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Better Approaches for *n*-Times Differentiable Convex Functions

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Abstract: In this work, by using an integral identity together with the Hölder–İşcan inequality we establish several new inequalities for n-times differentiable convex and concave mappings. Furthermore, various applications for some special means as arithmetic, geometric, and logarithmic are given.

Keywords: convex function; Hölder–İşcan integral inequality

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1. Introduction

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$.

The above is a well known definition in the literature. Convexity theory has been appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences (see, for example [1–4]). Recently, in the literature many researchers contributed their research on n-times differentiable functions on several kinds of convexities (see, for example [1–3,5–8]) and the references within these papers.

The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex or concave function.

Definition 2. *$f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a concave function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f\left(\frac{a+b}{2}\right)$$

is known in the literature as a Hermite–Hadamard's inequality for convex functions. Both inequalities holds if f is concave.

Hadamard's type inequalities for convex or concave functions has received the attention of the many researchers in recent years due to their remarkable variety of refinements and generalizations (see, for example see [1–3,5–7,9–12]).

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1 (Hölder–İşcan Integral Inequality [13]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on interval $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad \leq \quad & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\end{aligned}$$

In this paper, by using the Hölder–İşcan integral inequality (better approach than Hölder integral inequality) and together with an integral identity, we present a rather generalization of Hadamard type inequalities for functions whose derivative is absolute value at the certain power are convex and concave.

Let $0 < a < b$, and throughout this paper we will use

$$\begin{aligned} A(a,b) &= \frac{a+b}{2} \\ G(a,b) &= \sqrt{ab} \\ L_p(a,b) &= \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ a \neq b,\ p \in \mathbb{R},\ p \neq -1, 0 \end{aligned}$$

for the arithmetic, geometric, generalized logarithmic mean, respectively.

2. Main Results

We will use the following Lemma to obtain our main results.

Lemma 1 ([8]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx \tag{1}$$

where an empty sum is understood to be nil.

Theorem 2. *For $n \in \mathbb{N}$; let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \le \frac{1}{n!}\left(\frac{b-a}{3}\right)^{\frac{1}{q}}\\&\left\{\left[bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right.\\&\left.+\left[L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,2\left|f^{(n)}(b)\right|^q\right)\right\},\end{aligned} \tag{2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on interval $[a,b]$, then by using Lemma 1, the Hölder–İşcan integral inequality and from the following inequality

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\le\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q,$$

we have

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\\le\;&\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx\\\le\;&\frac{1}{n!}\frac{1}{b-a}\left(\int_a^b(b-x)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\&+\frac{1}{n!}\frac{1}{b-a}\left(\int_a^b(x-a)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\\le\;&\frac{1}{n!}\frac{1}{b-a}\left(\int_a^b(b-x)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}\\&+\frac{1}{n!}\frac{1}{b-a}\left(\int_a^b(x-a)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}\\=\;&\frac{1}{n!}\left(\frac{b-a}{3}\right)^{\frac{1}{q}}\left\{\left[bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right.\\&\left.+\left[L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,2\left|f^{(n)}(b)\right|^q\right)\right\}.\end{aligned}$$

This completes the proof of Theorem 2. □

Corollary 1. *Under the conditions of Theorem 2 for $n = 1$ we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\\le\;&\left(\frac{1}{3}\right)^{\frac{1}{q}}(b-a)^{\frac{1}{q}-1}\left\{\left[bL_p^p(a,b)-L_{p+1}^{p+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right.\\&\left.+\left[L_{p+1}^{p+1}(a,b)-aL_p^p(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,2\left|f'(b)\right|^q\right)\right\}.\end{aligned}$$

Proposition 1. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m, n \in \mathbb{N}$ with $m \geq n$, then we have*

$$\begin{aligned}\left|L_m^m(a,b)\left[(m+1)\sum_{k=0}^{n-1}\frac{(-1)^k P(m,k)}{(k+1)!}-1\right]\right| &\leq \frac{1}{n!}\left(\frac{1}{3}\right)^{\frac{1}{q}}(b-a)^{\frac{1}{q}-1}\\ &\times\left\{\left[bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2a^{(m-n)q},b^{(m-n)q}\right)\right.\\ &\left.+\left[L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(a^{(m-n)q},2b^{(m-n)q}\right)\right\},\end{aligned} \tag{3}$$

where

$$P(m,n)=\begin{cases} m(m-1)...(m-n+1), & m>n\\ n!, & m=n\\ 1, & n=0.\end{cases}$$

Proof. Under the assumption of the Proposition 1, let $f(x)=x^m$, $x\in(0,\infty)$. Then

$$\left|f^{(n)}(x)\right|^q=\left[P(m,n)x^{m-n}\right]^q$$

is convex on $(0,\infty)$ and the result follows directly from Theorem 2. □

Example 1. *If we take $m=2, n=1, p=q=2$ in the inequality (3), then we have the following inequality:*

$$\begin{aligned}2A\left(a^2,b^2\right)+G^2(a,b) &\leq \frac{1}{2\sqrt{2}}\left\{\left(\left[A\left(3a^2,b^2\right)+G^2(a,b)\right]A\left(2a^2,b^2\right)\right)^{\frac{1}{2}}\right.\\ &\left.+\left(\left[A\left(a^2,3b^2\right)+G^2(a,b)\right]A\left(a^2,2b^2\right)\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

Proposition 2. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $n \in \mathbb{N}$, then we have*

$$\begin{aligned}1 &\leq \left(\frac{1}{3}\right)^{\frac{1}{q}}(b-a)^{\frac{1}{q}-1}\left\{\left[bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2a^{-nq},b^{-nq}\right)\right.\\ &\left.+\left[L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(a^{-nq},2b^{-nq}\right)\right\}.\end{aligned} \tag{4}$$

Proof. Under the assumption of the Proposition 2, let $f(x)=\ln x$, $x\in(0,\infty)$. Then

$$\left|f^{(n)}(x)\right|^q=\left[(n-1)!x^{-n}\right]^q$$

is convex on $(0,\infty)$ and the result follows directly from Theorem 2. □

Example 2. *If we take $n=1, p=q=2$ in the inequality (4), then we have the following inequality:*

$$1 \leq \frac{1}{3\sqrt{2}}\left\{\left[A\left(3a^2,b^2\right)+G^2(a,b)\right]^{\frac{1}{2}}\frac{A^{\frac{1}{2}}\left(a^2,2b^2\right)}{G^2(a,b)}+\left[A\left(a^2,3b^2\right)+G^2(a,b)\right]^{\frac{1}{2}}\frac{A^{\frac{1}{2}}\left(2a^2,b^2\right)}{G^2(a,b)}\right\}.$$

Proposition 3. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-\infty,0]\cup[1,\infty)\setminus\{-2q,-q\}$, then we have*

$$\begin{aligned}L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) &\leq \left(\frac{1}{3}\right)^{\frac{1}{q}}(b-a)^{\frac{1}{q}-1}\left\{\left[bL_p^p(a,b)-L_{p+1}^{p+1}(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(2a^m,b^m\right)\right.\\ &\left.+\left[L_{p+1}^{p+1}(a,b)-aL_p^p(a,b)\right]^{\frac{1}{p}}A^{\frac{1}{q}}\left(a^m,2b^m\right)\right\}.\end{aligned} \tag{5}$$

Proof. Under the assumption of the Proposition 3, let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then

$$\left|f'(x)\right|^q = x^m$$

is convex on $(0,\infty)$ and the result follows directly from Corollary 1. □

Example 3. *If we take $m = 2$, $p = q = 2$ in the inequality (5), then we have the following inequality:*

$$\begin{aligned} 2A\left(a^2,b^2\right) + G^2(a,b) \quad &\leq \quad \frac{1}{\sqrt{2}}\left\{\left[A\left(3a^2,b^2\right) + G^2(a,b)\right]^{\frac{1}{2}} A^{\frac{1}{2}}\left(2a^2,b^2\right)\right. \\ &\quad \left. + \left[A\left(a^2,3b^2\right) + G^2(a,b)\right]^{\frac{1}{2}} A^{\frac{1}{2}}\left(a^2,2b^2\right)\right\}. \end{aligned}$$

Theorem 3. *For $n \in \mathbb{N}$; let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leq \frac{1}{n!}\left(\frac{1}{2}\right)^{\frac{1}{p}}(b-a)^{\frac{2}{p}-1} \\ &\left\{\left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right] \\ + \left|f^{(n)}(a)\right|^q\left[L_{nq+2}^{nq+2}(a,b) - 2bL_{nq+1}^{nq+1}(a,b) + b^2L_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[L_{nq+2}^{nq+2}(a,b) - 2aL_{nq+1}^{nq+1}(a,b) + a^2L_{nq}^{nq}(a,b)\right] \\ + \left|f^{(n)}(a)\right|^q\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right\}, \end{aligned} \tag{6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on interval $[a,b]$, by using Lemma 1 and the Hölder–İşcan integral inequality, we obtain the following inequality:

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
\leq\ & \frac{1}{n!}\int_a^b 1.x^n\left|f^{(n)}(x)\right|dx \\
\leq\ & \frac{1}{n!}\frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{1}{n!}\frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,dx\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{b-a}\int_a^b (b-x)\,(x-a)\,x^{nq}dx+\frac{\left|f^{(n)}(a)\right|^q}{b-a}\int_a^b (b-x)^2\,x^{nq}dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,dx\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{b-a}\int_a^b (x-a)^2\,x^{nq}dx+\frac{\left|f^{(n)}(a)\right|^q}{b-a}\int_a^b (x-a)\,(b-x)\,x^{nq}dx\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^{\frac{2}{p}-1}}{n!2^{\frac{1}{p}}}\left\{\left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[-L_{nq+2}^{nq+2}(a,b)+(a+b)L_{nq+1}^{nq+1}(a,b)-abL_{nq}^{nq}(a,b)\right]\\+\left|f^{(n)}(a)\right|^q\left[L_{nq+2}^{nq+2}(a,b)-2bL_{nq+1}^{nq+1}(a,b)+b^2L_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[L_{nq+2}^{nq+2}(a,b)-2aL_{nq+1}^{nq+1}(a,b)+a^2L_{nq}^{nq}(a,b)\right]\\+\left|f^{(n)}(a)\right|^q\left[-L_{nq+2}^{nq+2}(a,b)+(a+b)L_{nq+1}^{nq+1}(a,b)-abL_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof of Theorem 3, after a little simplifications. □

Remark 1. *In [8], Maden et al. obtained the following inequality using Hölder inequality and similar proof method of Theorem 3.*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \leq \frac{1}{n!}(b-a)^{\frac{1}{p}} \qquad (7)\\
&\times\left\{\left|f^{(n)}(b)\right|^q\left[L_{nq+1}^{nq+1}(a,b)-aL_{nq}^{nq}(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_{nq}^{nq}(a,b)-L_{nq+1}^{nq+1}(a,b)\right]\right\}^{\frac{1}{q}}
\end{aligned}$$

The inequality (6) gives better results than the inequality (7). Indeed, using the inequality $\left(x^\lambda+y^\lambda\right)/2 \leq \left((x+y)/2\right)^\lambda$, $x,y \geq 0$, $0 < \lambda \leq 1$ *by simple calculation we get*

$$\begin{aligned}
&\left\{\frac{1}{2}\left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[-L_{nq+2}^{nq+2}(a,b)+(a+b)L_{nq+1}^{nq+1}(a,b)-abL_{nq}^{nq}(a,b)\right]\\+\left|f^{(n)}(a)\right|^q\left[L_{nq+2}^{nq+2}(a,b)-2bL_{nq+1}^{nq+1}(a,b)+b^2L_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{1}{2}\left(\begin{array}{c}\left|f^{(n)}(b)\right|^q\left[L_{nq+2}^{nq+2}(a,b)-2aL_{nq+1}^{nq+1}(a,b)+a^2L_{nq}^{nq}(a,b)\right]\\+\left|f^{(n)}(a)\right|^q\left[-L_{nq+2}^{nq+2}(a,b)+(a+b)L_{nq+1}^{nq+1}(a,b)-abL_{nq}^{nq}(a,b)\right]\end{array}\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}}\left\{\left|f^{(n)}(b)\right|^q\left[L_{nq+1}^{nq+1}(a,b)-aL_{nq}^{nq}(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_{nq}^{nq}(a,b)-L_{nq+1}^{nq+1}(a,b)\right]\right\}^{\frac{1}{q}}.
\end{aligned}$$

which shows that the inequality (6) gives better results than the inequality (7).

Corollary 2. *Under the conditions of Theorem 3 for $n = 1$, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \left(\frac{1}{2}\right)^{\frac{1}{p}} (b-a)^{\frac{2}{p}-2} \left\{ \left(\begin{array}{c} |f'(b)|^q \left[-L_{q+2}^{q+2}(a,b) + (a+b)L_{q+1}^{q+1}(a,b) - abL_q^q(a,b)\right] \\ +|f'(a)|^q \left[L_{q+2}^{q+2}(a,b) - 2bL_{q+1}^{q+1}(a,b) + b^2 L_q^q(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\begin{array}{c} |f'(b)|^q \left[L_{q+2}^{q+2}(a,b) - 2aL_{q+1}^{q+1}(a,b) + a^2 L_q^q(a,b)\right] \\ +|f'(a)|^q \left[-L_{q+2}^{q+2}(a,b) + (a+b)L_{q+1}^{q+1}(a,b) - abL_q^q(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proposition 4. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m, n \in \mathbb{N}$ with $m \geq n$, then we have*

$$\begin{aligned}
&\left|L_m^m(a,b)\left[(m+1)\sum_{k=0}^{n-1}\frac{(-1)^k P(m,k)}{(k+1)!} - 1\right]\right| \leq \frac{P(m,n)}{n!}\left(\frac{1}{2}\right)^{\frac{1}{p}} (b-a)^{\frac{2}{p}-1} \\
&\left\{ \left(\begin{array}{c} b^{(m-n)q}\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right] \\ +a^{(m-n)q}\left[L_{nq+2}^{nq+2}(a,b) - 2bL_{nq+1}^{nq+1}(a,b) + b^2 L_{nq}^{nq}(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\begin{array}{c} b^{(m-n)q}\left[L_{nq+2}^{nq+2}(a,b) - 2aL_{nq+1}^{nq+1}(a,b) + a^2 L_{nq}^{nq}(a,b)\right] \\ +a^{(m-n)q}\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

where

$$P(m,n) = \begin{cases} m(m-1)...(m-n+1), & m > n \\ n!, & m = n \\ 1, & n = 0. \end{cases}$$

Proof. Under the assumption of the Proposition 4, let $f(x) = x^m$, $x \in (0,\infty)$. Then

$$\left|f^{(n)}(x)\right|^q = \left[P(m,n)x^{m-n}\right]^q$$

is convex on $(0,\infty)$ and the result follows directly from Theorem 3, respectively. □

Proposition 5. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $n \in \mathbb{N}$, then we have*

$$\begin{aligned}
1 \leq\ & \frac{(b-a)^{\frac{2}{p}-2}}{2^{\frac{1}{p}}} \left\{ \left(\begin{array}{c} b^{-nq}\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right] \\ +a^{-nq}\left[L_{nq+2}^{nq+2}(a,b) - 2bL_{nq+1}^{nq+1}(a,b) + b^2 L_{nq}^{nq}(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\begin{array}{c} b^{-nq}\left[L_{nq+2}^{nq+2}(a,b) - 2aL_{nq+1}^{nq+1}(a,b) + a^2 L_{nq}^{nq}(a,b)\right] \\ +a^{-nq}\left[-L_{nq+2}^{nq+2}(a,b) + (a+b)L_{nq+1}^{nq+1}(a,b) - abL_{nq}^{nq}(a,b)\right] \end{array} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. Under the assumption of the Proposition 5, let $f(x) = \ln x$, $x \in (0,\infty)$. Then

$$\left|f^{(n)}(x)\right|^q = \left[(n-1)!x^{-n}\right]^q$$

is convex on $(0,\infty)$ and the result follows directly from Theorem 3. □

Proposition 6. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-\infty, 0] \cup [1,\infty) \setminus \{-2q, -q\}$, we have*

$$L^{\frac{m}{q}+1}_{\frac{m}{q}+1}(a,b) \leq \frac{(b-a)^{\frac{2}{p}-2}}{2^{\frac{1}{p}}} \left\{ \left(\begin{array}{c} b^m \left[-L^{q+2}_{q+2}(a,b) + (a+b)L^{q+1}_{q+1}(a,b) - abL^{q}_{q}(a,b) \right] \\ +a^m \left[L^{q+2}_{q+2}(a,b) - 2bL^{q+1}_{q+1}(a,b) + b^2 L^{q}_{q}(a,b) \right] \end{array} \right)^{\frac{1}{q}} + \left(\begin{array}{c} b^m \left[L^{q+2}_{q+2}(a,b) - 2aL^{q+1}_{q+1}(a,b) + a^2 L^{q}_{q}(a,b) \right] \\ +a^m \left[-L^{q+2}_{q+2}(a,b) + (a+b)L^{q+1}_{q+1}(a,b) - abL^{q}_{q}(a,b) \right] \end{array} \right)^{\frac{1}{q}} \right\}.$$

Proof. The result follows directly from Corollary 2 for the function

$$f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}, x \in (0,\infty).$$

This completes the proof of Proposition. □

Corollary 3. *For $m = 1$ from Proposition 6, we obtain the following inequality:*

$$L^{\frac{1}{q}+1}_{\frac{1}{q}+1}(a,b) \leq \frac{(b-a)^{\frac{1}{p}-1}}{2^{\frac{1}{p}}} \left\{ \left[-L^{q+2}_{q+2}(a,b) - L^{q+1}_{q+1}(a,b) + bL^{q}_{q}(a,b) \right]^{\frac{1}{q}} + \left[L^{q+2}_{q+2}(a,b) + L^{q+1}_{q+1}(a,b) - aL^{q}_{q}(a,b) \right]^{\frac{1}{q}} \right\}.$$

Theorem 4. *For $n \in \mathbb{N}$; let $f : I \subset (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q > 1$ is concave on $[a,b]$, then the following inequality holds*

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \\ \leq\ & \frac{(b-a)^{\frac{1}{q}} \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|}{n!} \left\{ \left(bL^{np}_{np}(a,b) - L^{np}_{np+1}(a,b) \right)^{\frac{1}{p}} + \left(L^{np+1}_{np+1}(a,b) - aL^{np+1}_{np}(a,b) \right)^{\frac{1}{p}} \right\}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since the function $\left|f^{(n)}\right|^q$ for $q > 1$ is concave on interval $[a,b]$, with respect to Hermite–Hadamard integral inequality, we get

$$\int_a^b \left|f^{(n)}(x)\right|^q dx \leq (b-a) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q$$

and thus we have

$$\frac{1}{b-a} \int_a^b (b-x) \left|f^{(n)}(x)\right|^q dx \leq \int_a^b \left|f^{(n)}(x)\right|^q dx \leq (b-a) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q,$$

similarly

$$\frac{1}{b-a} \int_a^b (x-a) \left|f^{(n)}(x)\right|^q dx \leq (b-a) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q.$$

Using Lemma 1 and the Hölder–İşcan integral inequality, we obtain

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
\leq\ & \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
\leq\ & \frac{1}{n!}\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b(x-a)\,x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{1}{n!}\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,x^{np}dx\right)^{\frac{1}{p}}\left((b-a)^2\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b(x-a)\,x^{np}dx\right)^{\frac{1}{p}}\left((b-a)^2\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^{\frac{1}{q}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|}{n!}\left\{\left(bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right)^{\frac{1}{p}}+\left(L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right)^{\frac{1}{p}}\right\}
\end{aligned}$$

This completes the proof of Theorem 4. □

Corollary 4. *Under the conditions of the Theorem 4 for $n=1$, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)^{1/q}\left|f'\left(\frac{a+b}{2}\right)\right|\left\{\left(bL_p^p(a,b)-L_{p+1}^{p+1}(a,b)\right)^{\frac{1}{p}}+\left(L_{p+1}^{p+1}(a,b)-aL_p^p(a,b)\right)^{\frac{1}{p}}\right\}.
\end{aligned}$$

Proposition 7. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m,n\in\mathbb{N}$ with $m\geq n$, then we have*

$$\begin{aligned}
&\left|L_m^m(a,b)\left[(m+1)\sum_{k=0}^{n-1}\frac{(-1)^k P(m,k)}{(k+1)!}-1\right]\right| \\
\leq\ & \frac{P(m,n)}{n!}(b-a)^{\frac{1}{q}}\left(\frac{a+b}{2}\right)^{m-n}\left\{\left(bL_{np}^{np}(a,b)-L_{np+1}^{np+1}(a,b)\right)^{\frac{1}{p}}+\left(L_{np+1}^{np+1}(a,b)-aL_{np}^{np}(a,b)\right)^{\frac{1}{p}}\right\},
\end{aligned}$$

where

$$P(m,n)=\begin{cases} m(m-1)...(m-n+1), & m>n \\ n!, & m=n \\ 1, & n=0. \end{cases}$$

Proof. Under the assumption of the Proposition 7, let $f(x)=x^m$, $x\in(0,\infty)$. Then

$$\left|f^{(n)}(x)\right|^q=\left[P(m,n)x^{m-n}\right]^q$$

is convex on $(0,\infty)$ and the result follows directly from the Theorem 4. □

Proposition 8. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $n \in \mathbb{N}$, then we have*

$$1 \leq (b-a)^{\frac{1}{q}} A^{-n}(a,b) \left\{ \left(bL_{np}^{np}(a,b) - L_{np+1}^{np}(a,b) \right)^{\frac{1}{p}} + \left(L_{np+1}^{np+1}(a,b) - aL_{np}^{np+1}(a,b) \right)^{\frac{1}{p}} \right\}.$$

Proof. Under the assumption of the Proposition 8, let $f(x) = \ln x$, $x \in (0,\infty)$. Then

$$\left| f^{(n)}(x) \right|^q = \left[(n-1)! x^{-n} \right]^q$$

is convex on $(0,\infty)$ and the result follows directly from the Theorem 4. □

Proposition 9. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in [0,1]$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq (b-a)^{\frac{1}{q}} A^{\frac{m}{q}}(a,b) \left\{ \left(bL_p^p(a,b) - L_{p+1}^{p+1}(a,b) \right)^{\frac{1}{p}} + \left(L_{p+1}^{p+1}(a,b) - aL_p^p(a,b) \right)^{\frac{1}{p}} \right\}.$$

Proof. Under the assumption of the Proposition 9, let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then

$$\left| f'(x) \right|^q = x^m$$

is concave on $(0,\infty)$ and the result follows directly from the Corollary 4. □

3. Conclusions

By using an integral identity together with the Hölder–İşcan integral inequality (which is a better approach than Hölder integral inequality), we obtain several new inequalities for n-times differentiable convex and concave mappings. We would like to emphasize that some new integral inequalities can be obtained by using a similar method to different types of convex functions.

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