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On the Fractional Wave Equation

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Abstract: In this paper we study the time-fractional wave equation of order $1 < \nu < 2$ and give a probabilistic interpretation of its solution. In the case $0 < \nu < 1$, $d = 1$, the solution can be interpreted as a time-changed Brownian motion, while for $1 < \nu < 2$ it coincides with the density of a symmetric stable process of order $2/\nu$. We give here an interpretation of the fractional wave equation for $d > 1$ in terms of laws of stable $d-$dimensional processes. We give a hint at the case of a fractional wave equation for $\nu > 2$ and also at space-time fractional wave equations.

Keywords: Hankel contours; multivariate stable processes; contour integrals; fractional laplacian

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1. Introduction

In this paper we study in detail the solution of the time-fractional equation

$$\frac{\partial^\nu u}{\partial t^\nu} = c^2 \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \tag{1}$$

for $1 < \nu < 2$ under the initial conditions

$$\begin{cases} u(x,0) = \delta(x) \\ u_t(x,0) = 0. \end{cases} \tag{2}$$

The time-fractional derivative is hereafter understood in the Caputo sense:

$$\frac{\partial^\nu u}{\partial t^\nu} = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{\partial^m}{\partial t^m} u(x,s)\,(t-s)^{m-\nu-1}\,\mathrm{d}s \qquad m-1 < \nu < m. \tag{3}$$

We first prove that the Fourier transform of the solution of the Cauchy problem (1) and (2) is

$$\mathcal{U}(\gamma_1, \ldots, \gamma_d, t) = \mathcal{U}(\gamma, t) = E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) \tag{4}$$

where

$$E_{\nu,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + 1)} \qquad x \in \mathbb{R} \tag{5}$$

is the one-parameter Mittag-Leffler function, first introduced in [1]. The representation of (4) as a contour integral on the Hankel path H_a

$$E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = \frac{1}{2\pi i} \int_{H_a} \frac{e^w w^{\nu-1}}{w^\nu + t^\nu c^2 \|\gamma\|^2}\,\mathrm{d}w \tag{6}$$

permits us to obtain a representation of (6) as

$$E_{\nu,1}(-c^2\|\gamma\|^2t^\nu) = \frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{z^{\nu-1}e^{-tzc^{2/\nu}\|\gamma\|^{2/\nu}}}{z^{2\nu}+2z^\nu\cos\pi\nu+1}\mathrm{d}z + \frac{1}{\nu}\left[e^{c^{2/\nu}\|\gamma\|^{2/\nu}te^{i\pi/\nu}} + e^{c^{2/\nu}\|\gamma\|^{2/\nu}te^{-i\pi/\nu}}\right]. \tag{7}$$

Some details about the representation (6) and the Hankel path can be found in [2]. For $d = 1$, $1 < \nu < 2$ the inversion of (7) is presented in [3] with the conclusion that the solution of (1) is the distribution of a stable symmetric process of order $2/\nu$.

We here show that for $d > 1, 1 < \nu < 2$ the solution can be expressed in terms of the law of a $d-$ dimensional stable process $S_\alpha(t)$ with a suitable choice of the measure Γ appearing in

$$\mathbb{E}e^{i\gamma\cdot S_\alpha(t)} = e^{-t\int_{\mathbb{S}^{d-1}}\|\gamma\cdot s\|^\alpha\left(1-i\,\mathrm{sign}(\gamma\cdot s)\tan\frac{\pi\alpha}{2}\right)\Gamma(\mathrm{d}s)}. \tag{8}$$

In particular, for Γ uniform on the upper and lower hemispheres of $\mathbb{S}^{d-1} = \{s \in \mathbb{R}^d : \|s\| = 1\}$, we prove that (8) yields the characteristic functions in square brackets of Formula (7). We give also the explicit forms of $u(x,t)$ of the solution of (1) in terms of Bessel functions $J_{\frac{d}{2}-1}(\rho\|x\|)$, which for $d = 1$ can be reduced to Fujita's result. Some results concerning wave equations of fractional type can be found, e.g., in [4].

2. The Fractional Wave Equation

In this note we present some relationship between stable processes (and their inverses) with fractional equations. Stable processes are studied in depth in the monograph [5]. Some simple and well known results state that a symmetric stable process $S_\alpha(t), 0 < \alpha \leq 2$ with characteristic function

$$\mathbb{E}e^{i\gamma S_\alpha(t)} = e^{-|\gamma|^\alpha t} \qquad \gamma \in \mathbb{R}, t > 0 \tag{9}$$

has distribution $p_\alpha(x,t), x \in \mathbb{R}, t > 0$, satisfying the fractional equation

$$\frac{\partial p}{\partial t} = \frac{\partial^\alpha}{\partial|x|^\alpha}p \tag{10}$$

where $\frac{\partial^\alpha}{\partial|x|^\alpha}$ is the Riesz fractional derivative usually defined as

$$\frac{\partial^\alpha}{\partial|x|^\alpha}f(x) = \frac{1}{2\Gamma(m-\alpha)\cos(\pi\alpha/2)}\frac{d^m}{\mathrm{d}x^m}\int_{-\infty}^{+\infty}\frac{f(y)}{|x-y|^{\alpha+1-m}}\mathrm{d}y \qquad m-1<\alpha<m \tag{11}$$

with Fourier transform

$$\int_{-\infty}^{+\infty} e^{i\gamma x}\frac{\partial^\alpha}{\partial|x|^\alpha}f(x)\mathrm{d}x = -|\gamma|^\alpha\int_{-\infty}^{+\infty}e^{i\gamma x}f(x)\mathrm{d}x. \tag{12}$$

For the d-dimensional isotropic stable process $S_\alpha(t) = (S_\alpha^1(t), \ldots, S_\alpha^d(t))$ with characteristic function,

$$\mathbb{E}e^{i\gamma S_\alpha^d(t)} = e^{-\|\gamma\|^\alpha t} \qquad \gamma \in \mathbb{R}^d, t > 0 \tag{13}$$

The corresponding probability law $p_\alpha(x_1, \ldots, x_d, t) = p_\alpha(x,t)$ satisfies the equation

$$\frac{\partial p}{\partial t} = -(-\Delta)^\alpha p \tag{14}$$

where $-(-\Delta)^\alpha$ is the fractional Laplacian defined as the operator such that

$$- (-\Delta)^{\alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(\gamma,x)} \|\gamma\|^{\alpha} \hat{f}(\gamma) \mathrm{d}\gamma \, , \qquad f \in Dom((-\Delta)^{\alpha}) \tag{15}$$

where $\hat{f}(\gamma)$ is the Fourier transform of a function $f(x), x \in \mathbb{R}^d$ and the domain of the operator $Dom((-\Delta)^{\alpha})$ is

$$Dom((-\Delta)^{\alpha}) = \left\{ f \in L^1_{loc}(R^n) : \int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2 (1 + \|\gamma\|^{2\alpha}) \, \mathrm{d}\gamma < \infty \right\}$$

(on this point see for example [6]). The connection between fractional operators and stochastic processes is explored, e.g., in [7]. A detailed comparison of the several possible definitions of the fractional Laplacian can be found in [8]. For the time-fractional equation (see [9]),

$$\begin{cases} \frac{\partial^{\nu} p}{\partial t^{\nu}} = c^2 \frac{\partial^2 p}{\partial x^2} \qquad 0 < \nu \leq 2, \, x \in \mathbb{R}, \, t > 0 \\ u(x, 0) = \delta(x) \\ u_t(x, 0) = 0, \end{cases} \tag{16}$$

we have that the solution of the Cauchy problem is explicitly given by

$$u(x,t) = \frac{1}{2ct^{\nu/2}} W_{-\nu/2, 1-\nu/2} \left(-\frac{|x|}{ct^{\nu}} \right) \tag{17}$$

where

$$W_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \qquad \alpha > -1, b > 0, x \in \mathbb{R}$$

is the Wright function. The $d-$dimensional counterpart of (16) is

$$\begin{cases} \frac{\partial^{\nu} p}{\partial t^{\nu}} = c^2 \Delta u \qquad 0 < \nu \leq 2, \, x \in \mathbb{R}^d, \, t > 0 \\ u(x, 0) = \delta(x) \\ u_t(x, 0) = 0. \end{cases} \tag{18}$$

Some details about time-fractional derivatives can be found in [10]. For $0 < \nu < 1$ the solution of (18) corresponds to the distribution of the vector process

$$B\left(L_{\nu}(t)\right), \quad t \geq 0 \tag{19}$$

where $B(t) = (B_1(t), \ldots, B_d(t))$ is the $d-$dimensional Brownian motion and $L_{\nu}(t)$ is the inverse of the stable subordinator $H_{\nu}(t)$ (see [11]).

In the more general case

$$\begin{cases} \frac{\partial^{\nu} p}{\partial t^{\nu}} = -c^2 (-\Delta)^{\alpha} u \qquad 0 < \nu \leq 2, \, 0 < \alpha \leq 1, \, x \in \mathbb{R}^d, \, t > 0 \\ u(x, 0) = \delta(x) \\ u_t(x, 0) = 0. \end{cases} \tag{20}$$

The solution of the Cauchy problem (20) is the probability density of the process

$$S_{\alpha}(L_{\nu}(t)), \ t \geq 0 \tag{21}$$

where $S^{\alpha}(t) = (S^1_{\alpha}(t), \ldots, S^d_{\alpha}(t))$ is an isotropic stable process (see [11]). We here consider the case where in (18) and (20) the order of the fractional derivative is $1 < \nu \leq 2$.We start first with (18) and observe that the Laplace–Fourier transform of the solution $u_{\alpha}(x,t)$ is

$$\int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d} e^{i\gamma\cdot x}u(x,t)\mathrm{d}x = \frac{\lambda^{\nu-1}}{\lambda^\nu + c^2\|\gamma\|^2} \tag{22}$$

and the Fourier transform reads

$$\int_{\mathbb{R}^d} e^{i\gamma\cdot x}u(x,t)\,\mathrm{d}x = E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu). \tag{23}$$

The Mittag-Leffler function $E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu)$ can be represented as a contour integral on the Hankel path as

$$E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = \frac{1}{2\pi i}\int_{H_a} \frac{e^w w^{\nu-1}}{w^\nu + t^\nu c^2\|\gamma\|^2}\,\mathrm{d}w \tag{24}$$

where Ha is the contour in the complex plane represented in Figure 1.

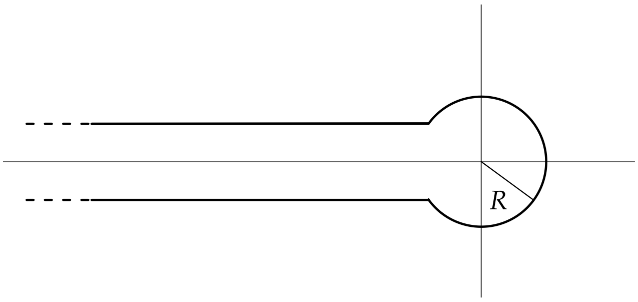

Figure 1. Hankel path in the complex plane.

The representation (24) is a consequence of the integral representation of the inverse of the Gamma function

$$\frac{1}{\Gamma(\nu)} = \frac{1}{2\pi i}\int_{Ha} e^w w^{-\nu}\mathrm{d}w.$$

The integral in (24) can be developed by inserting a ring of radius $\epsilon < R$.

The contour C is composed by the circumferences C_R and C_ϵ with two segments joining $Re^{-i\pi}$ with $\epsilon e^{-i\pi}$ and $Re^{i\pi}$ with $\epsilon e^{i\pi}$, and is run counterclockwise. See Figure 2.

In order to evaluate

$$\frac{1}{2\pi i}\int_C \frac{e^w w^{\nu-1}}{w^\nu + t^\nu c^2\|\gamma\|^2}\,\mathrm{d}w \tag{25}$$

we perform the transformation $w^\nu = z^{2m}$ for $2m-1 < \nu < 2m$.

The contour of Figure 2 after the transformation $w^\nu = z^{2m}$ takes the form shown in Figure 3.

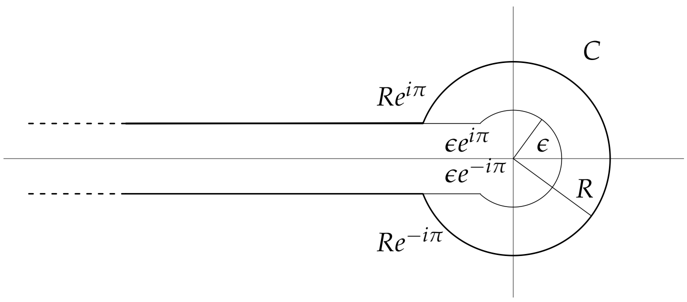

Figure 2. Representation of the contour C.

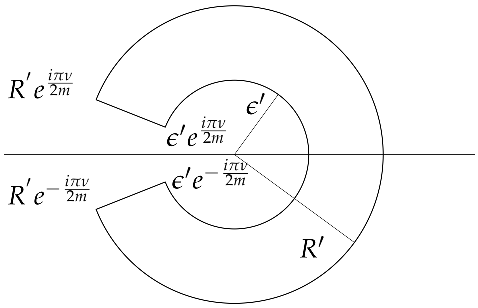

Figure 3. Representation of the contour C', with $R' = R^{\frac{\nu}{2m}}$ and $\epsilon' = \epsilon^{\frac{\nu}{2m}}$.

Therefore, the horizontal segments of Figure 2 are rotated by an angle of amplitude $\pm\pi\nu/2m$ and the radii are subject to contraction or dilation according to the value of ν. The integral on C' thus obtained from (25) is

$$\frac{1}{2\pi i}\int_C \frac{e^w w^{\nu-1}}{w^\nu + t^\nu c^2\|\gamma\|^2}\,\mathrm{d}w = \frac{2m}{2\pi\nu i}\int_{C'} \frac{e^{z^{2m/\nu}} z^{2m-1}}{z^{2m} + t^\nu c^2\|\gamma\|^2}\,\mathrm{d}z. \tag{26}$$

The integral on the right side of (26) can be evaluated by means of the Cauchy residue theorem. The function

$$f(z) = \frac{e^{z^{2m/\nu}} z^{2m-1}}{z^{2m} + t^\nu c^2\|\gamma\|^2} \qquad z \in \mathbb{C} \tag{27}$$

has $2m$ poles at points $z_k = e^{i\pi\frac{(2k+1)}{2m}}(c^2\|\gamma\|^2 t^\nu)^{\frac{1}{2m}}$ for $0 \le k \le 2m-1$. It is easy to show that the residues of (27) at the poles z_k are given by

$$\lim_{z\to z_k}(z - z_k)\frac{e^{z^{2m/\nu}} z^{2m-1}}{z^{2m} + t^\nu c^2\|\gamma\|^2} = \frac{e^{z_k^{2m/\nu}}}{2m}. \tag{28}$$

Thus the integral (26) can be written as

$$\frac{1}{2\pi i}\int_{C'} f(z)\,\mathrm{d}z = \frac{1}{2m}\sum_{k=0}^{2m-1} e^{z_k^{2m/\nu}} \qquad \text{where } z_k = e^{i\pi\frac{2k+1}{2m}}(c^2\|\gamma\|^2 t^\nu)^{\frac{1}{2m}} \tag{29}$$

By adding the contribution of the segments $(Re^{-i\pi}, \epsilon e^{-i\pi})$ and $(Re^{i\pi}, \epsilon e^{i\pi})$ for $R \to \infty$ and $\epsilon \to 0$ we obtain

$$E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = \frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{z^{\nu-1}e^{-tzc^{2/\nu}\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu\cos\pi\nu + 1}\mathrm{d}z + \frac{2m}{\nu}\left[\frac{1}{2m}\sum_{k=0}^{2m-1} e^{z_k^{2m/\nu}}\right]. \tag{30}$$

For $m = 1$ we must distinguish the cases $0 < \nu < 1$ where

$$E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = \frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{z^{\nu-1}e^{-tzc^{2/\nu}\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu\cos\pi\nu + 1}\mathrm{d}z \tag{31}$$

and $1 < \nu < 2$, where

$$\begin{aligned} E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = {} & \frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{z^{\nu-1}e^{-tzc^{2/\nu}\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu\cos\pi\nu + 1}\mathrm{d}z \\ & + \frac{1}{\nu}\left[e^{c^{2/\nu}\|\gamma\|^{2/\nu} t e^{i\pi/\nu}} + e^{c^{2/\nu}\|\gamma\|^{2/\nu} t e^{-i\pi/\nu}}\right]. \end{aligned} \tag{32}$$

In order to simplify the formulas involved in the analysis we take $c = 1$.

For $m = 2$ we have the subcases $2 < \nu < 3$ and $3 < \nu < 4$. In the first case the contour integral of Figure 3 involves two poles and thus yields two additional terms in the representation of the Mittag-Leffler function (32). In the second case we have the contribution of four poles in the contour integral of Figure 3, so that for $3 < \nu < 4$

$$E_{\nu,1}(-c^2\|\gamma\|^2 t^\nu) = \frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-tzc^{2/\nu}\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z \\ + \frac{1}{\nu}\left[e^{\|\gamma\|^{2/\nu} t e^{i\pi/4\nu}} + e^{\|\gamma\|^{2/\nu} t e^{i3\pi/4\nu}} + e^{\|\gamma\|^{2/\nu} t e^{i5\pi/4\nu}} + e^{\|\gamma\|^{2/\nu} t e^{i7\pi/4\nu}}\right]. \tag{33}$$

The contours for $2 < \nu < 3$ and $3 < \nu < 4$ are depicted below (Figure 4).

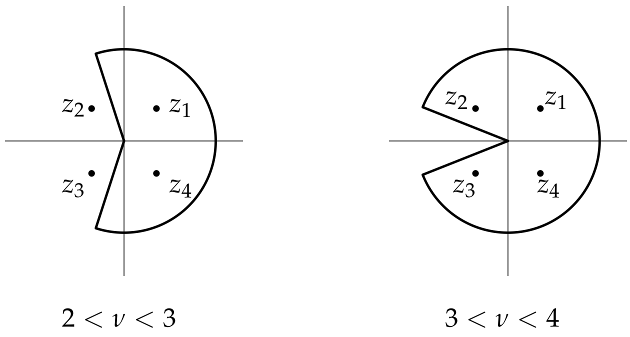

Figure 4. Representation of the contour C' for $m = 2$. The dots indicate the poles of $f(z)$.

The substantial difference between the cases $1 < \nu < 2$ and $\nu > 2$ is that in the first case we have that $\Re(e^{\pm i\pi/\nu})$ is negative and the contribution of the poles correspond to the characteristic function of stable processes, whereas for $2 < \nu < 3$, $\Re(e^{i\pi/\nu})$ and $\Re(e^{7i\pi/\nu})$ are positive and thus are not characteristic functions of random variables. Let us now concentrate our attention on the integrals in Equations (31) and (32) (which is also true in the general case for $\nu > 2$). If we write

$$\begin{aligned} &\frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-tz\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z \\ &= \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty \frac{e^{-tw^{1/\nu}\|\gamma\|^{2/\nu}}}{w^2 + 2w \cos \pi\nu + 1} \mathrm{d}w \\ &= \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty \frac{e^{-tw^{1/\nu}\|\gamma\|^{2/\nu}}}{(w + \cos \pi\nu)^2 + \sin^2 \pi\nu} \mathrm{d}w \\ &= \mathbb{E} e^{-t\|\gamma\|^{2/\nu} W^{1/\nu}} \end{aligned} \tag{34}$$

where W is a non-negative r.v. with density

$$f(w) = \frac{\sin \pi\nu}{\pi\nu} \frac{\mathrm{d}w}{(w + \cos \pi\nu)^2 + \sin^2 \pi\nu} \qquad w > 0,\ 0 < \nu < 1$$

Note that for $1 < \nu < 2$ the function (2) is negative on $(0, \infty)$. We note also that the r.v. $W_\nu, 0 < \nu < 1$ with density

$$P(W_\nu \in \mathrm{d}w)/\mathrm{d}w = \frac{\sin \pi\nu}{\pi} \frac{w^{\nu-1}}{1 + w^{2\nu} + 2w^\nu \cos \pi\nu} \qquad w > 0 \tag{35}$$

appearing in (35) has the same distribution as the ratio of two independent stable subordinators of degree $0 < \nu < 1$.

We now give the inverse Fourier transform of (34) for $0 < \nu < 1$, $d = 1$.

$$\begin{aligned} p_\nu(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma x} E_{\nu,1}(-\gamma^2 t^\nu)\mathrm{d}\gamma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma x}\frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{z^{\nu-1}e^{-tz\gamma^{2/\nu}}}{z^{2\nu}+2z^\nu\cos\pi\nu+1}\mathrm{d}z\,\mathrm{d}\gamma = (tz\gamma^{\frac{2}{\nu}}=w) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma x}\frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{e^{-w}w^{\nu-1}t^\nu\gamma^2}{w^{2\nu}+t^{2\nu}\gamma^4+2t^\nu\gamma^2 w^\nu\cos\pi\nu}\mathrm{d}w\,\mathrm{d}\gamma. \end{aligned} \tag{36}$$

We start by evaluating the following integral

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma x}\frac{t^\nu\gamma^2}{w^{2\nu}+t^{2\nu}\gamma^4+2t^\nu\gamma^2 w^\nu\cos\pi\nu}\mathrm{d}\gamma \\ &= \frac{1}{2\pi t^{\nu/2}}\int_{-\infty}^{\infty}\frac{e^{-ix\gamma' t^{-\nu/2}}\gamma'^2}{\gamma'^4+2\gamma'^2 w^\nu\cos\pi\nu+w^{2\nu}}\mathrm{d}\gamma' \\ &= \frac{1}{2\pi(wt)^{\nu/2}}\int_{-\infty}^{\infty}\frac{e^{-ix\gamma(w/t)^{\nu/2}}\gamma^2}{1+\gamma^4+2\gamma^2\cos\pi\nu}\mathrm{d}\gamma \\ &= \frac{1}{2\pi(wt)^{\nu/2}}\int_{-\infty}^{\infty}\frac{e^{-i\gamma A}\gamma^2}{1+\gamma^4+2\gamma^2\cos\pi\nu}\mathrm{d}\gamma \end{aligned}$$

where $A=(w/t)^{\nu/2}x$.

We must now evaluate the integral of

$$f(z)=\frac{z^2e^{-izA}}{z^4+2z^2\cos\pi\nu+1}\qquad z\in\mathbb{C}$$

on a suitable contour C_R. The four roots of $z^4+2z^2\cos\pi\nu+1=0$ are

$$\begin{cases} z_1 = e^{\frac{i\pi\nu}{2}-i\frac{\pi}{2}} = \sin\frac{\pi\nu}{2} - i\cos\frac{\pi\nu}{2} \\ z_2 = e^{i\frac{\pi\nu}{2}+i\frac{\pi}{2}} = -\sin\frac{\pi\nu}{2} + i\cos\frac{\pi\nu}{2} \\ z_3 = e^{-i\frac{\pi\nu}{2}-i\frac{\pi}{2}} = -\sin\frac{\pi\nu}{2} - i\cos\frac{\pi\nu}{2} \\ z_4 = e^{-i\frac{\pi\nu}{2}+i\frac{\pi}{2}} = \sin\frac{\pi\nu}{2} + i\cos\frac{\pi\nu}{2} \end{cases}$$

and are located in C as in Figure 5, because $1<\nu<2$.

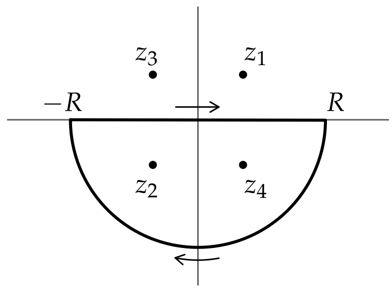

Figure 5. Integration contour for $x>0$.

We observe that $e^{-izA}=e^{-i(u+iv)x(w/t)^{\nu/2}}=e^{vA}e^{-iuA}$ for $x>0$ and $v<0$, the curvilinear integral on the half-circle $Re^{i\theta}$, $0\le\theta\le\pi$ tends to zero as $R\to\infty$. By the residue theorem we thus have

$$\int_{C_R} f(z)\,\mathrm{d}z = -2\pi i(R_{z_2}+R_{z_4}) \tag{37}$$

The minus sign is due to the fact that the contour in Figure 5 is run clockwise.

The residues R_{z_2} and R_{z_4} have the following values

$$R_{z_2} = -\frac{e^{i\frac{\pi\nu}{2}} e^{Ae^{i\pi\nu/2}}}{4\sin\pi\nu} \qquad R_{z_4} = \frac{e^{-i\frac{\pi\nu}{2}} e^{Ae^{-i\pi\nu/2}}}{4\sin\pi\nu} \tag{38}$$

and thus

$$\begin{aligned} \int_{C_R} f(z)\,\mathrm{d}z &= -\frac{2\pi i}{2^2\sin\pi\nu}\left(e^{-i\frac{\pi\nu}{2}} e^{Ae^{-i\pi\nu/2}} - e^{i\frac{\pi\nu}{2}} e^{Ae^{i\pi\nu/2}}\right) \\ &= -\frac{\pi}{\sin\pi\nu} e^{A\cos\frac{\pi\nu}{2}} \sin\left(\frac{\pi\nu}{2} + A\sin\frac{\pi\nu}{2}\right) \end{aligned} \tag{39}$$

For $x < 0$, the integration of $f(z)$ must be performed on the contour of Figure 6 and

$$\int_{C_R} f(z)\,\mathrm{d}z = -2\pi i(R_{z_1} + R_{z_3}) \tag{40}$$

the sign being in this case positive because the path is run counterclockwise.

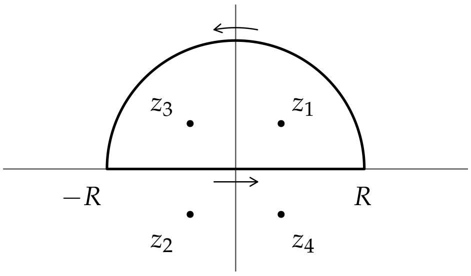

Figure 6. Integration contour for $x < 0$.

The residues in this case are

$$R_{z_1} = \frac{e^{i\pi\nu/2} e^{-Ae^{i\pi\nu/2}}}{4\sin\pi\nu} \qquad R_{z_3} = -\frac{e^{-i\pi\nu/2} e^{-Ae^{-i\pi\nu/2}}}{4\sin\pi\nu}$$

The integral (40), therefore, takes the form

$$\begin{aligned} \int_{C_R} f(z)\,\mathrm{d}z &= \frac{2\pi i}{4\sin\pi\nu}\left(e^{i\pi\nu/2} e^{-Ae^{i\pi\nu/2}} - e^{-i\pi\nu/2} e^{-Ae^{-i\pi\nu/2}}\right) \\ &= -\frac{\pi}{\sin\pi\nu} e^{-A\cos\frac{\pi\nu}{2}} \sin\left(\frac{\pi\nu}{2} - A\sin\frac{\pi\nu}{2}\right) \end{aligned} \tag{41}$$

In conclusion we have that

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{-ix\gamma(w/t)^{\nu/2}}\gamma^2}{1+\gamma^4+2\gamma^2\cos\pi\nu}\mathrm{d}\gamma \\ &= -\frac{\pi}{\sin\pi\nu} e^{|x|\cos\frac{\pi\nu}{2}(w/t)^{\nu/2}} \sin\left(\frac{\pi\nu}{2} + |x|\left(\frac{w}{t}\right)^{\frac{\nu}{2}} \sin\frac{\pi\nu}{2}\right). \end{aligned} \tag{42}$$

We now consider the integration with respect to w in (36). This leads to the evaluation of the following integral

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty e^{-w}w^{\nu-1}\frac{\sin\pi\nu}{\pi(wt)^{\nu/2}}\left[-\frac{\pi}{\sin\pi\nu}e^{|x|\cos\frac{\pi\nu}{2}(w/t)^{\nu/2}}\sin\left(\frac{\pi\nu}{2}+|x|\left(\frac{w}{t}\right)^{\frac{\nu}{2}}\sin\frac{\pi\nu}{2}\right)\right]\mathrm{d}w \\
&=-\frac{1}{2\pi t^{\frac{\nu}{2}}}\int_0^\infty e^{-w}w^{\frac{\nu}{2}-1}e^{|x|\cos\frac{\pi\nu}{2}(\frac{w}{t})^{\nu/2}}\sin\left(\frac{\pi\nu}{2}+|x|\left(\frac{w}{t}\right)^{\frac{\nu}{2}}\sin\frac{\pi\nu}{2}\right)\mathrm{d}w \\
&=-\frac{1}{\pi\nu t^{\frac{\nu}{2}}}\int_0^\infty e^{-tw^{\frac{2}{\nu}}}(w^{\frac{2}{\nu}}t)^{\frac{\nu}{2}-1}\,\frac{2}{\nu}tw^{\frac{2}{\nu}-1}e^{w|x|\cos\frac{\pi\nu}{2}}\sin\left(\frac{\pi\nu}{2}+w|x|\sin\frac{\pi\nu}{2}\right)\mathrm{d}w \\
&=-\frac{1}{\pi\nu}\int_0^\infty e^{-tw^{\frac{2}{\nu}}}e^{w|x|\cos\frac{\pi\nu}{2}}\sin\left(\frac{\pi\nu}{2}+w|x|\sin\frac{\pi\nu}{2}\right)\mathrm{d}w
\end{aligned} \tag{43}$$

The last step of (43) can be developed as follows

$$\begin{aligned}
&-\frac{1}{\pi\nu}\int_0^\infty e^{-tw^{2/\nu}}e^{w|x|\cos\frac{\pi\nu}{2}}\left[e^{i\left(\frac{\pi\nu}{2}+w|x|\sin\frac{\pi\nu}{2}\right)}-e^{-i\left(\frac{\pi\nu}{2}+w|x|\sin\frac{\pi\nu}{2}\right)}\right]\frac{\mathrm{d}w}{2i} \\
&=-\frac{e^{i\pi\nu/2}}{2i\pi\nu}\int_0^\infty e^{-tw^{2/\nu}}e^{w|x|e^{i\pi\nu/2}}\,\mathrm{d}w+\frac{e^{-i\pi\nu/2}}{2i\pi\nu}\int_0^\infty e^{-tw^{2/\nu}}e^{w|x|e^{-i\pi\nu/2}}\,\mathrm{d}w
\end{aligned} \tag{44}$$

We evaluate the first integral in (44) by taking the contour integral of the function

$$f(z)=e^{-tz^{2/\nu}+|x|ze^{i\pi\nu 2}} \qquad z\in\mathbb{C}$$

along the path C_R depicted in Figure 7.

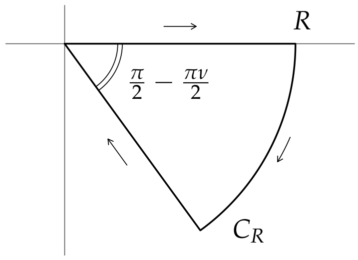

Figure 7. Path C_R corresponding to the change of variables $z'=e^{i\frac{\pi}{2}(1-\nu)}z$ in the first integral of (44).

By the Cauchy theorem we have that

$$\int_{C_R}f(z)\,\mathrm{d}z=\int_0^R f(w)\,\mathrm{d}w+\int_C f(Re^{i\theta})\,\mathrm{d}\theta+\int_R^0 f(ze^{i\frac{\pi}{2}-i\frac{\pi\nu}{2}})e^{i\frac{\pi}{2}-i\frac{\pi\nu}{2}}\,\mathrm{d}z=0 \tag{45}$$

The integral on the arc C tends to zero because

$$\left|\int_C f(Re^{i\theta})\,\mathrm{d}\theta\right|\le\int_C|f(Re^{i\theta})|\,\mathrm{d}\theta=\int_{\frac{(1-\nu)\pi}{2}}^0 e^{-tR^{\frac{2}{\nu}}\cos\frac{2\theta}{\nu}}e^{|x|R\cos\left(\theta+\frac{\pi\nu}{2}\right)}\,\mathrm{d}\theta \tag{46}$$

Since $\pi>\theta+\frac{\pi\nu}{2}>\frac{\pi}{2}$ as $\theta\in(\frac{(1-\nu)\pi}{2},0)$, the exponent of the second factor of (46) is negative as well as the first one because $\cos(2\theta/\nu)$ for θ ranging in the same interval.

We thus conclude that (46) converges to zero as $R\to\infty$ and thus (45) yields

$$\begin{aligned}
\int_0^\infty e^{-tw^{2/\nu}}e^{w|x|e^{i\frac{\pi\nu}{2}}}\,\mathrm{d}w&=\int_0^\infty e^{i\frac{\pi}{2}-i\frac{\pi\nu}{2}}e^{-t\left(ze^{i\frac{\pi}{2}-i\frac{\pi\nu}{2}}\right)^{2/\nu}+i|x|z}\,\mathrm{d}z \\
&=ie^{-i\frac{\pi\nu}{2}}\int_0^\infty e^{i|x|z}e^{tz^{2/\nu}e^{i\pi/\nu}}\,\mathrm{d}z
\end{aligned} \tag{47}$$

In order to evaluate the second integral of (44) we integrate

$$f(z)=e^{-tz^{2/\nu}+|x|ze^{-i\pi\nu 2}} \qquad z\in\mathbb{C}$$

along the contour of Figure 8.

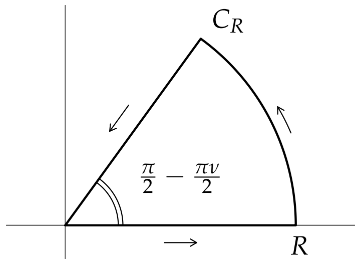

Figure 8. Path C_R corresponding to the change of variables $z' = e^{i\frac{\pi}{2}(\nu-1)}z$ in the second integral of (44).

By performing the same steps as above we obtain

$$\begin{aligned}\int_0^\infty e^{-tw^{2/\nu}} e^{w|x|e^{-i\frac{\pi\nu}{2}}}\,\mathrm{d}w &= \int_0^\infty e^{i\frac{\pi\nu}{2}-i\frac{\pi}{2}} e^{-t\left(ze^{i\frac{\pi\nu}{2}-i\frac{\pi}{2}}\right)^{2/\nu}+|x|ze^{i\frac{\pi\nu}{2}-i\frac{\pi}{2}}e^{-i\frac{\pi\nu}{2}}}\,\mathrm{d}z \\ &= -ie^{i\frac{\pi\nu}{2}}\int_0^\infty e^{-i|x|z}e^{tz^{2/\nu}e^{-i\pi/\nu}}\,\mathrm{d}z\end{aligned} \tag{48}$$

in view of (47) and (48) the integral (44) becomes

$$\begin{aligned}&-\frac{1}{2\pi\nu}\int_0^\infty e^{i|x|z}e^{tz^{2/\nu}e^{i\pi/\nu}}\,\mathrm{d}z - \frac{1}{2\pi\nu}\int_0^\infty e^{-i|x|z}e^{tz^{2/\nu}e^{-i\pi/\nu}}\,\mathrm{d}z \\ &= -\frac{1}{2\pi\nu}\int_0^\infty e^{i|x|z}e^{tz^{2/\nu}e^{i\pi/\nu}}\,\mathrm{d}z - \frac{1}{2\pi\nu}\int_{-\infty}^0 e^{i|x|z}e^{t(-z)^{2/\nu}e^{-i\pi/\nu}}\,\mathrm{d}z \\ &= -\frac{1}{2\pi\nu}\int_{-\infty}^{+\infty} e^{i|x|z}e^{t|z|^{2/\nu}e^{\frac{i\pi}{\nu}\,\mathrm{sign}z}}\,\mathrm{d}z \\ &= -\frac{1}{2\pi\nu}\int_{-\infty}^{+\infty} e^{-i|x|z}e^{t|z|^{\frac{2}{\nu}}e^{-\frac{i\pi}{\nu}\,\mathrm{sign}z}}\,\mathrm{d}z\end{aligned}$$

From (32), $d = 1$, we conclude that

$$\begin{aligned}p_\nu(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\gamma x}E_{\nu,1}(-\gamma^2 t^\nu)\,\mathrm{d}\gamma \\ &= \frac{1}{2\pi\nu}\left[-\int_{-\infty}^{+\infty} e^{-i|x|\gamma}e^{t|\gamma|^{2/\nu}e^{-\frac{i\pi}{\nu}\,\mathrm{sign}\gamma}}\,\mathrm{d}\gamma + \int_{-\infty}^{+\infty} e^{-ix\gamma}e^{t|\gamma|^{2/\nu}e^{\frac{i\pi}{\nu}\,\mathrm{sign}\gamma}}\,\mathrm{d}\gamma + \int_{-\infty}^{+\infty} e^{-ix\gamma}e^{t|\gamma|^{2/\nu}e^{-\frac{i\pi}{\nu}\,\mathrm{sign}\gamma}}\,\mathrm{d}\gamma\right] \\ &= \frac{1}{2\pi\nu}\int_{-\infty}^{+\infty} e^{i|x|\gamma}e^{t|\gamma|^{2/\nu}e^{-\frac{i\pi}{\nu}\,\mathrm{sign}\gamma}}\,\mathrm{d}\gamma\end{aligned} \tag{49}$$

Remark 1. *The function* $h(\gamma,\frac{2}{\nu}) = e^{t|\gamma|^{2/\nu}e^{-\frac{i\pi}{\nu}sign\gamma}} = e^{t|\gamma|^{\frac{2}{\nu}}\cos\frac{\pi}{\nu}(1-isign\gamma\tan\frac{\pi}{\nu})}$ *is the characteristic function of a stable random variable of order* $1 < 2/\nu < 2$ *with symmetry parameter* $\beta = 1$*. Many details about the properties of such densities can be found in [12]. The function* $p_{\frac{2}{\nu}}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\gamma x}h(\gamma, 2/\nu)\mathrm{d}\gamma$ *is unimodal with a positive maximal point and is such that* $\int_0^\infty p_{\frac{2}{\nu}}(x,t) = \mathrm{d}x = \nu/2$*. Analogously, the function* $h(\gamma,-2/\nu)$ *is the characteristic function of a negatively skewed random variable. This implies that the function* p_ν *can be seen as the superposition of the densities of stable random variables with index* $\beta = \pm 1$*, conditional to be respectively positive or negative.*

3. The Multidimensional Case for $1 < \nu < 2$

The Cauchy problem

$$\begin{cases}\frac{\partial^\nu u}{\partial t^\nu} = \sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2} & x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = \delta(x) \\ u_t(x,0) = 0.\end{cases} \tag{50}$$

has solution with Fourier transform

$$\int_{\mathbb{R}^d} e^{i\gamma\cdot x} u(x,t)\,\mathrm{d}x = E_{\nu,1}(-\|\gamma\|^2 t^\nu) \tag{51}$$
$$= \frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-tz\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z + \frac{1}{\nu}\left[e^{\|\gamma\|^{2/\nu} t e^{i\pi/\nu}} + e^{\|\gamma\|^{2/\nu} t e^{-i\pi/\nu}} \right].$$

as shown in the analysis presented above. Thus, the solution to the Cauchy problem (50) reads

$$u(x,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\gamma\cdot x} \tag{52}$$
$$\times \left\{ \frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-tz\|\gamma\|^{2/\nu}}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z + \frac{1}{\nu}\left[e^{\|\gamma\|^{2/\nu} t e^{i\pi/\nu}} + e^{\|\gamma\|^{2/\nu} t e^{-i\pi/\nu}} \right] \right\} \mathrm{d}\gamma$$

We must therefore evaluate the following three $d-$dimensional integrals, the first one being a function of z.

$$\int_{\mathbb{R}^d} e^{-i\gamma\cdot x} e^{-t\|\gamma\|^{2/\nu} z} \mathrm{d}\gamma\,, \qquad \int_{\mathbb{R}^d} e^{-i\gamma\cdot x} e^{\|\gamma\|^{2/\nu} t e^{i\pi/\nu}} \mathrm{d}\gamma\,, \qquad \int_{\mathbb{R}^d} e^{-i\gamma\cdot x} e^{\|\gamma\|^{2/\nu} t e^{-i\pi/\nu}} \mathrm{d}\gamma \tag{53}$$

Since the three integrals (53) are substantially similar, we restrict ourselves to the evaluation of the first one. In spherical coordinates we have that

$$\int_{\mathbb{R}^d} e^{-i\gamma\cdot x} e^{-t\|\gamma\|^{\frac{2}{\nu}} z} \mathrm{d}\gamma \tag{54}$$
$$= \int_0^\infty \rho^{d-1} \mathrm{d}\rho \int_0^\pi \mathrm{d}\theta_1 \cdots \int_0^\pi \mathrm{d}\theta_{d-2} \int_0^{2\pi} \mathrm{d}\phi \sin\theta_1^{d-2} \cdots \sin\theta_{d-2}$$
$$\times\, e^{-i\rho(x_d \sin\theta_1 \cdots \sin\theta_{d-2} \sin\phi + x_{d-1} \sin\theta_1 \cdots \sin\theta_{d-2} \cos\phi + \cdots + x_2 \sin\theta_1 \cos\theta_2 + x_1 \cos\theta_1)} e^{-t\rho^{2/\nu} z}$$
$$= \int_0^\infty \rho^{d-1} e^{-t\rho^{2/\nu} z} \frac{(2\pi)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho\|x\|)}{(\rho\|x\|)^{\frac{d}{2}-1}} \mathrm{d}\rho$$

The last step is the hyperspherical integral

$$\int_{\{\gamma_1,\ldots\gamma_d : \sum_{j=1}^d \gamma_j^2 = \rho^2\}} e^{-i\sum_{j=1}^d x_j\gamma_j} \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_d \tag{55}$$
$$= \int_0^\pi \mathrm{d}\theta_1 \cdots \int_0^\pi \mathrm{d}\theta_{d-2} \int_0^{2\pi} \mathrm{d}\phi \sin\theta_1^{d-2} \cdots \sin\theta_{d-2}$$
$$\times\, e^{-i\rho(x_d \sin\theta_1 \cdots \sin\theta_{d-2} \sin\phi + x_{d-1} \sin\theta_1 \cdots \sin\theta_{d-2} \cos\phi + \cdots + x_2 \sin\theta_1 \cos\theta_2 + x_1 \cos\theta_1)}$$
$$= \frac{(2\pi)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho\|x\|)}{(\rho\|x\|)^{\frac{d}{2}-1}}$$

which is proven in detail in [13], Formula (2.151).

By inserting (54) into (52) we have that

$$u(x,t) = \frac{1}{(2\pi)^d} \int_0^\infty \rho^{d-1} \frac{(2\pi)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho\|x\|)}{(\rho\|x\|)^{\frac{d}{2}-1}} \mathrm{d}\rho \tag{56}$$
$$\times \left\{ \frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-t\rho^{2/\nu} z}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z + \frac{1}{\nu} \left[e^{\rho^{2/\nu} t e^{i\pi/\nu}} + e^{\rho^{2/\nu} t e^{-i\pi/\nu}} \right] \right\}$$
$$= \frac{1}{(2\pi)^{\frac{d}{2}} \|x\|^{\frac{d}{2}-1}} \int_0^\infty \rho^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho\|x\|) \mathrm{d}\rho$$
$$\times \left\{ \frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{z^{\nu-1} e^{-t\rho^{2/\nu} z}}{z^{2\nu} + 2z^\nu \cos \pi\nu + 1} \mathrm{d}z + \frac{1}{\nu} \left[e^{\rho^{2/\nu} t e^{i\pi/\nu}} + e^{\rho^{2/\nu} t e^{-i\pi/\nu}} \right] \right\}$$

Note that the integral in z after the change of variable $z^\nu = z'$ becomes

$$\frac{\sin \pi\nu}{\pi\nu} \int_0^\infty \frac{e^{-t\rho^{2/\nu} z^{1/\nu}}}{z^2 + 2z\cos \pi\nu + 1} \mathrm{d}z = \frac{1}{\nu} \int_0^\infty e^{-t\rho^{2/\nu} z^{1/\nu}} \left[\frac{1}{\pi} \frac{\sin \pi\nu}{(z - \cos \pi\nu)^2 + \sin^2 \pi\nu} \right] \mathrm{d}z \tag{57}$$

since $1 < \nu < 2$ the function

$$f(z) = \frac{1}{\pi} \frac{\sin \pi\nu}{(z - \cos \pi\nu)^2 + \sin^2 \pi\nu} \tag{58}$$

has the form shown in Figure 9.

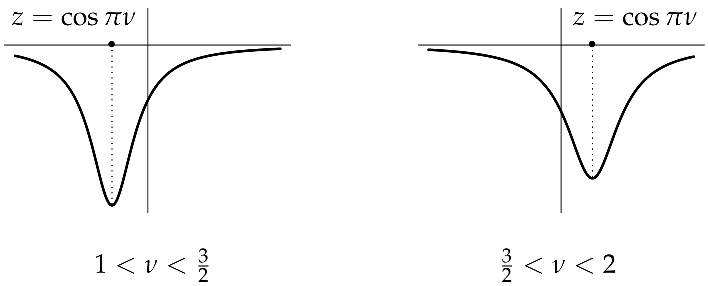

Figure 9. Plot of the function f given in (58).

In conclusion

$$\frac{1}{\nu} \int_0^\infty f(z)\, \mathrm{d}z = \begin{cases} -1 & \frac{3}{2} < \nu < 2 \\ -1 + \frac{1}{\nu} & 1 < \nu < \frac{3}{2}. \end{cases} \tag{59}$$

We recall now the definition of an $\alpha-$stable $d-$dimensional process $S^\alpha(t) = (S_1^\alpha(t), \ldots, S_d^\alpha(t))$, $0 < \alpha < 2$.

Its characteristic function has the following form [14]

$$\mathbb{E} e^{i\gamma \cdot S^\alpha(t)} = \begin{cases} e^{-t \int_{\mathbb{S}^{d-1}} |\gamma \cdot s|^\alpha (1 - i\, \mathrm{sign}(\gamma \cdot s) \tan \frac{\pi\alpha}{2}) \Gamma(\mathrm{d}s) + i\gamma\cdot\mu} & \alpha \neq 1 \\ e^{-t \int_{\mathbb{S}^{d-1}} |\gamma \cdot s| (1 - i \frac{2}{\pi} \mathrm{sign}(\gamma \cdot s) \log(\gamma \cdot s)) \Gamma(\mathrm{d}s) + i\gamma\cdot\mu} & \alpha = 1 \end{cases} \tag{60}$$

where $\mu \in \mathbb{R}^d$, Γ is a finite measure on the sphere $\mathbb{S}^{d-1} = \{s \in \mathbb{R}^d : \|s\| = 1\}$.

Since $\|\gamma \cdot s\| = \|\gamma\| \|s\| \cos\theta = \|\gamma\| \cos\theta$, where $s \in \mathbb{S}^{d-1}$ so that $\|s\| = 1$. Furthermore $\mathrm{sign}\,(\gamma \cdot s) = \mathrm{sign} \cos\theta$. We can assume γ oriented through the north pole of $\mathbb{S}^{d-1}$ and thus θ can be viewed as the latitude of vector s, as shown in Figure 10.

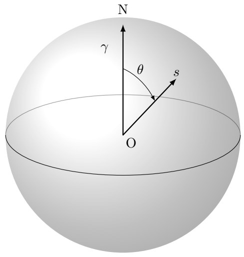

Figure 10. Domain of integration in (60), upon suitable rotation of the axis of the sphere.

We take the case $1 < \alpha < 2$ and rewrite the characteristic function as

$$\begin{aligned}\mathbb{E}e^{i\gamma\cdot S^\alpha(t)} &= e^{-t\|\gamma\|^\alpha \int_{\mathbb{S}^{d-1}} \cos^\alpha\theta(\cos\frac{\pi\alpha}{2} - i\,\mathrm{sign}(\cos\theta)\sin\frac{\pi\alpha}{2})\frac{\Gamma(\mathrm{d}s)}{\cos\frac{\pi\alpha}{2}} + i\gamma\cdot\mu} \\ &= e^{-t\|\gamma\|^\alpha\sigma^\alpha\int_{\mathbb{S}^{d-1}}\cos^\alpha\theta e^{-i\frac{\pi\alpha}{2}\mathrm{sign}(\cos\theta)}\Gamma(\mathrm{d}s) + i\gamma\cdot\mu}\end{aligned} \tag{61}$$

where $\sigma^\alpha = \frac{1}{\cos\frac{\pi\alpha}{2}}$.

For simplicity we assume $d = 3$ and suppose that Γ is a uniformly distributed measure on the upper hemisphere of the unit sphere. Thus

$$\Gamma(\mathrm{d}s) = \frac{\sin\theta}{2\pi}\,\mathrm{d}\theta\mathrm{d}\phi \qquad 0 < \theta < \frac{\pi}{2},\, 0 < \phi < 2\pi. \tag{62}$$

with $\mathrm{sign}(\cos\theta) = 1$. For $\mu = 0$ the integral in (61) becomes

$$\frac{e^{-i\frac{\pi\alpha}{2}}}{2\pi}\int_{\mathbb{S}^2}\cos^\alpha\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta = \frac{e^{-i\frac{\pi\alpha}{2}}}{2\pi}\int_0^{\frac{\pi}{2}}\sin\theta\cos^\alpha\theta\int_0^{2\pi}\mathrm{d}\phi = \frac{e^{-i\frac{\pi\alpha}{2}}}{\alpha+1} \tag{63}$$

The characteristic function (61) turns out to be

$$\mathbb{E}e^{i\gamma\cdot S^\alpha(t)} = e^{-t\|\gamma\|^\alpha\frac{1}{\cos\frac{\pi\alpha}{2}}\frac{e^{-i\frac{\pi\alpha}{2}}}{\alpha+1}} = e^{t\|\gamma\|^{\frac{2}{\nu}}\frac{1}{|\cos\frac{\pi}{\nu}|}\frac{e^{-i\frac{\pi}{\nu}}}{\alpha+1}} = e^{t\|\gamma\|^{\frac{2}{\nu}}\frac{1}{|\cos\frac{\pi}{\nu}|}e^{-i\frac{\pi}{\nu}}\sigma^\nu}$$

Since $\alpha = 2/\nu$ and $1 < \nu < 2$, we have $\pi > \pi/\nu > \pi/2$ so $\cos\frac{\alpha\pi}{2} = \cos\frac{\pi}{\nu}$ is negative.

If Γ is distributed on the lower hemisphere $\mathrm{sign}(\cos\theta) = -1$, and in the same way we have that

$$\mathbb{E}e^{i\gamma\cdot S^\alpha(t)} = e^{-t\|\gamma\|^\alpha\frac{1}{\cos\frac{\pi\alpha}{2}}\frac{e^{i\frac{\pi\alpha}{2}}}{\alpha+1}} = e^{t\|\gamma\|^{\frac{2}{\nu}}\frac{1}{|\cos\frac{\pi}{\nu}|}\frac{e^{i\frac{\pi}{\nu}}}{\alpha+1}} = e^{t\|\gamma\|^{\frac{2}{\nu}}\frac{1}{|\cos\frac{\pi}{\nu}|}e^{i\frac{\pi}{\nu}}\sigma^\nu}$$

The situation in the space $\mathbb{S}^{d-1}$, $d > 3$ is quite similar with the integral in (61) evaluated in hyperspherical coordinates.

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