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A Method of Generating Fuzzy Implications from n Increasing Functions and $n + 1$ Negations

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Abstract: In this paper, we introduce a new construction method of a fuzzy implication from n increasing functions $g_i : [0, 1] \rightarrow [0, \infty)$, ($g(0) = 0$) ($i = 1, 2, \dots, n$, $n \in \mathbb{N}$) and $n + 1$ fuzzy negations N_i ($i = 1, 2, \dots, n + 1$, $n \in \mathbb{N}$). Imagine that there are plenty of combinations between n increasing functions g_i and $n + 1$ fuzzy negations N_i in order to produce new fuzzy implications. This method allows us to use at least two fuzzy negations N_i and one increasing function g in order to generate a new fuzzy implication. Choosing the appropriate negations, we can prove that some basic properties such as the exchange principle (EP), the ordering property (OP), and the law of contraposition with respect to N are satisfied. The worth of generating new implications is valuable in the sciences such as artificial intelligence and robotics. In this paper, we have found a novel method of generating families of implications. Therefore, we would like to believe that we have added to the literature one more source from which we could choose the most appropriate implication concerning a specific application. It should be emphasized that this production is based on a generalization of an important form of Yager's implications.

Keywords: fuzzy implication; ordering property; least fuzzy negation; t-conditionality

1. Introduction

Fuzzy implications are the generalization of the classical (Boolean) implication in the interval of $[0, 1]$. They play an important role in the area of fuzzy logic, decision theory, and fuzzy control. We can generate fuzzy implications from aggregation functions and fuzzy negations [1–5]. Other ways of generating fuzzy implications can be achieved by additive generating functions or by some initial implications [6–11]. Fuzzy implications are used for the application of the ‘if-then’ rule in fuzzy systems and inference processes, through Modus Ponens and Modus Tollens [12].

This paper is inspired by Yager's f -generated implications where $f : [0, 1] \rightarrow [0, \infty]$ is a strictly decreasing and continuous function and $f(1) = 0$. In addition, a fuzzy implication $I : [0, 1]^2 \rightarrow [0, 1]$ is defined by: $I(x, y) = f^{-1}(xf(y))$, $x, y \in [0, 1]$ with the understanding $0 \cdot \infty = 0$ (see [1] Definition 3.1.1). In this paper, we use functions $g_i : [0, 1] \rightarrow [0, \infty)$, which are increasing and continuous, and also $g_i(0) = 0$. We present a new production machine of fuzzy implications. Such a type of generating fuzzy implications can be found in the literature [1, 2, 5–8], for example, $I_{RC} = 1 - x + xy$. The production of new fuzzy implications is accomplished with the help of any fuzzy negations and increasing functions. These generated fuzzy implications fulfill the necessary properties required to be fuzzy implications (see [1] Definition 1.1.1.). Moreover, if the negations are selected with certain properties, then the generated implications may also fulfill additional properties like the neutrality property (NP), exchange principle (EP), identity principle (IP), and some others. The worth of this production of implications could be estimated at artificial intelligence, robotics science, etc. [13–15]. This method of producing

implications gives us the possibility, in a fuzzy environment, to find a large number of implications, which could help any researcher choose the most appropriate one.

The paper is organized as follows. In Section 2, we recall the basic concepts and definitions used in the paper. In Section 3, we study the new constructed method of fuzzy implications. Firstly, we present a constructed method using one increasing function g and two negations N_1, N_2 , then a second method using two increasing functions g_1, g_2 and three negations N_1, N_2, N_3 . Finally, we generalize our constructed method using n functions $g_1, g_2, \dots, g_n$ and $n + 1$ negations $N_1, N_2, \dots, N_n, N_{n+1}$.

2. Preliminaries

In order to help the reader get familiar with the theory, we recall here some of the concepts and results employed in the rest of the paper.

Definition 1. (see [1] Definition 1.1.1). A function $I : [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy implication if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$x_1 \leq x_2 \text{ then } I(x_1, y) \geq I(x_2, y), \quad \text{i.e., } I \text{ is decreasing in the first variable} \quad (1)$$

$$y_1 \leq y_2 \text{ then } I(x, y_1) \leq I(x, y_2), \quad \text{i.e., } I \text{ is increasing in the second variable.} \quad (2)$$

$$I(0, 0) = 1 \quad (3)$$

$$I(1, 1) = 1 \quad (4)$$

$$I(1, 0) = 0 \quad (5)$$

The set of all fuzzy implications will be denoted by FI .

A fuzzy negation N is a generalization of the classical complement or negation $\neg$, whose truth table consists of the two conditions: $\neg 0 = 1$ and $\neg 1 = 0$.

Definition 2. (see [1] Definition 1.4.1). A function $N : [0, 1] \rightarrow [0, 1]$ is called a fuzzy negation if it satisfies the following conditions:

$$N(0) = 1, N(1) = 0 \quad (6)$$

$$N \text{ is decreasing} \quad (7)$$

Definition 3. (see [1] Definition 1.4.2 (i)). A fuzzy negation N is called strict if, in addition,

$$N \text{ is strictly decreasing} \quad (8)$$

$$N \text{ is continuous} \quad (9)$$

Definition 4. (see [1] Definition 1.4.2 (ii)). A fuzzy negation N is called strong if it is an involution, i.e.,

$$N(N(x)) = x, x \in [0, 1] \quad (10)$$

Definition 5. (see [1] Definition 1.4.2 (ii)). A fuzzy negation N is said to be non-vanishing if

$$N(x) = 0 \Leftrightarrow x = 1 \quad (11)$$

Definition 6. (see [1] Definition 1.4.2 (ii)). A fuzzy negation N is said to be non-filling if

$$N(x) = 1 \Leftrightarrow x = 0 \quad (12)$$

Definition 7. (see [1] Definition 1.4.15 (ii)). Let $I \in FI$. The function $N_I : [0, 1] \rightarrow [0, 1]$ defined by

$$N_I(x) := I(x, 0), \quad x \in [0, 1] \quad (13)$$

is called the natural negation of I .

Example 1. (see [1] example 1.4.4, [2] Section 2.1 Example 1). Important negations that will be used throughout this paper are the standard negation $N_C = 1 - x$, the least or Godel, and the greatest or dual Godel fuzzy negations given respectively by

$$N_{D_1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0, 1] \end{cases} \quad N_{D_2}(x) = \begin{cases} 1 & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1 \end{cases}$$

Definition 8. (see [1] Definitions 1.3.1, 1.5.1). A Fuzzy Implication I is said to satisfy

i. The left neutrality property if:

$$I(1, y) = y, \quad y \in [0, 1] \quad (14)$$

ii. The exchange principle if:

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1] \quad (15)$$

iii. The identity principle if:

$$I(x, x) = 1, \quad x \in [0, 1] \quad (16)$$

iv. The ordering property if:

$$I(x, y) = 1 \Leftrightarrow x \leq y, \quad x, y \in [0, 1] \quad (17)$$

v. The law of contraposition with respect to N if:

$$I(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1] \quad (18)$$

vi. The law of left contraposition with respect to N if:

$$I(N(x), y) = I(N(y), x), \quad x, y \in [0, 1] \quad (19)$$

vii. The law of right contraposition with respect to N if:

$$I(x, N(y)) = I(y, N(x)), \quad x, y \in [0, 1] \quad (20)$$

Definition 9. (see [1] Notations and Some Preliminaries). We say that functions $f, g : [0, 1]^n \rightarrow [0, 1]$ are Φ -conjugate if there exists a $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, x_2, \dots, x_n) = \varphi^{-1}(f(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n))), \quad x_1, x_2, \dots, x_n \in [0, 1]. \quad (21)$$

Definition 10. (see [1] Definition 2.2.1). A function $S : [0, 1]^2 \rightarrow [0, 1]$ is called a triangular conorm (t -conorm) if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$S(x, y) = S(y, x) \quad (22)$$

$$S(x, S(y, z)) = S(S(x, y), z) \quad (23)$$

$$\text{If } y \leq z, \text{ then } S(x, y) \leq S(x, z), \text{ i.e., } S(x, \cdot) \text{ is increasing} \quad (24)$$

$$S(x, 0) = x \quad (25)$$

Definition 11. (see [1] Definition 2.1.1). A function $S : [0, 1]^2 \rightarrow [0, 1]$ is called a triangular norm (*t-norm*) if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

$$T(x, y) = T(y, x) \quad (26)$$

$$T(x, T(y, z)) = T(T(x, y), z) \quad (27)$$

$$\text{If } y \leq z, \text{ then } T(x, y) \leq T(x, z), \text{ i.e., } T(x, \cdot) \text{ is increasing} \quad (28)$$

$$T(x, 1) = x \quad (29)$$

Remark 1. (see [1] Propositions 1.1.8, 1.4.8 and Remarks 2.1.4 (vii), 2.2.5 (vii)). It is proved that, if $\varphi \in \Phi$, T is a continuous *t-norm*, S is a continuous *t-conorm*, N is a fuzzy (strict, strong) negation, and I is a fuzzy implication, then T_φ is a *t-norm*, S_φ is a *t-conorm*, N_φ is a fuzzy (strict, strong) negation, and I_φ is a fuzzy implication.

Definition 12. (see [1] Definition 2.4.1). The equation $p \rightarrow q \equiv \neg p \vee q$ creates a new class of fuzzy implications.

A function $I : [0, 1]^2 \rightarrow [0, 1]$ is called an (S, N) -Implication if there exist a *t-conorm* S and a fuzzy negation N such that:

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1] \quad (30)$$

Definition 13. (see [1] Subsection 7.3). The equation $(p \wedge q) \rightarrow r \equiv (p \rightarrow (q \rightarrow r))$ is known as the law of importation and is a tautology in classical logic. The general form of the above equivalence is given by

$$I(T(x, y), z) = I(x, I(y, z)), \quad x, y, z \in [0, 1] \quad (31)$$

Definition 14. (see [1] Definition 7.4.1). An implication I and a *t-norm* T satisfy the *T-conditionality* property if and only if

$$T(x, I(x, y)) \leq y, \quad x, y \in [0, 1] \quad (32)$$

Proposition 1. (see [1] Definition 7.4.2). If $I \in FI$ is such that there exist $x, y \in (0, 1)$ such that $x > y$ and $I(x, y) = 1$, then I does not satisfy (32) with any *t-norm* T .

Proposition 2. (see [1] Definition 7.4.3). Let $I \in FI$, a *t-norm* T satisfy (32), N_I is the natural negation of I and N_T is the natural negation of T , then $N_I \leq N_T$, the natural negation of T .

Definition 15. (see [1] Definition 1.6.1). Let N be a fuzzy negation and I be a fuzzy implication. A function $I_N : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I_N(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1] \quad (33)$$

is called the *N-reciprocal* of I .

When N is the classical negation N_C , then I_N is called the reciprocal of I and is denoted by I' .

3. The Main Results

In this section, we give definitions of new generated implications and prove some useful properties of them.

3.1. Fuzzy Implications Generated by One Increasing Function g and Two fuzzy negations N_1, N_2

Theorem 1. If N_1, N_2 are two fuzzy negations and $g : [0, 1] \rightarrow [0, \infty)$ is an increasing and continuous function with $g(0) = 0$, then the function $I : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right), \quad x, y \in [0, 1] \quad (34)$$

is a fuzzy implication.

Proof. Let $g : [0, 1] \rightarrow [0, \infty)$ be an increasing and continuous function with $g(0) = 0$ and $x_1, x_2, y \in [0, 1]$.

If $x_1 \leq x_2$ then

$$g(x_1) \leq g(x_2) \Rightarrow \frac{g(x_1)}{g(1)} \cdot N_1(y) \leq \frac{g(x_2)}{g(1)} \cdot N_1(y) \Rightarrow N_2\left(\frac{g(x_1)}{g(1)} \cdot N_1(y)\right) \geq N_2\left(\frac{g(x_2)}{g(1)} \cdot N_1(y)\right) I(x_1, y) \geq I(x_2, y), \text{ i.e., } I(\cdot, y) \text{ is decreasing, i.e., } I \text{ satisfies (1)}$$

$$\text{Let } y_1, y_2, x \in [0, 1]. \text{ If } y_1 \leq y_2, \text{ then } N_1(y_1) \geq N_1(y_2) \Rightarrow \frac{g(x)}{g(1)} \cdot N_1(y_1) \geq \frac{g(x)}{g(1)} \cdot N_1(y_2) \Rightarrow N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y_1)\right) \leq N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y_2)\right) \Rightarrow I(x, y_1) \leq I(x, y_2), \text{ i.e., } I(x, \cdot) \text{ is increasing, i.e., } I \text{ satisfies (2)}$$

$$I(0, 0) = N_2\left(\frac{g(0)}{g(1)} \cdot N_1(0)\right) = N_2(0) = 1, \text{ i.e., } I \text{ satisfies (3)}$$

$$I(1, 1) = N_2\left(\frac{g(1)}{g(1)} \cdot N_1(1)\right) = N_2(0) = 1, \text{ i.e., } I \text{ satisfies (4)}$$

$$I(1, 0) = N_2\left(\frac{g(1)}{g(1)} \cdot N_1(0)\right) = N_2(1) = 0, \text{ i.e., } I \text{ satisfies (5)}$$

Therefore, $I \in \text{FI}$. $\square$

Proposition 3. Let I be the fuzzy implication of Theorem 1, then the fuzzy implication N -reciprocal of I is

$$I_N(x, y) = I(N(y), N(x)) = N_2\left(\frac{g(N(y))}{g(1)} \cdot N_1(N(x))\right) \quad (35)$$

Proposition 4. If $N_1 = N_2 = N$ are strong negations, then the fuzzy implication of Theorem 1 satisfies additionally the left neutrality property (14) and the exchange principle (15).

Proof. $I(1, y) = N\left(\frac{g(1)}{g(1)} \cdot N(y)\right) = N(N(y)) = y, y \in [0, 1], \text{ i.e., } I \text{ satisfies (14)}$

$$I(a, I(b, x)) = N\left(\frac{g(a)}{g(1)} \cdot N(I(b, x))\right) = N\left(\frac{g(a)}{g(1)} \cdot N\left(N\left(\frac{g(b)}{g(1)} \cdot N(x)\right)\right)\right) \stackrel{N: \text{strong negation}}{=} N\left(\frac{g(a)g(b)}{g(1)^2} \cdot N(x)\right) \\ I(b, I(a, x)) = N\left(\frac{g(b)}{g(1)} \cdot N(I(a, x))\right) = N\left(\frac{g(b)}{g(1)} \cdot N\left(N\left(\frac{g(a)}{g(1)} \cdot N(x)\right)\right)\right) \stackrel{N: \text{strong negation}}{=} N\left(\frac{g(a)g(b)}{g(1)^2} \cdot N(x)\right)$$

Thus, we have $I(a, I(b, x)) = I(b, I(a, x))$, i.e., I satisfies (15). $\square$

Theorem 2. If $\varphi \in \Phi$ and I is the fuzzy implication of Theorem 1, then I_φ is a fuzzy implication.

Proof. According to Remark 1, I_φ is a fuzzy implication. $\square$

Proposition 5. If $N_1(x) = N_{D_1}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in (0, 1] \end{cases}$ (the least fuzzy negation), then the fuzzy implication of Theorem 1 satisfies the Identity Principle (16).

Proof.

$$I(x, x) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(x)\right) \stackrel{x=0}{=} N_2\left(\frac{g(0)}{g(1)} \cdot N_1(0)\right) = N_1(0) = 1$$

$$I(x, x) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(x)\right) \stackrel{x \in (0, 1]}{=} N_2\left(\frac{g(0)}{g(1)} \cdot N_1(0)\right) = N_1(0) = 1$$

i.e., I satisfies (16). $\square$

Proposition 6. If the fuzzy implication of Theorem 1 satisfies the Identity Principle (16), then it satisfies the Ordering Property (17).

Proof. Let $x, y \in [0, 1]$ and $x \leq y$, then $I(x, y) \geq I(y, y) = 1$. Thus, $I(x, y) = 1$.

If $I(x, y) = 1 \Leftrightarrow$

$$I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right) = 1 \Leftrightarrow$$

$$\frac{g(x)}{g(1)} \cdot N_1(y) = 0 \Leftrightarrow$$

$$g(x) = 0 \text{ or } N_1(y) = 0 \Leftrightarrow$$

$$x = 0 \leq y \text{ or } y = 1 \geq x.$$

Thus, we have $x \leq y$. $\square$

Proposition 7. The natural negation N_I of the fuzzy implication of Theorem 1 is

$$N_I(x) = N_2\left(\frac{g(x)}{g(1)}\right)$$

Proof.

$$N_I(x) = I(x, 0) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(0)\right) = N_2\left(\frac{g(x)}{g(1)}\right).$$

$\square$

Proposition 8. When $N_1(x) = N_2(x) = N(x)$ are strong negations, then the fuzzy implication of Theorem 1 is an (S, N) -implication.

Proof. When $N_1(x) = N_2(x) = N(x)$ are strong negations, according to Theorem 1 and Proposition 4, the fuzzy implication $I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right)$ satisfies (I1) and (EP). Moreover, if $N_1(x), N_2(x)$ are continuous negations, then N_I is also a continuous fuzzy negation. We deduce that I is an (S, N) -implication (see [1] Theorem 2.4.10). $\square$

Example 2. Let $g(x) = x$, $N_2(x) = 1 - x$, $N_1(x) = 1 - x^2$.

$$\text{Then, } I(x, y) = N_2\left(x(1 - y^2)\right) = 1 - x(1 - y^2) = 1 - x + xy^2$$

The graph of the above surface is plotted in Figure 1.

Figure 1. I implication of example 2.

Example 3. Let $g(x) = x$, $N_1(x) = N_2(x) = 1 - x$. Thus, $I(x, y) = N_2(x(1 - y)) = 1 - x(1 - y) = 1 - x + xy$. Then, it is the Reinchenbach Implication.

The graph of the above surface is plotted in Figure 2.

Figure 2. I implication of example 3.

Example 4. Let $g(x) = x$, $N_2(x) = 1 - x$, $N_1(x) = \frac{1-x}{1+x}$. Then, $I(x, y) = N_2\left(x \cdot \frac{1-y}{1+y}\right) = \frac{1+y-x+xy}{1+y}$. The graph of the above surface is plotted in Figure 3.

Figure 3. I implication of example 4.

Proposition 9. Let N_1, N_2, N_3 be three fuzzy negations. Let us suppose also that N_3 is a strong fuzzy negation. If $g: [0,1] \rightarrow [0,\infty)$ is an increasing and continuous function with $g(0) = 0$ and $g(x) = N_1(N_3(x))$, then the fuzzy implication $I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right)$ defined in Theorem 1 satisfies the law of contraposition (18) with respect to N_3 .

Proof. $g: [0,1] \rightarrow [0,\infty)$ is an increasing and continuous function

$$\begin{aligned} I(N_3(y), N_3(x)) &= N_2\left(\frac{g(N_3(y))}{g(1)} \cdot N_1(N_3(x))\right) = N_2\left(\frac{N_1(N_3(N_3(y)))}{g(1)} \cdot N_1(N_3(x))\right) \stackrel{N_3: \text{strong negation}}{=} \\ N_2\left(\frac{N_1(y)}{g(1)} \cdot N_1(N_3(x))\right) &= N_2\left(\frac{N_1(y)}{g(1)} \cdot g(x)\right) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right) = I(x, y), \text{ i.e., } I \text{ satisfies (18). } \quad \square \end{aligned}$$

Lemma 1. (see [1] Proposition 1.5.3). Let N_1, N_2, N_3 be three fuzzy negations with the properties N_1, N_3 being strict ones and N_3 additionally being a strong negation. If $g: [0,1] \rightarrow [0,\infty)$ is an increasing and continuous function with $g(0) = 0$ and $g(x) = N_1(N_3(x))$, then the fuzzy implication $I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right)$, $x, y \in [0,1]$ defined by Theorem 1 satisfies the left (L-CP) and the right (R-CP) law of the contraposition.

Proof. According to Proposition 1.5.3 [1], I satisfies the left (19) and the right (20) law of the contraposition. $\square$

Using Definition 14 and Proposition 2 of Section 2, we prove the following:

Proposition 10. Let I be the fuzzy implication defined by Theorem 1, $I(x, y) = N_2\left(\frac{g(x)}{g(1)} \cdot N_1(y)\right)$, $x, y \in [0,1]$. Let us suppose that T is a t -norm and T satisfies (32), then:

- $N_2(x) \leq N_1(x)$, if N_2 is a strong negation.
- $g(x) \geq g(1) \cdot N_1(N_T(x, y))$, if N_1 is a strong negation.

Proof. As I and T satisfy (TC), then $T(x, I(x, y)) \leq y$ for all $x, y \in [0,1]$.

- Let $x = 1$, we have $I(1, y) \leq y$, $y \in [0,1]$ then $N_2\left(\frac{g(1)}{g(1)} \cdot N_1(y)\right) \leq y \Rightarrow N_2(N_1(y)) \leq y \Rightarrow N_2(N_2(N_1(y))) \geq N_2(y) \xrightarrow{N_2: \text{strong negation}} N_2(y) \leq N_1(y).$

- ii. From Proposition 2, we have $N_I \leq N_T \Rightarrow N_2\left(\frac{g(x)}{g(1)}\right) \leq N_T(x, y)$
 $\xrightarrow{N_2 : \text{strong negation}} N_2\left(N_2\left(\frac{g(x)}{g(1)}\right)\right) \geq N_2(N_T(x, y)) \Rightarrow \frac{g(x)}{g(1)} \geq N_2(N_T(x, y)) \Rightarrow g(x) \geq g(1) \cdot N_2(N_T(x, y))$

□

3.2. Fuzzy Implications Generated by Two Increasing Functions g_1, g_2 and Three Fuzzy Negations N_1, N_2, N_3

Theorem 3. If $g_1(x), g_2(x) : [0, 1] \rightarrow [0, \infty)$ are increasing and continuous functions with $g_1(0) = g_2(0) = 0$ and N_1, N_2, N_3 are fuzzy negations, then the function $I : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I(x, y) = N_3\left(\frac{\frac{g_1(x)}{g_1(1)}N_1(y) + \frac{g_2(x)}{g_2(1)}N_2(y)}{2}\right), x, y \in [0, 1] \quad (36)$$

is a fuzzy implication.

Proof.

Let $x_1, x_2, y \in [0, 1]$.

If

$$\begin{aligned} x_1 \leq x_2 &\Rightarrow \frac{g_1(x_1)}{g_1(1)} \leq \frac{g_1(x_2)}{g_1(1)} \text{ and } \frac{g_2(x_1)}{g_2(1)} \leq \frac{g_2(x_2)}{g_2(1)}, \\ \text{then, } N_1(y) \frac{g_1(x_1)}{g_1(1)} &\leq N_1(y) \frac{g_1(x_2)}{g_1(1)} \text{ and } N_2(y) \frac{g_2(x_1)}{g_2(1)} \leq N_2(y) \frac{g_2(x_2)}{g_2(1)} \\ \Rightarrow \frac{N_1(y) \frac{g_1(x_1)}{g_1(1)} + N_2(y) \frac{g_2(x_1)}{g_2(1)}}{2} &\leq \frac{N_1(y) \frac{g_1(x_2)}{g_1(1)} + N_2(y) \frac{g_2(x_2)}{g_2(1)}}{2} \Rightarrow \\ N_3\left(\frac{N_1(y) \frac{g_1(x_1)}{g_1(1)} + N_2(y) \frac{g_2(x_1)}{g_2(1)}}{2}\right) &\leq N_3\left(\frac{N_1(y) \frac{g_1(x_2)}{g_1(1)} + N_2(y) \frac{g_2(x_2)}{g_2(1)}}{2}\right) \end{aligned}$$

$I(x_1, y) \geq I(x_2, y)$, i.e., $I(\cdot, y)$ is decreasing, i.e., I satisfies (1).

Let $y_1, y_2, x \in [0, 1]$

$$\begin{aligned} \text{If } y_1 \leq y_2, \text{ then } N_1(y_1) \geq N_2(y_2) &\Rightarrow \frac{g_1(x)}{g_1(1)}N_1(y_1) \geq \frac{g_1(x)}{g_1(1)}N_1(y_2) \text{ and } \frac{g_2(x)}{g_2(1)}N_2(y_1) \geq \frac{g_2(x)}{g_2(1)}N_2(y_2) \\ \Rightarrow \frac{\frac{g_1(x)}{g_1(1)}N_1(y_1) + \frac{g_2(x)}{g_2(1)}N_2(y_1)}{2} &\geq \frac{\frac{g_1(x)}{g_1(1)}N_1(y_2) + \frac{g_2(x)}{g_2(1)}N_2(y_2)}{2} \Rightarrow N_3\left(\frac{\frac{g_1(x)}{g_1(1)}N_1(y_1) + \frac{g_2(x)}{g_2(1)}N_2(y_1)}{2}\right) \\ &\leq N_3\left(\frac{\frac{g_1(x)}{g_1(1)}N_1(y_2) + \frac{g_2(x)}{g_2(1)}N_2(y_2)}{2}\right) \end{aligned}$$

$\Rightarrow I(x, y_1) \geq I(x, y_2)$, i.e., $I(x, \cdot)$ is increasing, i.e., I satisfies (2).

$$I(0, 0) = N(0) = 1 \text{ i.e., } I \text{ satisfies (3)}$$

$$I(1, 1) = N(0) = 1 \text{ i.e., } I \text{ satisfies (4)}$$

$$I(1, 0) = N(1) = 0 \text{ i.e., } I \text{ satisfies (5).}$$

□

Proposition 11. Let I be the fuzzy implication of Theorem 3. If $N_1(x) = N_2(x) = N_3(x) = N(x)$ are strong negations, then the neutrality property (14) and the exchange principle (15) are satisfied.

Proof. $I(1, x) = N\left(\frac{N(x) + N(x)}{2}\right) = N\left(\frac{2N(x)}{2}\right) = N(N(x)) = x$, i.e., I satisfies (14)

$$\begin{aligned} I(a, I(b, x)) &= N\left(\frac{\frac{g_1(a)}{g_1(1)}N(I(b, x)) + \frac{g_2(a)}{g_2(1)}N(I(b, x))}{2}\right) = \\ &= N\left(\frac{\frac{g_1(a)}{g_1(1)}N\left(N\left(\frac{\frac{g_1(b)}{g_1(1)}N(x) + \frac{g_2(b)}{g_2(1)}N(x)}{2}\right)\right) + \frac{g_2(a)}{g_2(1)}N\left(N\left(\frac{\frac{g_1(b)}{g_1(1)}N(x) + \frac{g_2(b)}{g_2(1)}N(x)}{2}\right)\right)}{2}\right) \quad \text{N:strong negation} \\ &= N\left(\frac{\frac{g_1(a)}{g_1(1)} \cdot \frac{\frac{g_1(b)}{g_1(1)}N(x) + \frac{g_2(b)}{g_2(1)}N(x)}{2} + \frac{g_2(a)}{g_2(1)} \cdot \frac{\frac{g_1(b)}{g_1(1)}N(x) + \frac{g_2(b)}{g_2(1)}N(x)}{2}}{2}\right) = \\ &= N\left(\frac{\frac{g_1(a)}{g_1(1)} \cdot \frac{g_1(b)}{g_1(1)}N(x) + \frac{g_1(a)}{g_1(1)} \cdot \frac{g_2(b)}{g_2(1)}N(x) + \frac{g_2(a)}{g_2(1)} \cdot \frac{g_1(b)}{g_1(1)}N(x) + \frac{g_2(a)}{g_2(1)} \cdot \frac{g_2(b)}{g_2(1)}N(x)}{4}\right). \\ \\ I(b, I(a, x)) &= N\left(\frac{\frac{g_1(b)}{g_1(1)}N(I(a, x)) + \frac{g_2(b)}{g_2(1)}N(I(a, x))}{2}\right) = \\ &= N\left(\frac{\frac{g_1(b)}{g_1(1)}N\left(N\left(\frac{\frac{g_1(a)}{g_1(1)}N(x) + \frac{g_2(a)}{g_2(1)}N(x)}{2}\right)\right) + \frac{g_2(b)}{g_2(1)}N\left(N\left(\frac{\frac{g_1(a)}{g_1(1)}N(x) + \frac{g_2(a)}{g_2(1)}N(x)}{2}\right)\right)}{2}\right) \quad \text{N:strong negation} \\ &= N\left(\frac{\frac{g_1(b)}{g_1(1)} \cdot \frac{\frac{g_1(a)}{g_1(1)}N(x) + \frac{g_2(a)}{g_2(1)}N(x)}{2} + \frac{g_2(b)}{g_2(1)} \cdot \frac{\frac{g_1(a)}{g_1(1)}N(x) + \frac{g_2(a)}{g_2(1)}N(x)}{2}}{2}\right) \\ &= N\left(\frac{\frac{g_1(b)}{g_1(1)} \cdot \frac{g_1(a)}{g_1(1)}N(x) + \frac{g_1(b)}{g_1(1)} \cdot \frac{g_2(a)}{g_2(1)}N(x) + \frac{g_2(b)}{g_2(1)} \cdot \frac{g_1(a)}{g_1(1)}N(x) + \frac{g_2(b)}{g_2(1)} \cdot \frac{g_2(a)}{g_2(1)}N(x)}{4}\right) \end{aligned}$$

We conclude that (15) is satisfied. $\square$

Proposition 12. Let I be the fuzzy implication of Theorem 3 and N_1, N_2, N_3 be fuzzy negations. If $N_1(x) = N_2(x) = N_{D_1}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in (0, 1] \end{cases}$ (the least fuzzy negation), then the identity principle (16) is satisfied.

Proof.

$$\begin{aligned} I(x, y) &= N_3\left(\frac{\frac{g_1(x)}{g_1(1)}N_1(y) + \frac{g_2(x)}{g_2(1)}N_2(y)}{2}\right) \stackrel{x=0}{=} N_3(0) = 1 \\ I(x, y) &= N_3\left(\frac{\frac{g_1(x)}{g_1(1)}N_1(y) + \frac{g_2(x)}{g_2(1)}N_2(y)}{2}\right) \stackrel{x \in (0, 1]}{=} N_3(0) = 1 \end{aligned}$$

Thus, I satisfies (16). $\square$

Proposition 13. If the fuzzy implication of Theorem 3 satisfies the identity principle (16), then it satisfies the ordering property (17).

Proof. Let $x, y \in [0, 1]$ and $x \leq y$, then $I(x, y) \geq I(y, y) = 1$. Thus, $I(x, y) = 1$.

If $I(x, y) = 1 \Leftrightarrow$

$$I(x, y) = N_3 \left(\frac{\frac{g_1(x)}{g_1(1)} N_1(y) + \frac{g_2(x)}{g_2(1)} N_2(y)}{2} \right) = 1 \Leftrightarrow \frac{\frac{g_1(x)}{g_1(1)} N_1(y) + \frac{g_2(x)}{g_2(1)} N_2(y)}{2} = 0 \Leftrightarrow$$

$g_1(x)N_1(x) = 0$ and $g_2(x)N_2(x) = 0$ ($g_1, g_2, N_1, N_2 \geq 0$)

$g_1(x) = 0$ or $N_1(y) = 0$ and $g_2(x) = 0$ or $N_2(x) = 0$

$x = 0 \leq y$ or $y = 1 \geq x$.

Thus, we have $x \leq y$. $\square$

Proposition 14. Let N_1, N_2, N_3, N_4 be four fuzzy negations. Let us suppose also that N_4 is a strong fuzzy negation. If $g_1, g_2: [0, 1] \rightarrow [0, \infty)$ are increasing and continuous functions with $g_1(0) = g_2(0) = 0$ and $g_1(x) = N_1(N_4(x))$, and $g_2(x) = N_2(N_4(x))$, then the fuzzy implication defined in Theorem 3 satisfies the law of contraposition (16) with respect to N_4 .

Proof.

$$\begin{aligned} I(N_4(y), N_4(x)) &= N_3 \left(\frac{\frac{g_1(N_4(y))}{g_1(1)} N_1(N_4(x)) + \frac{g_2(N_4(y))}{g_2(1)} N_2(N_4(x))}{2} \right) = \\ &= N_3 \left(\frac{\frac{N_1(N_4(N_4(y)))}{g_1(1)} N_1(N_4(x)) + \frac{N_2(N_4(N_4(y)))}{g_2(1)} N_2(N_4(x))}{2} \right) \\ &\stackrel{N_4: \text{strong negation}}{=} N_3 \left(\frac{\frac{N_1(y)}{g_1(1)} N_1(N_4(x)) + \frac{N_2(y)}{g_2(1)} N_2(N_4(x))}{2} \right) \\ &= N_3 \left(\frac{\frac{g_1(x)}{g_1(1)} N_1(y) + \frac{g_2(x)}{g_2(1)} N_2(y)}{2} \right) = I(x, y) \end{aligned}$$

Thus, I satisfies (16). $\square$

Proposition 15. The natural negation N_I of the fuzzy implication of Theorem 3 is

$$N_I(x) = N_3 \left(\frac{\frac{g_1(x)}{g_1(1)} + \frac{g_2(x)}{g_2(1)}}{2} \right)$$

Proof.

$$N_I(x) = I(x, 0) = N_3 \left(\frac{\frac{g_1(x)}{g_1(1)} N_1(0) + \frac{g_2(x)}{g_2(1)} N_2(0)}{2} \right) = N_3 \left(\frac{\frac{g_1(x)}{g_1(1)} + \frac{g_2(x)}{g_2(1)}}{2} \right).$$

$\square$

Example 5. Let $g_1(x) = g_2(x) = x$, $N_3(x) = 1 - x = N_1(x)$, $N_2(x) = 1 - x^2$

Then, $I(x, y) = N_3 \left(\frac{x(1-y) + x(1-y^2)}{2} \right) = N_3 \left(\frac{2x - xy - xy^2}{2} \right) = \frac{2-2x+xy+xy^2}{2}$

The graph of the above surface is plotted in Figure 4.

Figure 4. I implication of example 5.

Example 6. Let $g_1(x) = x^2$, $g_2(x) = x$, $N_3(x) = 1 - \sqrt{x}$, $N_1(x) = 1 - x$, $N_2(x) = 1 - x^2$.

$$\text{Then, } I(x, y) = N_3\left(\frac{x^2(1-y) + x(1-y^2)}{2}\right) = 1 - \sqrt{\frac{x^2 - x^2y + x - xy^2}{2}}.$$

The graph of the above surface is plotted in Figure 5.

Figure 5. I implication of example 6.

3.3. Fuzzy Implications Generated by n Increasing Function $g_i (i = 1, 2, \dots, n, n \in \mathbb{N})$ and $n + 1$ Fuzzy Negations N_i , ($i = 1, 2, \dots, n + 1, n \in \mathbb{N}$)

Theorem 4. If $g_i(x): [0,1] \rightarrow [0,\infty)$ are increasing and continuous functions, where $g_i(0) = 0$ ($i = 1, 2, \dots, n, n \in \mathbb{N}$) and N_i are fuzzy negations ($i = 1, 2, \dots, n + 1, n \in \mathbb{N}$), then the function $I : [0,1]^2 \rightarrow [0,1]$ defined by

$$I(x, y) = N_{n+1}\left(\sum_{i=1}^n \left(\frac{g_i(x) \cdot N_i(y)}{g_i(1)}\right)\right), x, y \in [0, 1], \quad (37)$$

is a fuzzy implication.

Proof. Let $x_1, x_2, y \in [0, 1]$.

If $x_1 \leq x_2 \Rightarrow g_1(x_1) \leq g_1(x_2), g_2(x_1) \leq g_2(x_2), \dots, g_n(x_1) \leq g_n(x_2) \Rightarrow \sum_{i=1}^n \frac{g_i(x_1)}{g_i(1)} \leq \sum_{i=1}^n \frac{g_i(x_2)}{g_i(1)}$, then $N_1(y)g_1(x_1) \leq N_1(y)g_1(x_2), N_2(y)g_2(x_1) \leq N_2(y)g_2(x_2), \dots, N_n(y)g_n(x_1) \leq N_n(y)g_n(x_2)$
 $\sum_{i=1}^n \frac{1}{n} N_i(y) \frac{g_i(x_1)}{g_i(1)} \leq \sum_{i=1}^n \frac{1}{n} N_i(y) \frac{g_i(x_2)}{g_i(1)} \Rightarrow N_{n+1}\left(\sum_{i=1}^n \left(\frac{1}{n} N_i(y) \frac{g_i(x_1)}{g_i(1)}\right)\right) \geq N_{n+1}\left(\sum_{i=1}^n \left(\frac{1}{n} N_i(y) \frac{g_i(x_2)}{g_i(1)}\right)\right)$
 $\Rightarrow I(x_1, y) \geq I(x_2, y)$, i.e., $I(y)$ is decreasing, i.e., I satisfies (1)

Let $y_1, y_2, x \in [0, 1]$. If $y_1 \leq y_2$, then $N_i(y_1) \geq N_i(y_2) \Rightarrow \frac{g_i(x)}{g_i(1)} N_i(y_1) \geq \frac{g_i(x)}{g_i(1)} N_i(y_2) \Rightarrow$
 $\sum_{i=1}^n \frac{1}{n} \frac{g_i(x)}{g_i(1)} N_i(y_1) \geq \sum_{i=1}^n \frac{1}{n} \frac{g_i(x)}{g_i(1)} N_i(y_2) \Rightarrow$

$$\begin{aligned}
N_{n+1} \left(\sum_{i=1}^n \left(\frac{1}{n} N(y_1) \frac{g_i(x)}{g_i(1)} \right) \right) &\leq N_{n+1} \left(\sum_{i=1}^n \left(\frac{1}{n} N(y_2) \frac{g_i(x)}{g_i(1)} \right) \right) \\
\Rightarrow I(x, y_1) &\leq I(x, y_2), \text{ i.e., } I(x, \cdot) \text{ is increasing, i.e., } I \text{ satisfies (2)} \\
I(0, 0) = N(0) = 1, \text{ i.e., } I &\text{ satisfies (3)} \quad I(1, 1) = N(0) = 1, \text{ i.e., } I \text{ satisfies (4)} \\
I(1, 0) = N \left(\frac{1+1+\dots+1}{n} \right) &= N \left(\frac{n}{n} \right) = N(1) = 0, \text{ i.e., } I \text{ satisfies (5).} \quad \square
\end{aligned}$$

Proposition 16. Let I be the fuzzy implication of Theorem 4. If $N_{n+1}(x) = N_1(x) = \dots = N_n(x) = N(x)$ are strong negations, then the left neutrality property (14) and the exchange principle (15) are satisfied.

Proof. $I(1, y) = N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(1)}{g_i(1)} \cdot N(y)}{n} \right) \right) = N(N(y)) = y, y \in [0, 1], \text{ i.e., } I \text{ satisfies (14)}$

$$\begin{aligned}
I(a, I(b, x)) &= N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(a)}{g_i(1)} \cdot N(I(b, x))}{n} \right) \right) \\
&= N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(a)}{g_i(1)} \cdot N \left(N \left(\sum_{j=1}^n \left(\frac{\frac{g_j(b)}{g_j(1)} \cdot N(x)}{n} \right) \right) \right)}{n} \right) \right) \\
&\stackrel{\text{N: strong negation}}{=} N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(a)}{g_i(1)} \cdot \sum_{j=1}^n \left(\frac{\frac{g_j(b)}{g_j(1)} \cdot N(x)}{n} \right)}{n} \right) \right) \\
I(b, I(a, x)) &= N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(b)}{g_i(1)} \cdot N(I(a, x))}{n} \right) \right) = \\
&= N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(b)}{g_i(1)} \cdot N \left(N \left(\sum_{j=1}^n \left(\frac{\frac{g_j(a)}{g_j(1)} \cdot N(x)}{n} \right) \right) \right)}{n} \right) \right) \\
&\stackrel{\text{N: strong negation}}{=} N \left(\sum_{i=1}^n \left(\frac{\frac{g_i(b)}{g_i(1)} \cdot \sum_{j=1}^n \left(\frac{\frac{g_j(a)}{g_j(1)} \cdot N(x)}{n} \right)}{n} \right) \right)
\end{aligned}$$

We conclude that (15) is satisfied, $I(a, I(b, x)) = I(b, I(a, x))$. $\square$

Proposition 17. Let $I(x, y) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{\frac{g_i(x)}{g_i(1)} \cdot N_i(y)}{n} \right) \right)$, $x, y \in [0, 1]$, the fuzzy implication defined by Theorem 4. Then, if $N_1 = N_2 = \dots = N_n = N_{D_1} = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in (0, 1] \end{cases}$ (the least fuzzy negation), the identity principle (16) is satisfied.

Proof.

$$I(x, x) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x) \cdot N_i(x)}{g_i(1)} \right) \right) \stackrel{x=0}{=} N_{n+1}(0) = 1$$

$$I(x, x) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x) \cdot N_i(x)}{g_i(1)} \right) \right) \stackrel{x \in (0,1]}{=} N_{n+1}(0) = 1$$

Thus, I satisfies (16). $\square$

Proposition 18. Let $N_1, \dots, N_{n+1}, N_{n+2}$ $n + 2$ be fuzzy negations. Let us suppose also that N_{n+2} is a strong fuzzy negation. If $g_i: [0,1] \rightarrow [0,\infty)$ ($i = 1, \dots, n, n \in \mathbb{N}$) are increasing and continuous functions with $g_i(0) = \dots = g_n(0) = 0$ and $g_i(x) = N_i(N_{n+2}(x))$, then the function $I : [0,1]^2 \rightarrow [0,1]$ defined by Theorem 4, $I(x, y) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x) \cdot N_i(y)}{g_i(1)} \right) \right)$, $x, y \in [0,1]$, satisfies the law of contraposition (18) with respect to N_{n+2} .

Proof.

$$I(N_{n+2}(y), N_{n+2}(x)) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(N_{n+2}(y)) \cdot N_i(N_{n+2}(x))}{g_i(1)} \right) \right) =$$

$$= N_{n+1} \left(\sum_{i=1}^n \left(\frac{N_i(N_{n+2}(N_{n+2}(y))) \cdot N_i(N_{n+2}(x))}{g_i(1)} \right) \right)$$

$$\stackrel{N_{n+2}: \text{strong negation}}{=} N_{n+1} \left(\sum_{i=1}^n \left(\frac{N_i(y) \cdot N_i(N_{n+2}(x))}{g_i(1)} \right) \right) = I(x, y).$$

Thus, I satisfies (16). $\square$

Proposition 19. The natural negation N_I of the fuzzy negation of Theorem 4 is

$$N_I(x) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x)}{g_i(1)} \right) \right)$$

Proof.

$$N_I(x) = I(x, 0) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x) \cdot N_i(0)}{g_i(1)} \right) \right) = N_{n+1} \left(\sum_{i=1}^n \left(\frac{g_i(x)}{g_i(1)} \right) \right).$$

$\square$

4. Conclusions

In this paper, a new production machine of fuzzy implications from n continuous increasing functions and $n + 1$ negation are introduced. We studied certain properties of these new fuzzy implications, as the left neutrality property (14), exchange principle (15), identity principle (16), ordering property (17), law of contraposition (18), and T-Conditionality (32), where some results are obtained if the fuzzy negations are strong or the least fuzzy negations. The advance of this method relies on the fact that we can combine a lot of fuzzy negations N_i and increasing functions g_i in order to generate fuzzy implications.

Finally, we believe that this production machine needs to be investigated further. It has been observed that in order to be satisfied, certain desirable properties by the implications generated by this method must use strong fuzzy negations or the least fuzzy negation. A question that arises is the following one:

Are there non-strong fuzzy negations that satisfy the left neutrality property (14) or the exchange principle (15)? In addition, in a future paper, we will study the behavior of non-continuous functions in terms of the validity of certain basic properties.

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References

1. Baczynski, M.; Jayaram, B. *Fuzzy Implications*; Springer: Berlin/Heidelberg, Germany, 2008. [\[CrossRef\]](#)
2. Baczynski, M.; Jayaram, B. (S, N)-and R-implications; a state-of-the-art survey. *Fuzzy Sets Syst.* **2008**, *159*, 1836–1859. [\[CrossRef\]](#)
3. Baczynski, M.; Jayaram, B. QL-implications: Some properties and intersections. *Fuzzy Sets Syst.* **2010**, *161*, 158–188. [\[CrossRef\]](#)
4. Baczynski, M.; Jayaram, B. (U, N)-implications and their characterizations. *Fuzzy Sets Syst.* **2009**, *160*, 2049–2062. [\[CrossRef\]](#)
5. Durante, F.; Klement, E.P.; Meriar, R.; Sempi, C. Conjunctors and their residual implicators: Characterizations and construction methods. *Mediterr. J. Math.* **2007**, *4*, 343–356. [\[CrossRef\]](#)
6. Massanet, S.; Torrens, J. An overview of construction methods of fuzzy implications. In *Advances in Fuzzy Implication Functions. Studies in Fuzziness and Soft Computing*; Springer: Berlin/Heidelberg, 2013; Volume 300, pp. 1–30. [\[CrossRef\]](#)
7. Baczynski, M.; Jayaram, B.; Massanet, S.; Torrens, J. Fuzzy implications: Past, present, and future. In *Springer Handbook of Computational Intelligence. Springer Handbooks*; Springer: Berlin/Heidelberg, Germany, 2015; pp. 183–202. [\[CrossRef\]](#)
8. Sainio, E.; Turunen, E.; Mesiar, R. A characterization of fuzzy implications generated by generalized quantifiers. *Fuzzy Sets Syst.* **2008**, *159*, 491–499. [\[CrossRef\]](#)
9. Baczynski, M.; Jayaram, B. On the characterization of (S, N)-implications. *Fuzzy Sets Syst.* **2007**, *158*, 1713–1727. [\[CrossRef\]](#)
10. Massanet, S.; Torrens, J. Threshold generation method of construction of a new implication from two given ones. *Fuzzy Sets Syst.* **2012**, *205*, 50–75. [\[CrossRef\]](#)
11. Balasubramanian, J. Yager's new class of implications J_f and some classical tautologies. *Inf. Sci.* **2007**, *177*, 930–946. [\[CrossRef\]](#)
12. Zadeh, L.A. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. Syst. Man Cybern.* **1973**, *SMC-3*, 28–44. [\[CrossRef\]](#)
13. Bogiatzis, A.C.; Papadopoulos, B.K. Producing fuzzy inclusion and entropy measures and their application on 277 global image thresholding. *Evol. Syst.* **2018**, *9*, 331–353. [\[CrossRef\]](#)
14. Bogiatzis, A.C.; Papadopoulos, B.K. Local Thresholding of degraded or unevenly illuminated documents using fuzzy inclusion and entropy measures. *Evol. Syst.* **2019**, *10*, 593–619. [\[CrossRef\]](#)
15. Bogiatzis, A.C.; Papadopoulos, B. Global Image Thresholding Adaptive Neuro- Fuzzy Inference System Trained with Fuzzy Inclusion and Entropy Measures. *Symmetry* **2019**, *11*, 286. [\[CrossRef\]](#)

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