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Article

New Sharp Bounds for the Modified Bessel Function of the First Kind and Toader-Qi Mean †

Zhen-Hang Yang [1,2], **Jing-Feng Tian** [3,*] and **Ya-Ru Zhu** [3]

1 Engineering Research Center of Intelligent Computing for Complex Energy Systems of Ministry of Education, North China Electric Power University, Yonghua Street 619, Baoding 071003, China; yzhkm@163.com
2 Zhejiang Society for Electric Power, Hangzhou 310014, China
3 Department of Mathematics and Physics, North China Electric Power University, Yonghua Street 619, Baoding 071003, China; zhuyaru1982@126.com
* Correspondence: tianjf@ncepu.edu.cn
† Dedicated to the 60th Anniversary of Zhejiang Electric Power Company Research Institute.

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Abstract: Let $I_v(x)$ be he modified Bessel function of the first kind of order v. We prove the double inequality $\sqrt{\frac{\sinh t}{t}\cosh^{1/q}(qt)} < I_0(t) < \sqrt{\frac{\sinh t}{t}\cosh^{1/p}(pt)}$ holds for $t > 0$ if and only if $p \geq 2/3$ and $q \leq (\ln 2)/\ln \pi$. The corresponding inequalities for means improve already known results.

Keywords: modified Bessel function of the first kind; hyperbolic function; mean; inequality

MSC: 39B62; 33B10

1. Introduction

The modified Bessel function of the first kind of order v, denoted by $I_v(x)$, is a particular solution of the second-order differential equation ([1], p. 77)

$$x^2y''(x) + xy'(x) - \left(x^2+v^2\right)y(x) = 0, \tag{1}$$

which can be represented explicitly by the infinite series as

$$I_v(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+v}}{n!\Gamma(v+n+1)},\ x \in \mathbb{R},\ v \in \mathbb{R}\backslash\{-1,-2,\ldots\}, \tag{2}$$

where $\Gamma(x)$ is the gamma function [2–4]. There are many properties of $I_v(x)$, see for example, [5–11].

In this paper, we are interested in a special case of $I_v(x)$, that is, $I_0(x)$, which is related to Toader-Qi mean of positive numbers a and b defined by

$$TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta = \sqrt{ab}\, I_0\left(\ln\sqrt{\frac{a}{b}}\right)$$

(see [12–14]), where and in what follows $a, b > 0$ with $a \neq b$. It is undoubted that Toader-Qi mean $TQ(a,b)$ is a new newcomer. Recall that some classical means including the arithmetic mean, geometric mean, logarithmic mean, exponential mean and power mean of order p defined by

$$A \equiv A(a,b) = \frac{a+b}{2}, \qquad G \equiv G(a,b) = \sqrt{ab},$$
$$L \equiv L(a,b) = \frac{a-b}{\ln a - \ln b}, \quad I \equiv I(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$A_p \equiv A_p\left(a,b\right) = \left(\frac{a^p+b^p}{2}\right)^{1/p} \text{ if } p \neq 0 \text{ and } A_0 \equiv A_0\left(a,b\right) = \sqrt{ab},$$

respectively. Clearly, $A\left(a,b\right) = A_1\left(a,b\right)$ and $G\left(a,b\right) = A_0\left(a,b\right)$. It is known that $p \mapsto A_p\left(a,b\right)$ is increasing on $\mathbb{R}$. A simple relation among these elementary means is the following inequalities:

$$G < L < A_{1/3} < \frac{A+2G}{3} < A_{1/2} < \frac{2A+G}{3} < A_{2/3} < I < A_{\ln 2} < A_1 \tag{3}$$

(see [15–21]). Another interesting relation proven in [22] is that:

$$\sqrt{AG} < \sqrt{LI} < \frac{L+I}{2} < \frac{A+G}{2}. \tag{4}$$

Let $b > a > 0$ and $t = \ln\sqrt{a/b}$. Then those means mentioned above can be represented in terms of hyperbolic functions:

$$\begin{aligned} \frac{L\left(a,b\right)}{\sqrt{ab}} &= \frac{\sinh t}{t}, \quad \frac{I\left(a,b\right)}{\sqrt{ab}} = \exp\left(\frac{t}{\tanh t} - 1\right), \\ \frac{TQ\left(a,b\right)}{\sqrt{ab}} &= I_0\left(t\right), \quad \frac{A_p\left(a,b\right)}{\sqrt{ab}} = \cosh^{1/p}\left(pt\right) \quad \text{for } p \neq 0. \end{aligned}$$

Correspondingly, the inequalities mentioned above are equivalent to

$$\begin{aligned} 1 &< \frac{\sinh t}{t} < \cosh^3\left(\frac{t}{3}\right) < \frac{\cosh t + 2}{3} < \cosh^2\left(\frac{t}{2}\right) \\ &< \frac{2\cosh t + 1}{3} < \cosh^{3/2}\left(\frac{2t}{3}\right) < \exp\left(\frac{t}{\tanh t} - 1\right) < \cosh^{1/\ln 2}\left(t \ln 2\right), \end{aligned}$$

$$\sqrt{\cosh t} < \sqrt{\frac{\sinh t}{t}\exp\left(\frac{t}{\tanh t} - 1\right)} < \frac{1}{2}\left[\frac{\sinh t}{t} + \exp\left(\frac{t}{\tanh t} - 1\right)\right] < \frac{\cosh t + 1}{2}.$$

for $t > 0$.

Let us return to Toader-Qi mean. In 2015, Qi, Shi, Liu and Yang [13] proved that the inequalities

$$L\left(a,b\right) < TQ\left(a,b\right) < \frac{A\left(a,b\right) + G\left(a,b\right)}{2} < \frac{2A\left(a,b\right) + G\left(a,b\right)}{3} < I\left(a,b\right) \tag{5}$$

hold. Yang and Chu (Theorem 3.3 of [23]) established a series of sharp inequalities for $TQ\left(a,b\right)$ and $I_0\left(t\right)$, for example, the inequalities

$$\sqrt{\frac{\sinh\left(2t\right)}{\pi t}} < I_0\left(t\right) < \sqrt{\frac{\sinh\left(2t\right)}{2t}}, \tag{6}$$

$$\sqrt{\left(\frac{2}{\pi}\cosh t + 1 - \frac{2}{\pi}\right)\frac{\sinh t}{t}} < I_0\left(t\right) < \sqrt{\left(\lambda_0 \cosh t + 1 - \lambda_0\right)\frac{\sinh t}{t}}, \tag{7}$$

$$\left(\frac{\sinh t}{t}\right)^{3/4}\left(\cosh t\right)^{1/4} < I_0\left(t\right) < \frac{3}{4}\frac{\sinh t}{t} + \frac{1}{4}\cosh t, \tag{8}$$

hold for $t > 0$ with $\lambda_0 = 0.6766\ldots$. Inspired by the inequalities (3) and (4), Yang and Chu conjectured further that the inequality

$$TQ\left(a,b\right) < \sqrt{L\left(a,b\right) I\left(a,b\right)} \tag{9}$$

holds, which was proven in Theorem 3.1 of [24] by Yang, Chu and Song. In fact, they proved the following double inequality

$$\sqrt{\frac{e}{\pi}}\sqrt{L(a,b)\,I(a,b)} < TQ(a,b) < \sqrt{L(a,b)\,I(a,b)} \tag{10}$$

holds with the best coefficients $\sqrt{e/\pi} = 0.930\ldots$ and 1. More inequalities for $TQ(a,b)$ can be seen in [25,26].

Motivated by the inequalities (9) and $A_{2/3} < I$ listed in (3), the aim of this paper is to find the best constants p and q such that double inequality

$$\sqrt{L(a,b)\,A_q(a,b)} < TQ(a,b) < \sqrt{L(a,b)\,A_p(a,b)} \tag{11}$$

holds, or equivalently,

$$\sqrt{\frac{\sinh t}{t}\cosh^{1/q}(qt)} < I_0(t) < \sqrt{\frac{\sinh t}{t}\cosh^{1/p}(pt)} \tag{12}$$

for $t > 0$. Our main results are as follows.

Theorem 1. *The function*

$$F(t) = \frac{tI_0(t)^2}{\cosh^{3/2}(2t/3)\sinh t}$$

is strictly decreasing from $(0,\infty)$ *onto* $\left(\sqrt{8}/\pi, 1\right)$*. Therefore, the double inequality*

$$\frac{2^{3/4}}{\sqrt{\pi}}\sqrt{\frac{\sinh t}{t}\cosh^{3/2}\left(\frac{2t}{3}\right)} < I_0(t) < \sqrt{\frac{\sinh t}{t}\cosh^{3/2}\left(\frac{2t}{3}\right)}$$

holds for $t > 0$*, or equivalently,*

$$\frac{2^{3/4}}{\sqrt{\pi}}\sqrt{L(a,b)\,A_{2/3}(a,b)} < TQ(a,b) < \sqrt{L(a,b)\,A_{2/3}(a,b)} \tag{13}$$

holds, where the coefficients $2^{3/4}/\sqrt{\pi} = 0.94885\ldots$ *and* 1 *are the best.*

Theorem 2. *The double inequality (12) holds for* $t > 0$*, or equivalently, (11) holds for* $a, b > 0$ *with* $a \neq b$*, if and only if* $p \geq 2/3$ *and* $q \leq p_0 = (\ln 2)/\ln\pi = 0.605\ldots$*.*

2. Tools and Lemmas

To prove our results, we need two tools. The first tool was due to Biernacki and Krzyz [27], which play an important role in dealing with the monotonicity of the ratio of power series.

Lemma 1 ([27])**.** *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r, r)$ *(*$r > 0$*) with* $b_k > 0$ *for all k. If the sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all k, then the function* $t \mapsto A(t)/B(t)$ *is also increasing (decreasing) on* $(0, r)$*.*

Remark 1. *Recently, another monotonicity rule in the case when the sequence* $\{a_k/b_k\}_{k\geq 0}$ *is piecewise monotonic was presented in Theorem 1 of [28], which is now applied preliminarily, see for example, [29–32].*

The second tool is the so-called "L'Hospital Monotone Rule" (or, for short, LMR), which is very effective in studying the monotonicity of ratios of two functions.

Lemma 2 ([33], Theorem 2). *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b), with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each x in (a, b). If f'/g' is increasing (decreasing) on (a, b) then so is f/g.*

The following two lemmas will be used to prove Proposition 1.

Lemma 3 ([23], Lemma 2.8). *We have*

$$I_0(t)^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n}. \tag{14}$$

Lemma 4 ([34], Problems 85, 94). *The two given sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ satisfy the conditions*

$$b_n > 0; \quad \sum_{n=0}^{\infty} b_n t^n \text{ converges for all values of } t; \quad \lim_{n\to\infty} \frac{a_n}{b_n} = s.$$

Then $\sum_{n=0}^{\infty} a_n t^n$ converges too for all values of t and in addition

$$\lim_{t\to\infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s.$$

3. Three Propositions

The proofs of Theorems 1 and 2 rely on the following propositions.

Proposition 1. *Let*

$$f_0(t) = \theta \frac{2\cosh t + 1}{3} \frac{\sinh t}{t} + (1-\theta)\left(1 + \frac{1}{2}t^2 + \frac{229}{6720}t^4\right), \tag{15}$$

where $\theta = 11,009/10,449$. The function

$$F_0(t) = \frac{I_0(t)^2}{f_0(t)}$$

is strictly decreasing from $(0, \infty)$ onto $(3/(\theta\pi), 1)$.

Proof. Expanding in power series yields

$$\begin{aligned} f_0(t) &= \theta \frac{\sinh 2t + \sinh t}{3t} + (1-\theta)\left(1 + \frac{1}{2}t^2 + \frac{229}{6720}t^4\right) \\ &= \theta \sum_{n=0}^{\infty} \frac{2^{2n+1}+1}{3(2n+1)!} t^{2n} + (1-\theta)\left(1 + \frac{1}{2}t^2 + \frac{229}{6720}t^4\right) \\ &= 1 + \frac{1}{2}t^2 + \frac{387\theta + 229}{6720}t^4 + \sum_{n=3}^{\infty} \frac{\theta\left(2^{2n+1}+1\right)}{3(2n+1)!} t^{2n} := \sum_{n=0}^{\infty} v_n t^{2n}, \end{aligned}$$

where $v_0 = 1$, $v_1 = 1/2$,

$$v_2 = \frac{387\theta + 229}{6720} \text{ and } v_n = \frac{\theta\left(2^{2n+1}+1\right)}{3(2n+1)!} \text{ for } n \geq 3.$$

By Lemma 3, we see that

$$I_0(t)^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n} := \sum_{n=0}^{\infty} u_n t^{2n}.$$

Direct calculations gives

$$\begin{aligned}\frac{u_0}{v_0} &= \frac{u_1}{v_1} = 1, \ \frac{u_2}{v_2} = \frac{630}{3870 + 229}\\ \frac{u_n}{v_n} &= \frac{(2n)!}{2^{2n} n!^4} \Big/ \frac{\theta\left(2^{2n+1}+1\right)}{3\left(2n+1\right)!} \text{ for } n \geq 3,\end{aligned}$$

then

$$\begin{aligned}\frac{u_1}{v_1} - \frac{u_0}{v_0} &= 0, \quad \frac{u_2}{v_2} - \frac{u_1}{v_1} = -\frac{3870 - 401}{3870 + 229} < 0,\\ \frac{u_3}{v_3} - \frac{u_2}{v_2} &= -\frac{35}{172}\frac{11610 - 1145}{\theta\left(3870 + 229\right)} < 0,\end{aligned}$$

$$\frac{u_{n+1}}{v_{n+1}} \Big/ \frac{u_n}{v_n} - 1 = -\frac{2^{2n+1} - \left(3n^2+6n+2\right)}{\left(n+1\right)^2\left(2^{2n+3}+1\right)} < 0 \text{ for } n \geq 3,$$

where the last inequality holds due to

$$\begin{aligned}2^{2n+1} - \left(3n^2+6n+2\right) &> 1 + \left(2n+1\right) + \frac{\left(2n+1\right)\left(2n\right)}{2!} + \frac{\left(2n+1\right)\left(2n\right)\left(2n-1\right)}{3!}\\ -\left(3n^2+6n+2\right) &= \frac{1}{3}n\left(4n+5\right)\left(n-2\right) > 0 \text{ for } n \geq 3.\end{aligned}$$

This shows that the sequence $\{u_n/v_n\}_{n\geq 0}$ is strictly decreasing, so is $I_0\left(t\right)^2/f_0\left(t\right)$ on $(0,\infty)$ by Lemma 1. It is easy to check that

$$\lim_{t\to 0}\frac{I_0\left(t\right)^2}{f_0\left(t\right)} = \frac{u_0}{v_0} = 1 \text{ and } \lim_{t\to\infty}\frac{I_0\left(t\right)^2}{f_0\left(t\right)} = \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{3}{\pi\theta},$$

where the second limits holds due to Lemma 4, thereby completing the proof. □

Proposition 2. *Let $f_0\left(t\right)$ be defined by (15). The function*

$$F_1\left(t\right) = \frac{t f_0\left(t\right)}{\cosh^{3/2}\left(2t/3\right)\sinh t}$$

is strictly decreasing from $(0,\infty)$ onto $\left(\sqrt{8}\theta/3, 1\right)$, where $\theta = 11{,}009/10{,}449$.

Proof. Let

$$\begin{aligned}f_1\left(t\right) &= \ln F_1\left(t\right) = \ln\left[\theta\frac{2\cosh t + 1}{3}\frac{\sinh t}{t} + \left(1-\theta\right)\left(1 + \frac{1}{2}t^2 + \frac{229}{6720}t^4\right)\right]\\ &\quad -\frac{3}{2}\ln\left(\cosh\frac{2t}{3}\right) - \ln\frac{\sinh t}{t}.\end{aligned}$$

Differentiation yields

$$f_1'\left(t\right) = -\frac{1}{6t\sinh t\cosh\left(2t/3\right)}\frac{f_2\left(t\right)}{f_0\left(t\right)},$$

where

$$f_2\left(t\right) = t^5 f_{25}\left(t\right) + t^4 f_{24}\left(t\right) + t^3 f_{23}\left(t\right) + t^2 f_{22}\left(t\right) + t f_{21}\left(t\right) + f_{20}\left(t\right), \tag{16}$$

$$
\begin{aligned}
f_{25}(t) &= \frac{229}{1120}(1-\theta)\left(\cosh\frac{2t}{3}\cosh t+3\sinh\frac{2t}{3}\sinh t\right),\\
f_{24}(t) &= \frac{229}{224}(\theta-1)\cosh\frac{2t}{3}\sinh t,
\end{aligned}
$$

$$
\begin{aligned}
f_{23}(t) &= \frac{3}{2}(1-\theta)\left(2\cosh\frac{2t}{3}\cosh t+3\sinh\frac{2t}{3}\sinh t\right)\\
f_{22}(t) &= 9(\theta-1)\cosh\frac{2t}{3}\sinh t,
\end{aligned}
$$

$$
\begin{aligned}
f_{21}(t) &= 3(1-\theta)\left(2\cosh\frac{2t}{3}\cosh t+3\sinh\frac{2t}{3}\sinh t\right),\\
f_{20}(t) &= 6(\theta-1)\cosh\frac{2t}{3}\sinh t-4\theta\cosh\frac{2t}{3}\sinh^3 t\\
&\quad +3\theta\sinh\frac{2t}{3}\sinh^2 t+6\theta\sinh\frac{2t}{3}\cosh t\sinh^2 t.
\end{aligned}
$$

Expanding in power series gives

$$
f_{25}(3s)=-\frac{229}{4480}(\theta-1)(5\cosh 5s-\cosh s)=-\frac{229}{4480}(\theta-1)\sum_{n=2}^{\infty}\frac{5^{2n-3}-1}{(2n-4)!}s^{2n-4},
$$

$$
f_{24}(3s)=\frac{229}{448}(\theta-1)(\sinh 5s+\sinh s)=\frac{229}{448}(\theta-1)\sum_{n=2}^{\infty}\frac{5^{2n-3}+1}{(2n-3)!}s^{2n-3},
$$

$$
f_{23}(3s)=-\frac{3}{4}(\theta-1)(5\cosh 5s-\cosh s)=-\frac{3}{4}(\theta-1)\sum_{n=1}^{\infty}\frac{5^{2n-1}-1}{(2n-2)!}s^{2n-2},
$$

$$
f_{22}(3s)=\frac{9}{2}(\theta-1)(\sinh 5s+\sinh s)=\frac{9}{2}(\theta-1)\sum_{n=1}^{\infty}\frac{5^{2n-1}+1}{(2n-1)!}s^{2n-1},
$$

$$
f_{21}(3s)=-\frac{3}{2}(\theta-1)(5\cosh 5s-\cosh s)=-\frac{3}{2}(\theta-1)\sum_{n=0}^{\infty}\frac{5^{2n+1}-1}{(2n)!}s^{2n},
$$

$$
\begin{aligned}
f_{20}(3s) &= \frac{1}{4}\theta\sinh 11s+\frac{3}{4}\theta\sinh 8s-\frac{5}{4}\theta\sinh 7s+\left(\frac{15}{4}\theta-3\right)\sinh\ 5s\\
&\quad -\frac{3}{4}\theta\sinh 4s-\frac{3}{2}\theta\sinh 2s+\left(\frac{21}{4}\theta-3\right)\sinh s
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\Bigg[\frac{\theta}{4}11^{2n+1}+\frac{3\theta}{4}8^{2n+1}-\frac{5\theta}{4}7^{2n+1}+\left(\frac{15\theta}{4}-3\right)5^{2n+1}\\
&\quad -\frac{3\theta}{4}4^{2n+1}-\frac{3\theta}{2}2^{2n+1}+\left(\frac{21}{4}\theta-3\right)\Bigg]\frac{s^{2n+1}}{(2n+1)!}
\end{aligned}
$$

Then $f_2(3s)$ defined by (16) can be written as

$$
f_2(3s)=243s^5 f_{25}(3s)+81s^4 f_{24}(3s)+27s^3 f_{23}(3s)+9s^2 f_{22}(3s)+3s f_{21}(3s)+f_{20}(3s)
$$

$$
=54s^3+(\theta-1)\sum_{n=2}^{\infty}a_n^{[1]}\frac{s^{2n+1}}{(2n+1)!}+\sum_{n=2}^{\infty}a_n^{[2]}s^{2n+1}+\frac{3\theta}{4}\sum_{n=2}^{\infty}a_n^{[3]}\frac{s^{2n+1}}{(2n+1)!},
$$

where

$$\begin{aligned} a_n^{[1]} &= -\frac{55,647}{4480}\frac{(2n+1)!}{(2n-4)!}5^{2n-3}+\frac{18,549}{448}\frac{(2n+1)!}{(2n-3)!}5^{2n-3} \\ &\quad -\frac{81}{4}\frac{(2n+1)!}{(2n-2)!}5^{2n-1}+\frac{81}{2}\frac{(2n+1)!}{(2n-1)!}5^{2n-1} \\ &\quad -\frac{9}{2}\frac{(2n+1)!}{(2n)!}5^{2n+1}+\frac{3}{4}\frac{5\theta-4}{\theta-1}5^{2n+1}+\frac{1}{4}\frac{\theta}{\theta-1}11^{2n+1}, \end{aligned}$$

$$\begin{aligned} a_n^{[2]} &= \frac{55,647}{4480}\frac{\theta-1}{(2n-4)!}+\frac{18,549}{448}\frac{\theta-1}{(2n-3)!}+\frac{81}{4}\frac{\theta-1}{(2n-2)!} \\ &\quad +\frac{81}{2}\frac{\theta-1}{(2n-1)!}+\frac{9}{2}\frac{\theta-1}{(2n)!}+\frac{1}{4}\frac{3(7\theta-4)}{(2n+1)!}, \\ a_n^{[3]} &= 8^{2n+1}-\frac{5}{3}\times 7^{2n+1}-4^{2n+1}-2^{2n+2}. \end{aligned}$$

It remains to prove $a_n^{[i]} > 0$ for $i = 1,2,3$ and $n \geq 2$. It is clear that $a_n^{[2]} > 0$ due to $\theta = 11,009/10,449 > 1$. For $a_n^{[3]}$, it is easy to check that

$$a_{n+1}^{[3]} - 49a_n^{[3]} = 12\left(10\times 2^{4n}+11\times 2^{2n}+15\right)\times 2^{2n} > 0,$$

which together with $a_2^{[3]} = 11,005 > 0$ yields $a_n^{[3]} > 0$ for all $n \geq 2$. For $a_n^{[1]}$, since $(5\theta-4) > 5(\theta-1)$ and

$$\frac{\theta}{\theta-1} = \frac{11,009}{560} = 19.659\ldots > 18,$$

we have

$$\begin{aligned} a_n^{[1]} &> -\frac{55,647}{4480}\frac{(2n+1)!}{(2n-4)!}5^{2n-3}+\frac{18,549}{448}\frac{(2n+1)!}{(2n-3)!}5^{2n-3} \\ &\quad -\frac{81}{4}\frac{(2n+1)!}{(2n-2)!}5^{2n-1}+\frac{81}{2}\frac{(2n+1)!}{(2n-1)!}5^{2n-1} \\ &\quad -\frac{9}{2}\frac{(2n+1)!}{(2n)!}5^{2n+1}+\frac{15}{4}\times 5^{2n+1}+\frac{9}{2}\times 11^{2n+1} \end{aligned}$$

$$\begin{aligned} &= \frac{9}{2}\times 11^{2n+1}-3\left(\frac{18,549}{28}n^5-\frac{154,575}{56}n^4+\frac{138,915}{16}n^3\right. \\ &\quad \left.-\frac{1,357,425}{224}n^2+\frac{848,523}{224}n+\frac{3125}{4}\right)\times 5^{2n-4} := a_n^{[0]}. \end{aligned}$$

The sequence $\left\{a_n^{[0]}\right\}_{n\geq 2}$ satisfies the recurrence relation

$$\begin{aligned} \frac{a_{n+1}^{[0]}-121a_n^{[0]}}{9\times 5^{2n-4}} &= \frac{148,392}{7}n^5-\frac{463,725}{4}n^4+\frac{2,202,435}{7}n^3 \\ &\quad -\frac{36,754,425}{112}n^2+\frac{3,895,809}{56}n-\frac{21,875}{2}, \end{aligned}$$

which can be written as

$$\begin{aligned}&\frac{148,392}{7}(n-2)^5+\frac{2,689,605}{28}(n-2)^4+\frac{1,645,965}{7}(n-2)^3\\&+\frac{52,997,895}{112}(n-2)^2+\frac{29,042,799}{56}(n-2)+\frac{312,145}{2}>0\end{aligned}$$

for $n\geq 2$. This in combination with $a_2^{[0]}=10,126,407/16>0$ leads to $a_n^{[0]}>0$ for $n\geq 2$, and so is $a_n^{[1]}$. Therefore, $f_1(t)>0$ for $t>0$, so $f_0(t)$ is strictly increasing on $(0,\infty)$. An easy computation yields

$$\lim_{t\to 0}f_1(t)=0\quad\text{and}\quad\lim_{t\to\infty}f_1(t)=\ln\frac{\sqrt{8\theta}}{3},$$

which completes the proof. □

Using Lemma 2 we can prove the following lemma, which will be use to prove Theorem 2.

Proposition 3. *Let $q\neq 0,1/2,1$. The ratio*

$$t\mapsto\frac{\cosh^{1/q}(qt)-1}{\cosh t-1}$$

is strictly increasing on $(0,\infty)$ if $q\in(-\infty,0)\cup(1/2,1)$ and strictly decreasing on $(0,\infty)$ if $q\in(0,1/2)\cup(1,\infty)$. Consequently, the double inequality

$$q\cosh t+1-q<\cosh^{1/q}(qt)<c_q\cosh t+1-c_q\tag{17}$$

holds for $t>0$ if $q\in(-\infty,0)\cup(1/2,1)$, where the weights q and $c_q=2^{1-1/q}$ if $q>0$ and $c_q=0$ if $q<0$ are the best possible. If $q\in(0,1/2)\cup(1,\infty)$, then the double inequality (17) is reversed.

Proof. Let

$$g_1(t)=\cosh^{1/q}(qt)-1\ \text{ and }\ g_2(t)=\cosh t-1.$$

Clearly, $g_1(0^+)=g_2(0^+)=0$, and

$$\lim_{t\to 0}\frac{g_1(t)}{g_2(t)}=q\ \text{ and }\ \lim_{t\to\infty}\frac{g_1(t)}{g_2(t)}=c_q=\begin{cases}2^{1-1/q} & \text{if } q>0,\\ 0 & \text{if } q<0.\end{cases}$$

Differentiation yields

$$\begin{aligned}\frac{g_1'(t)}{g_2'(t)}&=\frac{\cosh^{1/q-1}(qt)\sinh(qt)}{\sinh t},\\ \left[\frac{g_1'(t)}{g_2'(t)}\right]'&=\frac{(1-2q)t}{2\sinh^2 t\cosh^{2-1/q}(qt)}\left(\frac{\sinh|(1-2q)t|}{|(1-2q)t|}-\frac{\sinh t}{t}\right).\end{aligned}$$

Since the function $(\sinh x)/x$ is strictly increasing on $(0,\infty)$, we find that

$$\left[\frac{g_1'(t)}{g_2'(t)}\right]'\begin{cases}>0 & \text{if } (|1-2q|-1)(1-2q)>0,\text{ i.e., } q\in(-\infty,0)\cup\left(\frac{1}{2},1\right),\\ <0 & \text{if } (|1-2q|-1)(1-2q)<0,\text{ i.e., } q\in(1,\infty)\cup\left(0,\frac{1}{2}\right).\end{cases}$$

By Lemma 2, the desired monotonicity follows. The double inequality (17) and its reverse follow from the monotonicity of $g_1(t)/g_2(t)$ on $(0,\infty)$. This completes the proof. □

Remark 2. *Taking $q = p_0 = (\ln 2) / \ln \pi$ in the double inequality (17) we obtain the double inequality*

$$p_0 \cosh t + 1 - p_0 < \cosh^{1/p_0}(p_0 t) < \frac{2}{\pi} \cosh t + 1 - \frac{2}{\pi} \tag{18}$$

for $t > 0$.

Remark 3. *The generalized Heronian mean [35] is defined by*

$$H_w(a,b) = \frac{a + b + w\sqrt{ab}}{w+2}.$$

Let $t = \ln \sqrt{a/b}$ with $b > a > 0$ and $q = w/(w+2) > 0$. Then Proposition 3 give a best approximation for $H_w(a,b)$ by power means:

$$\begin{aligned} H_w(a,b) &< A_{w/(w+2)}(a,b) \text{ if } w \in (2,\infty), \\ H_w(a,b) &> A_{w/(w+2)}(a,b) \text{ if } q \in (0,2). \end{aligned}$$

Our proof is clearly concise than Li, Long and Chu's given in [35].

4. Proofs of Theorem 1 and 2

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1. We have

$$F(t) = \frac{t I_0(t)^2}{\cosh^{3/2}(2t/3)\sinh t} = \frac{I_0(t)^2}{f_0(t)} \times \frac{t f_0(t)}{\cosh^{3/2}(2t/3)\sinh t} = F_0(t) \times F_1(t).$$

As shown in Propositions 1 and 2, the functions $F_0(t)$ and $F_1(t)$ are both strictly positive and decreasing on $(0,\infty)$, so is $F(t)$. And, we easily obtain

$$\begin{aligned} \lim_{t\to 0} F(t) &= \lim_{t\to 0} F_0(t) \times \lim_{t\to 0} F_1(t) = 1, \\ \lim_{t\to \infty} F(t) &= \lim_{t\to \infty} F_0(t) \times \lim_{t\to \infty} F_1(t) = \frac{3}{\pi\theta} \frac{\sqrt{8}\theta}{3} = \frac{\sqrt{8}}{\pi}. \end{aligned}$$

Using the monotonicity of $F(t)$, the desired double inequality follows. This completes the proof. □

Proof of Theorem 2. (i) The necessary condition for the right hand side inequality of (12) to hold follows from the limit relation

$$\lim_{t\to 0} \frac{I_0(t)^2 - \cosh^{1/p}(pt)(\sinh t)/t}{t^2} = -\frac{1}{6}(3p-2) \le 0.$$

The sufficiency follow from Theorem 1 and the increasing property of $p \mapsto \cosh^{1/p}(pt)$ on $\mathbb{R}$.

(ii) The necessary condition for the left hand side inequality of (12) to hold follows from the limit relation

$$\lim_{t\to\infty} \frac{(\cosh(qt))^{1/q}(\sinh t)/t}{I_0(t)^2} \le 1.$$

Since $I_0(t) \sim e^t/\sqrt{2\pi t}$ as $t \to \infty$ (see [36], 9.7.1) and

$$\begin{aligned} \cosh^{1/q}(qt)\frac{\sinh t}{t} &\le \frac{\sinh t}{t} \sim \frac{e^t}{2t} \text{ if } q \le 0, \\ \cosh^{1/q}(qt)\frac{\sinh t}{t} &= e^t\left(\frac{1+e^{-2qt}}{2}\right)^{1/q} e^t \frac{1-e^{-2t}}{2t} \sim \frac{1}{2^{1/q}}\frac{e^{2t}}{2t}, \end{aligned}$$

we have

$$\lim_{t\to\infty}\frac{\cosh^{1/q}(qt)(\sinh t)/t}{I_0(t)^2}=\begin{cases}0 & \text{if } q\le 0,\\ \dfrac{\pi}{2^{1/q}} & \text{if } q>0.\end{cases}$$

Therefore, the necessary condition is that $\pi/2^{1/q}\le 1$ if $q>0$ and $q\le 0$, that is, $q\le(\ln 2)/\ln\pi=p_0$.

By the increasing property of $q\mapsto\cosh^{1/q}(qt)$, to prove the sufficiency, it suffices to prove the left hand side inequality of (12) holds when $q=p_0$. From the first inequality of (7) and the second inequality of (18) it follows that

$$I_0(t)>\sqrt{\frac{\sinh t}{t}\left(\frac{2}{\pi}\cosh t+1-\frac{2}{\pi}\right)}>\sqrt{\frac{\sinh t}{t}\cosh^{1/p_0}(p_0t)}$$

for $t>0$, which proves the sufficiency, and the proof is completed. □

5. Concluding Remarks

In this paper, we obtained the best constants p and q such that the double inequality (12) holds for $t>0$, or equivalently, (11) holds for $a,b>0$ with $a\ne b$. This improved the result in [24]. We close the paper by giving two remarks on our results.

Remark 4. *It was shown in ([20], 5.25) that*

$$A_{2/3}(a,b)<I(a,b)<\frac{2\sqrt{2}}{e}A_{2/3}(a,b).$$

Then the double inequality (11) can be extended as

$$\begin{aligned}\sqrt{\frac{e}{\pi}}\sqrt{L(a,b)I(a,b)} &< \frac{2^{3/4}}{\sqrt{\pi}}\sqrt{L(a,b)A_{2/3}(a,b)}<TQ(a,b)\\ &< \sqrt{L(a,b)A_{2/3}(a,b)}<\sqrt{L(a,b)I(a,b)}.\end{aligned}$$

Remark 5. *As a computable bound, the upper bound* $\sqrt{t^{-1}\sinh t\cosh^{3/2}(2t/3)}$ *for* $I_0(t)$ *is superior to those given (6) and (8). In fact, we have*

$$I_0(t)<\sqrt{\frac{\sinh t}{t}\cosh^{3/2}\left(\frac{2t}{3}\right)}<\sqrt{\frac{\sinh t}{t}\cosh t}=\sqrt{\frac{\sinh(2t)}{2t}}\tag{19}$$

and

$$I_0(t)<\sqrt{\frac{\sinh t}{t}\cosh^{3/2}\left(\frac{2t}{3}\right)}<\frac{3}{4}\frac{\sinh t}{t}+\frac{1}{4}\cosh t\tag{20}$$

for $t>0$. The inequalities (19) are clear, and we have to check (20). Let

$$h(t)=\ln\sqrt{\frac{\sinh(3t/2)}{3t/2}\cosh^{3/2}(t)}-\ln\left[\frac{3}{4}\frac{\sinh(3t/2)}{3t/2}+\frac{1}{4}\cosh(3t/2)\right].$$

Differentiation yields

$$h'(t)=\frac{1}{6}\frac{h_1(t)}{t\,[\sinh(3t)+t\cosh(3t)]\cosh(2t)\sinh(3t)},$$

where

$$\begin{aligned} h_1(t) &= t^2\left(3\cosh 2t\cosh^2 3t + 3\sinh 2t\cosh 3t\sinh 3t - 6\cosh 2t\sinh^2 3t\right) \\ &+ \left(3\sinh 2t\sinh^2 3t - 4\cosh 2t\cosh 3t\sinh 3t\right)t + \cosh 2t\sinh^2 3t. \end{aligned}$$

Using "product into sum" formulas for hyperbolic functions and expanding in power series give

$$\begin{aligned} h_1(t) &= \frac{9}{2}t^2\cosh 2t - \frac{3}{2}t^2\cosh 4t - \frac{3}{2}t\sinh 2t - \frac{7}{4}t\sinh 4t - \frac{1}{4}t\sinh\ 8t \\ &+\frac{1}{4}\cosh 4t - \frac{1}{2}\cosh 2t + \frac{1}{4}\cosh 8t \end{aligned}$$

$$\begin{aligned} h_1(t) &= \frac{9}{2}\sum_{n=1}^{\infty}\frac{2^{2n-2}}{(2n-2)!}t^{2n} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{4^{2n-2}}{(2n-2)!}t^{2n} \\ &-\frac{3}{2}\sum_{n=1}^{\infty}\frac{2^{2n-1}}{(2n-1)!}t^{2n} - \frac{7}{4}\sum_{n=1}^{\infty}\frac{4^{2n-1}}{(2n-1)!}t^{2n} \\ &-\frac{1}{4}\sum_{n=1}^{\infty}\frac{8^{2n-1}}{(2n-1)!}t^{2n} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{4^{2n}}{(2n)!}t^{2n} \\ &-\frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{8^{2n}}{(2n)!}t^{2n} = -\frac{1}{16}\sum_{n=1}^{\infty}\frac{b_n(2t)^{2n}}{(2n)!}, \end{aligned}$$

where

$$b_n = (n-4)4^{2n} + \left(6n^2+11n-4\right)2^{2n} - 4\left(18n^2-15n-2\right).$$

Since $b_1 = b_2 = 0$, $b_3 = 756$ and for $n \geq 4$,

$$\begin{aligned} b_n &\geq \left(6n^2+11n-4\right)2^8 - 4\left(18n^2-15n-2\right) \\ &= 4\left(366n^2+719n-254\right) > 0, \end{aligned}$$

we have $h_1(t) < 0$ for $t > 0$, so is $h'(t)$. This leads to $h(t) < \lim_{t\to 0} h(t) = 0$, which proves the second inequality of (20) holds for $t > 0$.

Remark 6. *Due to*

$$\begin{aligned} L(a,b) &= \frac{a-b}{\ln a - \ln b} = \int_0^1 a^s b^{1-s} ds, \\ TQ(a,b) &= \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta = \frac{1}{\pi}\int_0^1 a^s b^{1-s}(s(1-s))^{-1/2} ds, \end{aligned}$$

the referee introduces a new family of means $L_\alpha(a,b)$ defined for $\alpha > 0$ by

$$L_\alpha(a,b) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\int_0^1 a^s b^{1-s}(s(1-s))^{\alpha-1} ds.$$

The referee also gives an interesting relation between this new mean and the modified Bessel functions of the first kind:

$$\frac{L_\alpha(a,b)}{\sqrt{ab}} = \Gamma\left(\alpha+\frac{1}{2}\right)\left(\frac{t}{2}\right)^{1/2-\alpha} I_{\alpha-1/2}(t), \quad t = \ln\sqrt{\frac{a}{b}}.$$

It is easy to check that

$$\lim_{\alpha\to 0} L_\alpha(a,b) = \frac{a+b}{2}$$

and

$$L_{\alpha}(a,b) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\int_0^1 a^s b^{1-s}(s(1-s))^{\alpha-1}\,ds$$
$$< \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\int_0^1 (sa+(1-s)b)(s(1-s))^{\alpha-1}\,ds = \frac{a+b}{2}.$$

However, more problems remain to be researched on this new family of means, for example: (i) checking the monotonicity of this mean with respect to the parameter α; (ii) finding the lower and upper bounds for this mean in terms of elementary means; (iii) comparing this new mean with others.

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