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Establishing New Criteria for Oscillation of Odd-Order Nonlinear Differential Equations

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Abstract: By establishing new conditions for the non-existence of so-called Kneser solutions, we can generate sufficient conditions to ensure that all solutions of odd-order equations are oscillatory. Our results improve and expand the previous results in the literature.

Keywords: odd-order differential equations; Kneser solutions; oscillation criteria

1. Introduction

In the 20th century, the extremely fast development of science led to applications in the fields of biology, population, chemistry, medicine dynamics, social sciences, genetic engineering, economics, and others. Many of these phenomena are modeled by delay differential equations. All these disciplines were promoted to a higher level and discoveries were made with the help of this kind of mathematical modeling.

The neutral differential equations are the differential equations in which the delayed argument occurs in the highest derivative of the state variable. The neutral equations appear in the modeling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits); see [1].

Recently, an increasing interest in establishing conditions for the oscillatory behavior of different order of differential equations has been observed; see [2–9].

It is known that determination of the signs of the derivatives of the solution is necessary and causes a significant effect before studying the oscillation of delay differential equations. The other essential thing is to establish relationships between derivatives of different orders, which may lead to additional restrictions during the study. In odd-order differential equations, in some cases, it is difficult to find relationships between derivatives of different orders, which in turn is central to the study of oscillatory behavior. Therefore, it can very easily be observed that differential equations of odd-order received less attention than differential equations with even-order. Additionally, most studies are concerned with finding sufficient conditions that guarantee that every non-oscillating solution tends to zero; see [4,10–20].

In this paper, in Section 2, we offer some auxiliary lemmas that define the different cases of signs of derivatives and the relationships between derivatives of different orders. In Section 3, we establish a set of new criteria that ensure that there are no non-oscillating solutions in each case of derivatives separately. In Section 4, we establish new criteria for the oscillation of all solutions of the studied equation. Finally, in conclusion, we discuss the results and compare them to the related works.

In detail, we investigate the oscillatory properties of solutions to the odd-order neutral equation

$$\left(r(t)\left(\left(x(t)+p(t)\,x(\tau(t))\right)^{(n-1)}\right)^{\alpha}\right)'+q(t)\,f(x(g(t)))=0, \tag{1}$$

where n is an odd natural number. Moreover, we suppose that

Hypothesis 1 (H1). *α is the ratio of odd positive integers, $r\in C^1(I_0,\mathbb{R}^+)$, $p\in C(I_0,[0,p_0])$, where p_0 is a positive constant, $\tau, g\in C^1(I_0,\mathbb{R})$, $q\in C(I_0,[0,\infty))$, $r'(t)\geq 0$, $g(t)\leq t$, $\lim_{t\to\infty} g(t)=\infty$, $\lim_{t\to\infty}\tau(t)=\infty$, $\int_{t_0}^{\infty} r^{-1/\alpha}(\rho)\,\mathrm{d}\rho=\infty$, q is not eventually zero on any half line I_* for $t_*\geq t_0$, and $I_s:=[t_s,\infty)$.*

Hypothesis 2 (H2). *$f\in C(\mathbb{R},\mathbb{R})$ and there exists a positive constant k such that $f(x)\geq kx^{\alpha}$.*

Next, we present the basic definitions.

Definition 1. *The function $z(t):=x(t)+p(t)\,x(\tau(t))$ is called the corresponding function of x, and*

$$\phi(s,t)=\int_s^t r^{-1/\alpha}(\varrho)\,\mathrm{d}\varrho$$

is called the canonical operator.

Definition 2. *Let x be a real-valued function defined for all t in a real interval I_x, $t_x\geq t_0$, and having a n^{th} derivative for all $t\in I_x$. The function x is called a* ***solution*** *of the differential equation (Equation (1)) on I if x is continuous; $r\left(z^{(n-1)}\right)^{\alpha}$ is continuously differentiable and x satisfies (1), for all t in I_x.*

Definition 3. *A nontrivial solution x of (1) is said to be* ***oscillatory*** *if it has arbitrary large zeros; that is, there exists a sequence of zeros $\{t_n\}_{n=0}^{\infty}$ (i.e., $x(t_n)=0$) of x such that $\lim_{n\to\infty} t_n=\infty$. Otherwise, it is said to be* ***non-oscillatory****.*

Notation 1. *The set of all eventually positive solutions of (1) is denoted by X^+.*

We restrict our discussion to those solutions x of (1) which satisfy $\sup\{|x(t)|:\ t_1\leq t_0\}>0$ for every $t_1\in I_x$. All functional inequalities and properties, such as increasing, decreasing, positive, and so on, are assumed to hold eventually; that is, they are satisfied for all t large enough.

2. Preliminary Results

During this part of the paper, we provide auxiliary lemmas. These lemmas will be the cornerstone of the main results.

Notation 2. *For the sake of convenience, we use the following notation:*

$$\eta(t):=\frac{\lambda}{(n-2)!}\frac{g^{n-2}(t)\,g'(t)}{r^{1/\alpha}(t)},$$

$$\Theta(t):=kq(t)\left(1-p(g(t))\right)^{\alpha},\quad \widetilde{\Theta}(t):=\int_t^{\infty}\Theta(\varrho)\,\mathrm{d}\varrho,$$

$$Q_1(t):=\min\{q(t),q(\tau(t))\},\quad Q_2(t):=\min\left\{q\left(g^{-1}(t)\right),q\left(g^{-1}(\tau(t))\right)\right\},$$

$$\psi_1(s,t) := \int_s^t \phi(\varrho,t)\,\mathrm{d}\varrho, \quad \psi_{k+1}(s,t) := \int_s^t \psi_k(\varrho,t)\,\mathrm{d}\varrho,\ k=1,2,...,n-2,$$

and

$$\mu := \begin{cases} 1 & \text{for } 0<\alpha\le 1; \\ 2^{\alpha-1} & \text{for } \alpha>1. \end{cases}$$

Lemma 1. *([21], Lemma 1, Lemma 2) Assume that* $u,\ v \in [0,\infty)$*. Then,*

$$(u+v)^\alpha \le \mu\left(u^\alpha + v^\alpha\right).$$

Lemma 2. *[22] Let* $F \in C^n([t_0,\infty),(0,\infty))$*. Assume that* $F^{(n)}(t)$ *is of fixed sign and not identically zero on* I_0 *and that there exists a* $t_1 \ge t_0$ *such that* $F^{(n-1)}(t)F^{(n)}(t) \le 0$ *for all* $t \ge t_1$*. If* $\lim_{t\to\infty} F(t) \ne 0$*; then for every* $\lambda \in (0,1)$ *there exists* $t_\mu \ge t_1$ *such that*

$$F(t) \ge \frac{\lambda}{(n-1)!} t^{n-1}\left|F^{(n-1)}(t)\right| \text{ for } t \ge t_\mu.$$

The following lemma is a well-known result; see ([20], Lemma 2.4, Lemma 2.5); also see ([22], Lemma 2.2.1).

Lemma 3. *Suppose that* $x \in X^+$*. Then, there exists a sufficiently large* $t_1 \ge t_0$ *such that, for all* $t \ge t_1$*,*

$$z(t)>0,\ z''(t)>0,\ z^{(n-1)}(t)>0 \text{ and } z^{(n)}(t)\le 0.$$

Furthermore, there are only two cases:

$$\mathbf{P}: z'(t)>0,$$

or

$$\mathbf{N}:\ (-1)^k z^{(k)}(t)>0, \text{ for } k=1,2,...,n-2.$$

Lemma 4. *Suppose that* $x \in X^+$ *and z satisfies* **N***. Then*

$$z(\rho) \ge r^{1/\alpha}(\sigma)\, z^{(n-1)}(\sigma)\, \psi_{n-2}(\rho,\sigma) \tag{2}$$

for $\rho \le \sigma$*.*

Proof. It follows from the monotonicity of $r(z^{(n-1)})(t)$ that

$$\begin{aligned} -z^{(n-2)}(\rho) &\ge z^{(n-2)}(\sigma) - z^{(n-2)}(\rho) = \int_\rho^\sigma \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s)\, z^{(n-1)}(s)\,\mathrm{d}s \\ &\ge r^{1/\alpha}(\sigma)\, z^{(n-1)}(\sigma)\, \phi(\rho,\sigma). \end{aligned} \tag{3}$$

Integrating (3) from ρ to σ, we have

$$-z^{(n-2)}(\sigma) + z^{(n-2)}(\rho) \ge r^{1/\alpha}(\sigma)\, z^{(n-1)}(\sigma) \int_\rho^\sigma \phi(s,\sigma)\,\mathrm{d}s$$

and so

$$z^{(n-3)}(\rho) \ge r^{1/\alpha}(\sigma)\, z^{(n-1)}(\sigma)\, \psi_1(\rho,\sigma). \tag{4}$$

Integrating (4) $n-3$ times from ρ to σ, we get

$$z(\rho) \geq r^{1/\alpha}(\sigma)\, z^{(n-1)}(\sigma)\, \psi_{n-2}(\rho,\sigma).$$

The proof is complete. □

Lemma 5. *Suppose that $x \in X^{+}$ and z satisfies* **P**. *If $p_0 < 1$, g is non-decreasing and*

$$w(t) := \delta(t)\, r(t) \left(\frac{z^{(n-1)}(t)}{z(g(t))} \right)^{\alpha}, \tag{5}$$

then

$$w'(t) \leq \frac{\delta'(t)}{\delta(t)} w(t) - \delta(t)\,\Theta(t) - \alpha\delta(t)\,\eta(t)\, w^{1+1/\alpha}(t), \tag{6}$$

where $\delta \in C^{1}(I_0,(0,\infty))$.

Proof. Assume that $x \in X^{+}$and z satisfies **P**. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. Since $z(t) > x(t)$ and $z'(t) > 0$, it follows from the Definition 1 that $x(t) > (1-p(t))\, z(t)$. Thus, (1) becomes

$$\begin{aligned} (r(t)(z^{(n-1)}(t))^{\alpha})' &= -q(t) f(x(g(t))) \leq -kq(t)\, x^{\alpha}(g(t)) \\ &\leq -kq(t)\,(1-p(g(t)))^{\alpha} z^{\alpha}(g(t)). \end{aligned} \tag{7}$$

Using Lemma 2 with $F = z'$, we obtain for every $\lambda \in (0,1)$,

$$(n-2)!z'(t) \geq \lambda t^{n-2} z^{(n-1)}(t)$$

which with the fact that $z^{(n)} \leq 0$ gives

$$z'(g(t)) \geq \frac{\lambda}{(n-2)!} g^{n-2}(t)\, z^{(n-1)}(g(t)) \geq \frac{\lambda}{(n-2)!} g^{n-2}(t)\, z^{(n-1)}(t). \tag{8}$$

Hence, from (5), (7) and (8), we get

$$\begin{aligned} w(t) &= \frac{\delta'(t)}{\delta(t)} w(t) + \delta(t) \frac{\left(r \left(z^{(n-1)} \right)^{\alpha} \right)'(t)}{z^{\alpha}(g(t))} - \delta(t) \frac{\left(r \left(z^{(n-1)} \right)^{\alpha} \right)(t)}{z^{\alpha+1}(g(t))} \alpha z'(g(t))\, g'(t) \\ &\leq \frac{\delta'(t)}{\delta(t)} w(t) - \delta(t)\,\Theta(t) - \frac{\alpha\lambda}{(n-2)!} \delta(t)\, r(t)\, g^{n-2}(t)\, g'(t) \left(\frac{z^{(n-1)}(t)}{z(g(t))} \right)^{\alpha+1} \\ &\leq \frac{\delta'(t)}{\delta(t)} w(t) - \delta(t)\,\Theta(t) - \alpha\delta(t)\,\eta(t)\, w^{1+1/\alpha}(t). \end{aligned}$$

The proof is complete. □

Lemma 6. *Suppose that $x \in X^{+}$. If*

$$\tau'(t) \geq \tau_0 > 0, \tag{9}$$

then

$$\left((r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0} \left(r \left(z^{(n-1)} \right)^{\alpha} \right)(\tau(t)) \right)' + kQ(t)\, z^{\alpha}(g(t)) \leq 0. \tag{10}$$

Moreover, if (9) holds and

$$\left(g^{-1}(t)\right)' \geq g_0 > 0, \tag{11}$$

then

$$\left(\frac{1}{g_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)\left(g^{-1}(t)\right) + \frac{p_0^{\alpha}}{g_0\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)\left(g^{-1}(\tau(t))\right)\right)' + \frac{k}{\mu}Q_2(t)\, z^{\alpha}(t) \leq 0. \tag{12}$$

Proof. Let $x \in X^+$. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. From (1), we get

$$\frac{1}{\tau'(t)}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'(\tau(t)) + kq(\tau(t))\, x^{\alpha}(g(\tau(t))) \leq 0, \tag{13}$$

Combining (1) and (13) and taking into account that $\tau'(t) \geq \tau_0$, we obtain

$$(r(z^{(n-1)})^{\alpha})'(t) + \frac{p_0^{\alpha}}{\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'(\tau(t)) + kq(t)\, x^{\alpha}(g(t)) + kp_0^{\alpha}q(\tau(t))\, x^{\alpha}(g(\tau(t))) \leq 0. \tag{14}$$

This implies that

$$\left((r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)(\tau(t))\right)' + kQ_1(t)\left(x^{\alpha}(g(t)) + p_0^{\alpha}x^{\alpha}(g(\tau(t)))\right) \leq 0.$$

Using Lemma 1, we obtain

$$\left((r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)(\tau(t))\right)' + \frac{k}{\mu}Q_1(t)\left((x(g(t)) + p_0x(g(\tau(t))))\right)^{\alpha} \leq 0.$$

From the definition of z, it is easy to conclude that

$$\left((r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)(\tau(t))\right)' + \frac{k}{\mu}Q_1(t)\, z^{\alpha}(g(t)) \leq 0.$$

Next, from (1), we get

$$\frac{1}{(g^{-1}(t))'}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'\left(g^{-1}(t)\right) + kq\left(g^{-1}(t)\right)x^{\alpha}(t) \leq 0 \tag{15}$$

and

$$\frac{1}{(g^{-1}(\tau(t)))'}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'\left(g^{-1}(\tau(t))\right) + kq\left(g^{-1}(\tau(t))\right)x^{\alpha}(\tau(t)) \leq 0. \tag{16}$$

Using (15) and (16) and taking into account (9) and (11), we obtain

$$\begin{aligned}\frac{1}{g_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'\left(g^{-1}(t)\right) + \frac{p_0^{\alpha}}{g_0\tau_0}\left(r\left(z^{(n-1)}\right)^{\alpha}\right)'\left(g^{-1}(\tau(t))\right) + kq\left(g^{-1}(t)\right)x^{\alpha}(t)\\ + kq\left(g^{-1}(\tau(t))\right)x^{\alpha}(\tau(t)) \leq 0.\end{aligned} \tag{17}$$

By replacing (14) with (17), this part of proof is similar to that of the previous case and so we omit it. □

3. Nonexistence Criteria of Non-Oscillatory Solutions

At the beginning of this section, we define the following classes:

Notation 3. *The set of all positive solutions of (1) whose corresponding function z satisfies* **P** *or* **N** *is denoted by* X_P^+ *or* X_N^+*, respectively.*

Now, we create various criteria that ensure that there are no positive solutions of (1) whose corresponding function satisfies **P**.

Theorem 1. *If*

$$\frac{1}{\widetilde{\Theta}(t)} \int_t^\infty \eta(\varrho) \widetilde{\Theta}^{1+1/\alpha}(\varrho) \, d\varrho > \frac{1}{(1+\alpha)^{1+1/\alpha}}, \tag{18}$$

then X_P^+ *is an empty class.*

Proof. Assume the contrary that $x \in X_P^+$. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. Using Lemma 5 with $\delta(t) := 1$, we arrive at

$$w'(t) \leq -\Theta(t) - \alpha\eta(t) w^{1+1/\alpha}(t) < 0.$$

By integrating the last inequality from t to ∞, we find

$$w(t) \geq \widetilde{\Theta}(t) + \alpha \int_t^\infty \eta(\varrho) w^{1+1/\alpha}(\varrho) \, d\varrho. \tag{19}$$

This implies that

$$\frac{w(t)}{\widetilde{\Theta}(t)} \geq 1 + \frac{\alpha}{\widetilde{\Theta}(t)} \int_t^\infty \eta(\varrho) \widetilde{\Theta}^{1+1/\alpha}(\varrho) \left(\frac{w(\varrho)}{\widetilde{\Theta}(\varrho)}\right)^{1+1/\alpha} d\varrho. \tag{20}$$

From (19), we note that $w(t) \geq \widetilde{\Theta}(t)$. Thus, we have

$$\beta := \inf \frac{w(t)}{\widetilde{\Theta}(t)} \geq 1. \tag{21}$$

Taking into account (18) and (21), (20) becomes

$$\beta \geq 1 + \alpha \left(\frac{\beta}{1+\alpha}\right)^{1+1/\alpha}$$

or

$$\frac{\beta}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1} \left(\frac{\beta}{\alpha+1}\right)^{1+1/\alpha},$$

which contradicts the expected value of $\beta > 1$ and $\alpha > 0$; therefore, the proof is complete. □

Now, let $\{S_m(t)\}_{m=0}^\infty$ be a sequence of continuous functions defined as follows: $S_0(t) = \widetilde{\Theta}(t)$ and

$$S_{m+1}(t) = S_0(t) + \alpha \int_t^\infty \eta(\varrho) S_m^{1+1/\alpha}(\varrho) \, d\varrho, \; m = 0, 1, ... \tag{22}$$

By using the definition of $\{S_m(t)\}_{m=0}^\infty$, we can infer more new criteria as follows:

Theorem 2. *If*

$$\int_{t_0}^{\infty} \varphi(\varrho)\,\Theta(\varrho)\,\mathrm{d}\varrho = \infty, \tag{23}$$

then X_P^+ is an empty class, where

$$\varphi(t) := \exp\left(\int_{t_1}^{t} \alpha\eta(\varrho)\,S_m^{1/\alpha}(\varrho)\,\mathrm{d}\varrho\right).$$

Proof. Assume the contrary that $x \in X_P^+$. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. From Theorem 1, we have that (19) holds. By induction, using (19), it is easy to see that the sequence $\{S_m(t)\}_{m=0}^{\infty}$ is non-decreasing and $w(t) \geq S_m(t)$. Thus the sequence $\{S_m(t)\}_{m=0}^{\infty}$ converges to $S(t)$. By the Lebesgue monotone convergence theorem and letting $m \to \infty$ in (22), we get

$$S(t) = S_0(t) + \alpha\int_t^{\infty} \eta(\varrho)\,S^{1+1/\alpha}(\varrho)\,\mathrm{d}\varrho$$

which with $S(t) \geq S_m(t)$, gives

$$\begin{aligned} S'(t) &= -\Theta(t) - \alpha\eta(\varrho)\,S^{1+1/\alpha}(\varrho) \\ &\leq -\Theta(t) - \alpha\eta(\varrho)\,S(\varrho)\,S_m^{1/\alpha}(\varrho), \end{aligned}$$

and so

$$S'(t) + \left(\alpha\eta(\varrho)\,S_m^{1/\alpha}(\varrho)\right)S(\varrho) \leq -\Theta(t).$$

Thus, we get that

$$\varphi(t)\,S'(t) + \varphi(t)\left(\alpha\eta(\varrho)\,S_m^{1/\alpha}(\varrho)\right)S(\varrho) \leq -\varphi(t)\,\Theta(t)$$

or

$$(\varphi(t)\,S(t))' \leq -\varphi(t)\,\Theta(t). \tag{24}$$

Integrating (24) from $t_1$to t, we obtain

$$\varphi(t)\,S(t) \leq \varphi(t_1)\,S(t_1) - \int_{t_1}^{t} \varphi(\varrho)\,\Theta(\varrho)\,\mathrm{d}\varrho.$$

However, letting $t \to \infty$ and using (23), the above inequality yields $\varphi(t)\,S(t) \to -\infty$, which contradicts the fact that $\varphi(t)\,S(t)$ is nonnegative. The proof is complete. □

Theorem 3. *If there exist some $\lambda \in (0,1)$ and $S_m(t)$ such that*

$$\limsup_{t\to\infty} \frac{1}{r(t)} g^{\alpha(n-1)}(t)\,S_m(t) > \left(\frac{(n-1)!}{\lambda}\right)^{\alpha}, \tag{25}$$

then X_P^+ is an empty class.

Proof. Assume the contrary that $x \in X_P^+$. Using Lemma 2 and taking into account the fact that $z^{(n-1)}(t)$ is non-increasing, we find

$$\begin{aligned} z(g(t)) &\geq \frac{\lambda}{(n-1)!} g^{n-1}(t) z^{(n-1)}(g(t)) \\ &\geq \frac{\lambda}{(n-1)!} g^{n-1}(t) z^{(n-1)}(t), \end{aligned}$$

for $\lambda \in (0,1)$. Then, from definition of $w(t)$ with $\delta(t) = 1$, we have

$$\frac{1}{w(t)} = \frac{1}{r(t)} \left(\frac{z(g(t))}{z^{(n-1)}(t)} \right)^{\alpha} \geq \frac{1}{r(t)} \left(\frac{\lambda g^{n-1}(t)}{(n-1)!} \right)^{\alpha},$$

and so

$$\left(\frac{(n-1)!}{\lambda} \right)^{\alpha} \geq \frac{1}{r(t)} g^{\alpha(n-1)}(t) w(t) \geq \frac{1}{r(t)} g^{\alpha(n-1)}(t) S_m(t),$$

which contradicts (25). The proof is complete. □

Corollary 1. *If there exist some $\lambda \in (0,1)$ such that*

$$\limsup_{t \to \infty} \frac{1}{r(t)} \left(g^{(n-1)}(t) \right)^{\alpha} \int_t^{\infty} \Theta(\varrho) \, d\varrho > \left(\frac{(n-1)!}{\lambda} \right)^{\alpha}, \tag{26}$$

then X_P^+ is an empty class.

Proof. Letting $m = 0$ in Theorem 3, we get (26). □

Next, by using comparison principles, we will create various criteria that ensure that there are no positive solutions of (1) whose corresponding function satisfies **N**.

Theorem 4. *If the first-order advanced inequality*

$$G'(t) + \frac{k\tau_0}{\tau_0 + p_0^{\alpha}} Q_1(t) \psi_{n-2}^{\alpha}(g(t), t) G\left(\tau^{-1}(t)\right) \leq 0, \tag{27}$$

then X_N^+ is an empty class.

Proof. Assume the contrary that $x \in X^+$ and z satisfy **N**. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. From Lemmas 4 and 6, we arrive at (2) and (10), respectively. Now from (2), we get

$$z(g(t)) \geq r^{1/\alpha}(t) z^{(n-1)}(t) \psi_{n-2}(g(t), t) \tag{28}$$

which, by virtue of (10) yields that

$$0 \geq \left((r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0} \left(r\left(z^{(n-1)}\right)^{\alpha} \right)(\tau(t)) \right)' + kQ_1(t) r(t) \left(z^{(n-1)}(t) \psi_{n-2}(g(t), t) \right)^{\alpha}. \tag{29}$$

Now, set

$$G(t) := (r(z^{(n-1)})^{\alpha})(t) + \frac{p_0^{\alpha}}{\tau_0} \left(r\left(z^{(n-1)}\right)^{\alpha} \right)(\tau(t)) > 0. \tag{30}$$

Using the fact that $r(t)\left(z^{(n-1)}(t)\right)$ is non-increasing, we obtain

$$G(t) \leq r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)$$

or equivalently,

$$r(t)\left(z^{(n-1)}(t)\right)^{\alpha} \geq \frac{\tau_0}{\tau_0+p_0^{\alpha}} G\left(\tau^{-1}(t)\right). \tag{31}$$

Using (31) in (29), we see that G is a positive solution of the inequality

$$G'(t)+\frac{k\tau_0}{\tau_0+p_0^{\alpha}} Q_1(t)\,\psi_{n-2}^{\alpha}(g(t),t)\,G\left(\tau^{-1}(t)\right) \leq 0.$$

This a contradiction, and thus the proof is complete. □

Theorem 5. *If there exists a function $\vartheta(t) \in C(I_0,(0,\infty))$ satisfying*

$$g(t) \leq \vartheta(t),\ \tau^{-1}(\vartheta(t)) < t \tag{32}$$

and the first-order delay equation

$$G'(t)+\frac{k\tau_0}{\tau_0+p_0^{\alpha}} Q_1(t)\,\psi_{n-2}^{\alpha}(g(t),\vartheta(t))\,G\left(\tau^{-1}(\vartheta(t))\right)=0 \tag{33}$$

is oscillatory, then X_N^+ is an empty class.

Proof. Assume the contrary that $x \in X^+$ and z satisfy **N**. Then, there exists a $t_1 \geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(g(t))>0$ for $t \in I_1$. From Lemma 4 and Lemma 6, we arrive at (2) and (10), respectively. Now from (2), we get

$$z(g(t)) \geq r^{1/\alpha}(\vartheta(t))\,z^{(n-1)}(\vartheta(t))\,\psi_{n-2}(g(t),\vartheta(t)). \tag{34}$$

By replacing (28) with (34) and proceeding as in proof of Theorem 4, we arrive at G (defined as in (30)) which is a positive solution of the inequality

$$G'(t)+\frac{k\tau_0}{\tau_0+p_0^{\alpha}} Q_1(t)\,\psi_{n-2}^{\alpha}(g(t),\vartheta(t))\,G\left(\tau^{-1}(\vartheta(t))\right) \leq 0.$$

In view of ([23], Theorem 1), we have that (33) also has a positive solution, a contradiction. Thus, the proof is complete. □

Corollary 2. *If there exists a function $\vartheta(t) \in C(I_0,(0,\infty))$ satisfying (32) and*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\vartheta(t))}^{t} Q_1(\varrho)\,\psi_{n-2}^{\alpha}(g(t),\vartheta(t))\,\mathrm{d}\varrho > \frac{\tau_0+p_0^{\alpha}}{\mathrm{e}k\tau_0}, \tag{35}$$

then X_N^+ is an empty class.

Proof. By using Theorem 2 in [15], conditions (35) imply that (33) is oscillatory. □

Theorem 6. *Assume that* $f(x(g(t))) := x^{\alpha}(t)$ *and* $p(t) < \widetilde{R}(t)$. *If there exists a function* $\theta \in C^1(I_0, (0,\infty))$ *satisfying*

$$\theta'(t) \geq 0, \;\; \theta(t) > t, \;\; \tau\left(\theta^{n-1}(t)\right) < t \tag{36}$$

and the first-order delay equation

$$\omega'(t) + B_{n-2}(t)\,\omega\left(\tau\left(\theta^{n-1}(t)\right)\right) = 0 \tag{37}$$

is oscillatory, then X_N^+ *is an empty class, where* $\theta^{m-1}(t) := \theta\left(\theta^{m-2}(t)\right), \theta^0(t) := \theta(t)$,

$$R_0(t) := \left(\frac{1}{r(t)}\int_t^{\infty} kq(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha}, \quad R_m(t) := \int_t^{\infty} R_{m-1}(\varrho)\,\mathrm{d}\varrho,$$

$$\widetilde{R}(t) := \exp\left(-\int_{\tau(t)}^{t} R_{n-2}(\varrho)\,\mathrm{d}\varrho\right),$$

$$B_0(t) := \left(\frac{1}{r(t)}\int_t^{\theta(t)} q(t)\left(\widetilde{R}(\varrho) - p(\varrho)\right)^{\alpha}\mathrm{d}\varrho\right)^{1/\alpha} \;\; \textit{and} \;\; B_m(t) := \int_t^{\theta(t)} B_{m-1}(\varrho)\,\mathrm{d}\varrho,$$

for $m = 1, 2, ..., n-2$.

Proof. Assume the contrary that $x \in X^+$ and z satisfy **N**. Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \in I_1$. It is easy to notice that $\lim_{t\to\infty} z^{(j)} = 0$ for $j = 1, 2, ..., n-2$ and $\lim_{t\to\infty} r(t)\left(z^{(n-1)}(t)\right)^{\alpha} = 0$. Hence, by integrating (1) from t to ∞, we obtain

$$\begin{aligned} r(t)\left(z^{(n-1)}(t)\right)^{\alpha} &= \int_t^{\infty} q(\varrho)\,x^{\alpha}(\varrho)\,\mathrm{d}\varrho \leq \int_t^{\infty} kq(\varrho)\,z^{\alpha}(\varrho)\,\mathrm{d}\varrho \\ &\leq z^{\alpha}(t)\int_t^{\infty} kq(\varrho)\,\mathrm{d}\varrho, \end{aligned}$$

and hence

$$z^{(n-1)}(t) \leq z(t)\left(\frac{1}{r(t)}\int_t^{\infty} kq(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha} = z(t)\,R_0(t).$$

Integrating the last inequality $n-2$ times from t to ∞, we obtain

$$-z'(t) \leq z(t)\,R_{n-2}(t).$$

Thus, we get

$$z(v) \geq z(u)\exp\left(-\int_u^{v} R_{n-2}(\varrho)\,\mathrm{d}\varrho\right),$$

for $u \leq v$. From the definition of z, we have

$$x(t) \geq z(t) - p(t)\,z(\tau(t)) \geq \left(\widetilde{R}(t) - p(t)\right)z(\tau(t))$$

which with (1) yields

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' = -q(t)\,x^{\alpha}(t) \leq -q(t)\left(\widetilde{R}(t) - p(t)\right)^{\alpha} z^{\alpha}(\tau(t)).$$

Integrating the last inequality from t to $\theta(t)$, we arrive at

$$\begin{aligned} z^{(n-1)}(t) &\geq \left(\frac{1}{r(t)}\int_t^{\theta(t)} q(t)\left(\widetilde{R}(\varrho)-p(\varrho)\right)^{\alpha} z^{\alpha}(\tau(\varrho))\,\mathrm{d}\varrho\right)^{1/\alpha} \\ &\geq z(\tau(\theta(t)))\,B_0(t). \end{aligned}$$

Integrating the last inequality $n-2$ times from t to $\theta(t)$, we get

$$z'(t)+z\left(\tau\left(\theta^{n-1}(t)\right)\right)B_{n-2}(t)\leq 0.$$

If we set

$$\omega(t) := \int_t^{\infty} z\left(\tau\left(\theta^{n-1}(t)\right)\right)B_{n-2}(t) > 0,$$

then ω is a positive solution of the inequality $\omega'(t)+B_{n-2}(t)\,\omega\left(\tau\left(\theta^{n-1}(t)\right)\right)\leq 0$. In view of ([23], Theorem 1), we have that (37) also has a positive solution, a contradiction. The proof is complete. □

Corollary 3. *Assume that $f(x(g(t))) := x^{\alpha}(t)$ and $p(t)<\widetilde{R}(t)$. If there exists a function $\theta\in C^1(I_0,(0,\infty))$ satisfying (36) and*

$$\liminf_{t\to\infty}\int_{\tau\left(\theta^{n-1}(t)\right)}^{t} B_{n-2}(\varrho)\,\mathrm{d}\varrho > \frac{1}{\mathrm{e}}, \tag{38}$$

then X_N^{+} is an empty class, where the functions $\widetilde{R}$, θ^{n-1} and B_{n-2} are defined as in Theorem 6.

Proof. By using Theorem 2 in [15], condition (38) implies that (37) is oscillatory. □

4. Asymptotic and Oscillatory Properties

Theorem 7. *Each non-oscillatory solution of (1) tends to zero if*

$$\lim_{\varrho\to\infty}\int_t^{\varrho}\left(\frac{1}{r(t)}\int_t^{\infty} q(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha} = \infty \tag{39}$$

and one of the conditions (18) or (26) is fulfilled.

Proof. Let x be a non-oscillatory solution of (1). Without loss of generality, we assume that $x\in X_{+}$. From Lemma 3, we have only two cases for z. Each of the conditions (18) or (26) contradicts that z fulfills **P**. Now, we suppose that z satisfies **N**. Since $z(t)>0$ and $z'(t)<0$, we get that $z\to c$ as $t\to\infty$, where $c\geq 0$. Suppose that $c>0$. Then, for every $\epsilon>0$, there exists a $T\geq t_0$ such that $c<z(t)<c+\epsilon$ for all $t>T$. By set $\varepsilon<(1-p)(c/p)$, we get that

$$\begin{aligned} x(t) &= z(t)-p(t)x(\tau(t)) > c-pz(\tau(t)) \\ &> M(c+\epsilon) > Mz(t), \end{aligned}$$

where $M=(c-p(c+\epsilon))\,/\,(c+\epsilon)>0$. Thus, integrating from t to ∞, we have

$$\begin{aligned} r(t)\left(z^{(n-1)}(t)\right)^{\alpha} &\geq k\int_t^{\infty} q(\varrho)\,x^{\alpha}(g(\varrho))\,\mathrm{d}\varrho \geq kM^{\alpha}\int_t^{\infty} q(\varrho)\,z^{\alpha}(g(\varrho))\,\mathrm{d}\varrho \\ &\geq kM^{\alpha}z^{\alpha}(t)\int_t^{\infty} q(\varrho)\,\mathrm{d}\varrho > kM^{\alpha}c^{\alpha}\int_t^{\infty} q(\varrho)\,\mathrm{d}\varrho \end{aligned}$$

or

$$z^{(n-1)}(t) > k^{1/\alpha} Mc \left(\frac{1}{r(t)} \int_t^\infty q(\varrho)\, d\varrho \right)^{1/\alpha}.$$

By integrating from t to ϱ, we find

$$z^{(n-2)}(t) < z^{(n-2)}(\varrho) - k^{1/\alpha} Mc \int_t^\varrho \left(\frac{1}{r(t)} \int_t^\infty q(\varrho)\, d\varrho \right)^{1/\alpha}.$$

Taking the limit of both sides as $t \to \infty$ and using (39), we get that $z^{(n-2)}(t) \to -\infty$ as $t \to \infty$. But, z^{n-2} is a negative increasing function, this a contradiction. Therefore, $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

In the following, based on the fact that there are only two cases for the corresponding function z, we infer new criteria for oscillation of all solutions of the Equation (1). In each of the following theorems, we refer to two conditions through which it is possible to exclude the existence of solutions in X_P^+ or X_N^+. Thus, we rule out the existence of non-oscillatory solutions.

Theorem 8. *Assume that (18) or (26) holds. If there exists a function $\vartheta(t) \in C(I_0, (0,\infty))$ satisfying (32) and the first-order delay Equation (33) is oscillatory, then every solution of (1) is oscillatory.*

Theorem 9. *Assume that $f(x(g(t))) := x^\alpha(t)$, $p(t) < \widetilde{R}(t)$ and (18) hold. If there exists a function $\theta \in C^1(I_0, (0,\infty))$ satisfying (36) and the first-order delay Equation (37) is oscillatory, then every solution of (1) is oscillatory, where the functions $\widetilde{R}$, θ^{n-1} and B_{n-2} are defined as in Theorem 6.*

Corollary 4. *Assume that (18) or (26) holds. If there exists a function $\vartheta(t) \in C(I_0, (0,\infty))$ satisfying (32) and (35), then every solution of (1) is oscillatory.*

Corollary 5. *Assume that $f(x(g(t))) := x^\alpha(t)$, $p(t) < \widetilde{R}(t)$ and (18) (or (26)) hold. If there exists a function $\theta \in C^1(I_0, (0,\infty))$ satisfying (36) and (38), then every solution of (1) is oscillatory, where the functions $\widetilde{R}$, θ^{n-1} and B_{n-2} are defined as in Theorem 6.*

Example 1. *Consider the third-order neutral differential equation*

$$\left(\left((x(t) + p_0 x(\tau_0 t))'' \right)^\alpha \right)' + \frac{q_0}{t^{2\alpha+1}} x^\alpha(g_0 t) = 0, \tag{40}$$

where $p_0, \tau_0, g_0 \in (0,1)$ and $q_0 > 0$. From (40), we note that $n = 3$, $p(t) := p_0$, $\tau(t) := \tau_0 t$, $q(t) := q_0/t^{2\alpha+1}$, $g(t) := g_0 t$ and $r(t) = 1$. It is easy to verify that

$$\eta(t) = \lambda g_0^2 t, \quad \Theta(t) = q_0 (1-p_0)^\alpha \frac{1}{t^{2\alpha+1}}, \quad Q_1(t) := q_0/t^{2\alpha+1},$$

$$\phi(s,t) = (t-s), \quad \psi_1(s,t) = \frac{1}{2}(s-t)^2$$

and

$$\widetilde{\Theta}(t) = \frac{1}{2\alpha} q_0 (1-p_0)^\alpha \frac{1}{t^{2\alpha}}.$$

Thus, the condition (18) becomes:

$$q_0 (1 - p_0)^\alpha > \frac{1}{g_0^{2\alpha}} \left(\frac{2\alpha}{1+\alpha} \right)^{\alpha+1}.$$

The condition (26) simplifies to

$$q_0 (1 - p_0)^\alpha > \frac{\alpha 2^{\alpha+1}}{\lambda^\alpha g_0^{2\alpha}}.$$

By choosing $\vartheta(t) := (g_0 + \tau_0)(t/2)$, *where* $g_0 < 1$, *the condition (35) extends to*

$$q_0 (\tau_0 - g_0)^{2\alpha} \ln \frac{2\tau_0}{g_0 + \tau_0} > 2^{2\alpha+1} \frac{\tau_0 + p_0^\alpha}{\mathrm{e}\tau_0}.$$

Using Corollary 4, Equation (40) is oscillatory if

$$q_0 > \max \left\{ \frac{1}{g_0^{2\alpha} (1 - p_0)^\alpha} \left(\frac{2\alpha}{1+\alpha} \right)^{\alpha+1}, \frac{2^{2\alpha+1} (\tau_0 + p_0^\alpha)}{\mathrm{e}\tau_0 (\tau_0 - g_0)^{2\alpha}} \left(\ln \frac{2\tau_0}{g_0 + \tau_0} \right)^{-1} \right\} \tag{41}$$

or

$$q_0 > \max \left\{ \frac{\alpha 2^{\alpha+1}}{g_0^{2\alpha} (1 - p_0)^\alpha}, \frac{2^{2\alpha+1} (\tau_0 + p_0^\alpha)}{\mathrm{e}\tau_0 (\tau_0 - g_0)^{2\alpha}} \left(\ln \frac{2\tau_0}{g_0 + \tau_0} \right)^{-1} \right\}.$$

Next, if we set $g(t) := t$, $\theta(t) := \gamma t$, $\gamma > 1$ *and* $p_0 < \tau_0^A$, *then the condition (38) becomes*

$$q_0^{1/\alpha} \left(\tau_0^A - p_0 \right) \frac{(\gamma - 1)^{-3}}{(2\alpha)^{1/\alpha}} \ln \left(\frac{1}{\gamma^2 \tau_0} \right) > \frac{1}{\mathrm{e}},$$

where $A = (q_0/2\alpha)^{1/\alpha}$. *When* $g_0 = 1$, *by using Corollary 5, Equation (40) is oscillatory if*

$$q_0 > \max \left\{ \frac{1}{(1 - p_0)^\alpha} \left(\frac{2\alpha}{1+\alpha} \right)^{\alpha+1}, \frac{2\alpha (\gamma - 1)^{3\alpha}}{\mathrm{e} (\tau_0^A - p_0)^\alpha (\ln 1/\gamma^2 \tau_0)^\alpha} \right\} \tag{42}$$

or

$$q_0 > \max \left\{ \frac{\alpha 2^{\alpha+1}}{(1 - p_0)^\alpha}, \frac{2\alpha (\gamma - 1)^{3\alpha}}{\mathrm{e} (\tau_0^A - p_0)^\alpha (\ln 1/\gamma^2 \tau_0)^\alpha} \right\}.$$

5. Conclusions

When studying the oscillatory behavior of solutions of differential equations with odd-order, it is customary to find conditions that ensure solutions are either oscillatory or tend to zero. Dzurina et al. [5] and Vidhyaa et al. [24] established criteria for the oscillation of all solutions of a third-order linear and half-linear neutral differential equation, respectively. As an extension and also an improvement of these results, we obtained new oscillation criteria for the odd-order non-linear neutral Equation (1).

If we consider the third order differential equation

$$\left(x(t) + \frac{1}{10} x\left(\frac{1}{2} t \right) \right)''' + \frac{q_0}{t^3} x^\alpha \left(\frac{1}{10} t \right) = 0. \tag{43}$$

From Example 1 in [5], Equation (43) is oscillatory if $q_0 > 120$. However, by using our criterion (41), we get that (43) is oscillatory if $q_0 > 111.11$. Moreover, we consider the equation

$$\left(x\left(t\right)+\frac{1}{3}x\left(\frac{1}{2}t\right)\right)^{\prime\prime\prime}+\frac{q_0}{t^3}x^{\alpha}\left(t\right)=0. \tag{44}$$

From Example 3 in [24], by choosing $\beta = 4/3$ Equation (44) is oscillatory if $q_0 > 4$. However, if we choose $\gamma = 4/3$, then our criterion (42) becomes $q_0 > 2$, and hence (44) is oscillatory. Thus, our results improve the results in [5,24]. In the future, we can try to study the advanced odd-order differential equations by the same approach.

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