}{4\delta^{3}} + \frac{B_{N}D_{S}}{\delta^{2}}. \end{split}$$

Appendix C. Proofs of Superadditivity Characteristic Functions

Appendix C.1. Additional Statement

Proposition A1.

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Proof. Let $x = e^{-\delta(T-t_0)}$, then we need to prove:

$$-x^2 + 4x + 2\delta(T - t_0) - 3 > 0.$$

We solve the quadratic equation:

$$-x^2 + 4x + 2\delta(T - t_0) - 3 = 0.$$

The roots of the equation are:

$$x_1 = 2 - \sqrt{1 + 2\delta(T - t_0)}, \quad x_2 = 2 + \sqrt{1 + 2\delta(T - t_0)}.$$

The branches of the parabola $y = -x^2 + 4x + 2\delta(T - t_0) - 3$ are directed downwards. This means that:

$$-x^2 + 4x + 2\delta(T - t_0) - 3 > 0$$

when:

$$x \in \left(2 - \sqrt{1 + 2\delta(T - t_0)}, \quad 2 + \sqrt{1 + 2\delta(T - t_0)}\right)$$

We change *x* to $e^{-\delta(T-t_0)}$. Let us prove that:

$$e^{-\delta(T-t_0)} \in \left(2 - \sqrt{1 + 2\delta(T-t_0)}, \quad 2 + \sqrt{1 + 2\delta(T-t_0)}\right).$$
 (A7)

We make a replacement $m = \delta(T - t_0) > 0$. Therefore, we prove:

$$e^{-m} \in \left(2 - \sqrt{1 + 2m}, \quad 2 + \sqrt{1 + 2m}\right).$$
 (A8)

First, let us show that:

$$2 + \sqrt{1 + 2m} - e^{-m} > 0. \tag{A9}$$

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We have:

$$\sqrt{1+2m} \ge 1,$$
$$e^{-m} \le 1.$$

Hence:

$$2 + \sqrt{1 + 2m - e^{-m}} \ge 2.$$

This completes the proof (A9).

Secondly, let us show that:

$$e^{-m} - 2 + \sqrt{1 + 2m} > 0. \tag{A10}$$

We transform the expression:

$$e^{-m} - 2 + \sqrt{1 + 2m} = e^{-m}(e^m\sqrt{1 + 2m} - 2e^m + 1).$$

We clearly have $e^{-m} > 0$. To prove $e^m \sqrt{1 + 2m} - 2e^m + 1 > 0$, we need to differentiate $(e^m \sqrt{1 + 2m} - 2e^m + 1)$ with respect to *m*:

$$\frac{d(e^m\sqrt{1+2m}-2e^m+1)}{dm} = \frac{2e^m(m-\sqrt{1+2m}+1)}{\sqrt{1+2m}}$$
(A11)

Obviously, $e^m > 0$, $\sqrt{1 + 2m} > 0$ for m > 0. Let us show:

$$m - \sqrt{1 + 2m} + 1 > 0. \tag{A12}$$

Assume the converse. Then:

$$m - \sqrt{1 + 2m} + 1 \le 0.$$

Hence:

$$m+1 \le \sqrt{1+2m}$$
$$(m+1)^2 \le 1+2m$$

because m > 0. Therefore:

$$m^2 \le 0,$$

$$m = 0.$$

This contradiction proves (A12). This means that in (A11):

$$\frac{d(e^m\sqrt{1+2m}-2e^m+1)}{dm} > 0$$

That is, the function $y = e^m \sqrt{1 + 2m} - 2e^m + 1$ is increasing for m > 0 and:

$$e^m \sqrt{1+2m} - 2e^m + 1 > 0.$$

This implies the inequality (A10). Finally, using (A9) and (A10), we get (A8). This yields

$$e^{-\delta(T-t_0)} \in \left(2 - \sqrt{1 + 2\delta(T-t_0)}, \quad 2 + \sqrt{1 + 2\delta(T-t_0)}\right).$$

and hence, $-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$ \Box

Appendix C.2. Superadditivity of the α -Characteristic Function

We prove that the α -characteristic function is superadditive using (2).

$$\begin{split} &V^{\alpha}\big(S_{1}\cup S_{2},t_{0},x_{0}\big)-V^{\alpha}\big(S_{1},t_{0},x_{0}\big)-V^{\alpha}\big(S_{2},t_{0},x_{0}\big)\\ &=-\frac{s_{2}D_{S_{1}}^{2}+s_{1}D_{S_{2}}^{2}+2D_{S_{1}}D_{S_{2}}\left(s_{1}+s_{2}\right)}{4\delta^{3}}e^{-2\delta(T-t_{0})}\\ &+\left(\frac{s_{2}D_{S_{1}}^{2}+s_{1}D_{S_{2}}^{2}+2D_{S_{1}}D_{S_{2}}\left(s_{1}+s_{2}\right)}{2\delta^{2}}\right)e^{-\delta(T-t_{0})}\\ &+\left(\frac{s_{2}D_{S_{1}}^{2}+s_{1}D_{S_{2}}^{2}+2D_{S_{1}}D_{S_{2}}\left(s_{1}+s_{2}\right)}{2\delta^{2}}\right)\left(T-t_{0}\right)\\ &-\frac{3\left(s_{2}D_{S_{1}}^{2}+s_{1}D_{S_{2}}^{2}+2D_{S_{1}}D_{S_{2}}\left(s_{1}+s_{2}\right)\right)}{4\delta^{3}}\\ &=\frac{s_{2}D_{S_{1}}^{2}+s_{1}D_{S_{2}}^{2}+2D_{S_{1}}D_{S_{2}}\left(s_{1}+s_{2}\right)}{4\delta^{3}}\left(-e^{-2\delta(T-t_{0})}+4e^{-\delta(T-t_{0})}+2\delta(T-t_{0})-3\right). \end{split}$$

Clearly,

$$\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2 D_{S_1} D_{S_2} \left(s_1 + s_2\right)}{4 \delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Hence, $V^{\alpha}(S_1 \cup S_2, t_0, x_0) - V^{\alpha}(S_1, t_0, x_0) - V^{\alpha}(S_2, t_0, x_0) > 0.$

Appendix C.3. Superadditivity of the δ -Characteristic Function

We verify the superadditivity of the δ -characteristic function using the definition (2).

$$\begin{split} &V^{\delta}(S_{1} \cup S_{2}, t_{0}, x_{0}) - V^{\delta}(S_{1}, t_{0}, x_{0}) - V^{\delta}(S_{2}, t_{0}, x_{0}) \\ &= -\frac{s_{2}D_{S_{1}}^{2} + s_{1}D_{S_{2}}^{2} + 2D_{S_{1}}D_{S_{2}}\left(s_{1} + s_{2} - 2\right)}{4\delta^{3}}e^{-2\delta(T - t_{0})} \\ &+ \left(\frac{s_{2}D_{S_{1}}^{2} + s_{1}D_{S_{2}}^{2} + 2D_{S_{1}}D_{S_{2}}\left(s_{1} + s_{2} - 2\right)}{2\delta^{2}}\right)e^{-\delta(T - t_{0})} \\ &+ \left(\frac{s_{2}D_{S_{1}}^{2} + s_{1}D_{S_{2}}^{2} + 2D_{S_{1}}D_{S_{2}}\left(s_{1} + s_{2} - 2\right)}{2\delta^{2}}\right)(T - t_{0}) \\ &- \frac{3\left(s_{2}D_{S_{1}}^{2} + s_{1}D_{S_{2}}^{2} + 2D_{S_{1}}D_{S_{2}}\left(s_{1} + s_{2} - 2\right)\right)}{4\delta^{3}} \\ &= \frac{s_{2}D_{S_{1}}^{2} + s_{1}D_{S_{2}}^{2} + 2D_{S_{1}}D_{S_{2}}\left(s_{1} + s_{2} - 2\right)}{4\delta^{3}}\left(-e^{-2\delta(T - t_{0})} + 4e^{-\delta(T - t_{0})} + 2\delta(T - t_{0}) - 3\right). \end{split}$$

Using $s_1 = |S_1| \ge 1$, $s_2 = |S_2| \ge 1$, we get $s_1 + s_2 - 2 \ge 0$. It is obvious that:

$$\frac{s_2 D_{S_1}^2 + s_1 D_{S_2}^2 + 2 D_{S_1} D_{S_2} \left(s_1 + s_2 - 2\right)}{4 \delta^3} > 0.$$

According to Proposition A1:

 $-e^{-2\delta(T-t_0)}+4e^{-\delta(T-t_0)}+2\delta(T-t_0)-3>0.$

Therefore, $V^{\delta}(S_1 \cup S_2, t_0, x_0) - V^{\delta}(S_1, t_0, x_0) - V^{\delta}(S_2, t_0, x_0) > 0.$

Appendix C.4. Superadditivity of the ζ -Characteristic Function

Let us show that the ζ -characteristic function is superadditive using (2).

$$\begin{split} &V^{\zeta}(S_1 \cup S_2, t_0, x_0) - V^{\zeta}(S_1, t_0, x_0) - V^{\zeta}(S_2, t_0, x_0) \\ &= -\frac{D_N(s_2D_{S_1} + s_1D_{S_2})}{2\delta^3} e^{-2\delta(T-t_0)} + \frac{2D_N(s_2D_{S_1} + s_1D_{S_2})}{\delta^3} e^{-\delta(T-t_0)} \\ &+ \frac{D_N(s_2D_{S_1} + s_1D_{S_2})}{\delta^2} - \frac{3D_N(s_2D_{S_1} + s_1D_{S_2})}{2\delta^3} (T-t_0) \\ &= \frac{D_N(s_2D_{S_1} + s_1D_{S_2})}{2\delta^3} \left(-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 \right). \end{split}$$

Trivially,

$$\frac{D_N(s_2D_{S_1}+s_1D_{S_2})}{2\delta^3} > 0.$$

According to Proposition A1:

 $-e^{-2\delta(T-t_0)}+4e^{-\delta(T-t_0)}+2\delta(T-t_0)-3>0.$

It follows that $V^{\zeta}(S_1 \cup S_2, t_0, x_0) - V^{\zeta}(S_1, t_0, x_0) - V^{\zeta}(S_2, t_0, x_0) > 0.$

Appendix C.5. Superadditivity of the η-Characteristic Function

Let us check that the η -characteristic function is superadditive using (2).

$$\begin{split} &V^{\eta}(S_{1} \cup S_{2}, t_{0}, x_{0}) - V^{\eta}(S_{1}, t_{0}, x_{0}) - V^{\eta}(S_{2}, t_{0}, x_{0}) \\ &= -\frac{D_{S_{1}}(s_{2}D_{N} - D_{S_{2}}) + D_{S_{2}}(s_{1}D_{N} - D_{S_{1}})}{2\delta^{3}}e^{-2\delta(T-t_{0})} \\ &+ \frac{2D_{S_{1}}(s_{2}D_{N} - D_{S_{2}}) + 2D_{S_{2}}(s_{1}D_{N} - D_{S_{1}})}{\delta^{3}}e^{-\delta(T-t_{0})} \\ &+ \frac{D_{S_{1}}(s_{2}D_{N} - D_{S_{2}}) + D_{S_{2}}(s_{1}D_{N} - D_{S_{1}})}{\delta^{2}}(T-t_{0}) \\ &- \frac{3\left(D_{S_{1}}(s_{2}D_{N} - D_{S_{2}}) + D_{S_{2}}(s_{1}D_{N} - D_{S_{1}})\right)}{2\delta^{3}} \\ &= \frac{D_{S_{1}}(s_{2}D_{N} - D_{S_{2}}) + D_{S_{2}}(s_{1}D_{N} - D_{S_{1}})}{2\delta^{3}}\left(-e^{-2\delta(T-t_{0})} + 4e^{-\delta(T-t_{0})} + 2\delta(T-t_{0}) - 3\right). \end{split}$$

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We obtain:

$$\frac{D_{S_1}(s_2D_N - D_{S_2}) + D_{S_2}(s_1D_N - D_{S_1})}{2\delta^3} \ge \frac{D_{S_1}(D_N - D_{S_2}) + D_{S_2}(D_N - D_{S_1})}{2\delta^3}$$
$$= \frac{D_{S_1}D_{N\setminus S_2} + D_{S_2}D_{N\setminus S_1}}{2\delta^3} > 0.$$

According to Proposition A1:

$$-e^{-2\delta(T-t_0)} + 4e^{-\delta(T-t_0)} + 2\delta(T-t_0) - 3 > 0.$$

Therefore, $V^{\eta}(S_1 \cup S_2, t_0, x_0) - V^{\eta}(S_1, t_0, x_0) - V^{\eta}(S_2, t_0, x_0) > 0.$

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