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Inventory Models with Defective Units and Sub-Lot Inspection

Han-Wen Tuan, Gino K. Yang and Kuo-Chen Hung *

Department of Computer Science and Information Management, Hungkuang University, Taichung 43302, Taiwan; dancathy@ms5.hinet.net (H.-W.T.); yangklung@yahoo.com.tw (G.K.Y.)

* Correspondence: kchung@hk.edu.tw; Tel. +886-426-318-652 (ext. 5411)

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Abstract: Inventory models must consider the probability of sub-optimal manufacturing and careless shipping to prevent the delivery of defective products to retailers. Retailers seeking to preserve a reputation of quality must also perform inspections of all items prior to sale. Inventory models that include sub-lot sampling inspections provide reasonable conditions by which to establish a lower bound and a pair of upper bounds in terms of order quantity. This should make it possible to determine the conditions of an optimal solution, which includes a unique interior solution to the problem of an order quantity satisfying the first partial derivative. The approach proposed in this paper can be used to solve the boundary. These study findings provide the analytical foundation for an inventory model that accounts for defective items and sub-lot sampling inspections. The numerical examples presented in a previous paper are used to demonstrate the derivation of an optimal solution. A counter-example is constructed to illustrate how existing iterative methods do not necessarily converge to the optimal solution.

Keywords: distribution-free inventory model; defective items; sub-lot sampling inspection; backordered rate; crashable lead time

1. Introduction

An effective inventory management policy is meant to enhance the efficiency of inventory control and minimize inventory costs. Conventional economic order quantity (EOQ) models assume that the items provided by suppliers are in perfect condition; however, variability in manufacturing and accidents during shipping can result in retailers receiving defective items. The subsequent sale of defective products can lead to complaints, returns, and damage to the retailer’s reputation. Retailers should perform careful inspections of all items prior to sale; however, rigorous assessments are not always possible. Thus, inventory models must account for sub-lot inspections and the possibility that defective units could end up in the inventory.

Inventory models have been developed to meet the ever-evolving challenges of real-world applications. There has been considerable research into the issue of defective units and sub-lot inspection in inventory models. Paknejad et al. [1] proposed a modified EOQ model in which the number of defective goods is treated as a random variable. Wu and Ouyang [2] presented a continuous review inventory model combining backorders and lost sales, under the assumption that each lot contained a random number of defective units. Ouyang et al. [3] presented an iterative algorithm aimed at deriving the optimal vendor–buyer strategy for three situations involving defective items, respectively using the crisp defect rate approach, a triangular fuzzy number approach, and a statistical fuzzy approach. Sarkar et al. [4] proposed an integrated inventory model that deals with defective items using an inspection policy such as those decision variables that are the lead time and the ordering quantity. Sarkar et al. [5] provided a distribution-free approach to generalize inventory systems where the quality improvement and setup cost reduction are examined. Khan et al. [6] studied a vendor-managed inventory of consignment stock for a supply chain with defective items. An economic production quantity (EPQ) inventory system with a reworking of all defective items and multiple shipments was developed by Taleizadeh et al. [7]. Kang et al. [8] incorporated the effects of random defects on lot size and expected total cost function to assess a work-in-process-based inventory assuming that manufacturing defects follow a random distribution. Manna et al. [9] presented an imperfect production inventory model in which the defect follows the production rate as dependent. An EOQ system with a returned policy of deteriorated products and sample quantity inspection was constructed by Cheikhrouhou et al. [10]. An integrated inventory system to consider ordering quantity, lead time, number of shipments, and safety factor was provided by Kim et al. [11]. A continuously reviewed inventory system with multiple items was provided by Malik and Sarkar [12] to consider limited storage space, fuzzy demand, and stochastic lead time demand. A coordinated supply chain with a reliable seller and an unreliable seller was constructed by Malik and Sarkar [13] to reduce the transportation lead time. A continuously reviewed inventory system was examined by Bhuiya et al. [14] for a partially known distribution of lead time demand where backlogs and the lead time-dependent lost sales are considered as decision variables. A supply chain to minimize the total cost by setup cost investment and crashable lead time was developed by Ganguly et al. [15]. A traditional EPQ model with one vendor and one buyer was examined by Malik and Kim [16] to obtain the optimal solution. A sustainable EPQ system with cap-based production and the carbon tax was considered by Mishra et al. [17] to control the carbon emission rate under several shortage environments to invest in green technology. A transportation model was developed by Hota et al. [18] to show that unequal lot size is the best policy to transfer items from the manufacturer to the retailer.

Wu and Ouyang [2] tried to show the existence and uniqueness of the interior optimal solution; the minimum boundary for this solution is the goal of the proposed approach. Wu et al. [19] tried to generalize Wu and Ouyang [2] by considering such that their first model has a mixed normal probability density function and their second model has a mixed cumulative distribution where the first moment and the second moment for two distributions are known. Wu et al. [19] claimed that the optimal solution can be derived by an iterative method for the system of first partial derivatives. Hence, Wu et al. [19] claimed that their optimal solution must be an interior solution. If we assume their two distributions are identical, then the second model of Wu et al. [19] will degenerate to the distribution-free model of Wu and Ouyang [2]. In this study, we will show that sometimes the optimal solution will occur on the boundary such that the iterative method of Wu et al. [19] will contain questionable results. In the current study, this proposed model questions the assertion provided by Wu and Ouyang [2]. A detailed derivation is provided to obtain the criteria to decide the location of the optimal solution. This study then develops a theorem to show the condition for the boundary minimum solution. There are two numerical examples such that the first one is demonstrated for the interior optimal solution, in which our result is superior to that of Wu and Ouyang [2]. The second one is a boundary optimal solution to indicate the questionable assertion proposed by Wu and Ouyang [2]. In the past, for inventory models under the distribution-free approach, researchers always believed that the two decision variables: ordering quantity and the safety factor are both interior solutions and they can be derived by the iterative method through the first partial derivative system of the objective function. In this study, conditions are provided to help decision-makers to search the optimal solution on the boundary.

This study compared with other inventory models from the above-mentioned literature review have been summarized as Table 1.

Table 1. A comparison for the inventory models by the literature review.

	Inspecting Cost	The Uninspected Defective Penalty Cost	The Lead Time Crashing Cost	EOQ Model	Numerical Approach	Analytical Derivation
Paknejad et al. [1]	✓				✓	
Wu and Ouyang [2]	✓	✓	✓	✓	✓	
Ouyang et al. [3]	✓				✓	
Sarkar et al. [4]	✓		✓			✓
Sarkar et al. [5]				✓		✓
Khan et al. [6]	✓				✓	
Taleizadeh et al. [7]					✓	
Kang et al. [8]	✓					✓
Manna et al. [9]	✓				✓	
Cheikhrouhou et al. [10]	✓			✓		✓
Kim et al. [11]	✓		✓	✓	✓	
Malik and Sarkar [12]			✓	✓	✓	
Malik and Sarkar [13]			✓	✓	✓	
Bhuiya et al. [14]			✓	✓	✓	
Ganguly et al. [15]	✓		✓	✓		✓
Malik and Kim [16]			✓	✓	✓	
Mishra et al. [17]					✓	
Hota et al. [18]		✓	✓	✓		✓
Wu et al. [19]	✓	✓	✓	✓	✓	
This study	✓	✓	✓	✓		✓

2. Notation and Assumptions

For convenience, this study uses the same notation and assumptions of Wu and Ouyang [2]. The relative definitions have been used as follows:

D:	It denotes the expected demand per year.
A:	It denotes setup cost per setup.
h:	It denotes non-defective (including uninspected defective items) holding cost per unit per year.
π:	It denotes a shortage cost per unit short.
π_0:	It denotes a marginal profit per unit.
v:	It denotes unit inspection costs.
w:	It denotes unit penalty costs for uninspected defective items.
β:	It denotes the backorder rate of the demand during the stockout period, $0 \le \beta \le 1$.
p:	It denotes the defective rate in an order lot (independent of lot size), $0 \le p \le 1$ and it is a random variable.
$g(p)$:	It denotes the probability density function (p.d.f.) of p.
f:	It denotes the proportion of order quantity inspected.
Q:	It denotes lot size (order quantity), a decision variable.
L:	It denotes the length of the lead time, a decision variable.
X:	It denotes the lead time demand with finite mean μL and standard deviation $\sigma\sqrt{L}$ for lead time L, where μ is the mean for the unit time, and σ is the standard derivation for the unit time.
r:	It denotes the reorder point, $r = \mu L + k\sigma\sqrt{L}$, where k is the safety factor that is a decision variable.

The lead time L has n mutually independent components. The i-th component has a minimum duration a_i, normal duration b_i, and a crashing cost per unit time c_i. Moreover, for convenience, the researchers rearrange c_i such that $c_1 \le c_2 \le \cdots \le c_n$. The components of lead time are crashed one at a time starting with component 1 (because it has the minimum unit crashing cost), and then component 2, etc.

If researchers let $L_0 = \sum_{j=1}^{n} b_j$ and $L_i = \sum_{i+1}^{n} b_j + \sum_{j=1}^{i} a_j$, the lead time crashing cost $R(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

3. Review of Distribution-Free Model

For the distribution-free model, this study recalls the minimum problem that is proposed by Wu and Ouyang [2], for $Q > 0$, $k \ge 0$ and $L \in [L_i, L_{i-1}]$, the expected total annual cost (EAC) is denoted as

$$
\begin{aligned}
\mathrm{EAC}^{u}(Q,k,L) = {} & \frac{AD}{Q[1-fE(p)]} + \frac{h}{2}Q[1-fE(p)] + \frac{Qf^2h[E(p^2)-E^2(p)]}{2[1-fE(p)]} \\
& + \frac{f(1-f)hE[p(1-p)]}{2[1-fE(p)]} + h\sigma\sqrt{L}\left[k + \frac{1-\beta}{2}\left(\sqrt{1+k^2}-k\right)\right] + \frac{Dvf}{1-fE(p)} \\
& + \frac{D\sigma\sqrt{L}}{2Q[1-fE(p)]}[\pi + \pi_0(1-\beta)]\left(\sqrt{1+k^2}-k\right) + \frac{Dw(1-f)E(p)}{1-fE(p)} + \frac{R(L)D}{Q[1-fE(p)]}
\end{aligned} \tag{1}
$$

where A is the ordering cost for one replenishment.

$$
\begin{aligned}
& \frac{h}{2D}Q^2[1-fE(p)]^2 + \frac{Q^2f^2h[E(p^2)-E^2(p)]}{2D} + \frac{f(1-f)QhE[p(1-p)]}{2D} \\
& + \frac{h\sigma\sqrt{L}Q[1-fE(p)]}{D}\left[k + \frac{1-\beta}{2}\left(\sqrt{1+k^2}-k\right)\right].
\end{aligned}
$$

is the holding cost for one replenishment. $\frac{\sigma\sqrt{L}}{2}[\pi + \pi_0(1-\beta)]\left(\sqrt{1+k^2}-k\right)$ is the stockout cost for one replenishment. Qvf is the inspecting cost for one replenishment. $wQ(1-f)E(p)$ is the uninspected defective penalty cost for one replenishment. $R(L)$ is the lead time crashing cost for one replenishment. The duration time for one replenishment is $\frac{Q[1-fE(p)]}{D}$.

Wu and Ouyang [2] took the sum of (a) the ordering cost, (b) the holding cost, (c) the stockout cost, (d) the inspecting cost, (e) the uninspected defective penalty cost, and (f) the lead time crashing cost for one replenishment to find the total cost for one replenishment. Wu and Ouyang [2] divided the total cost for one replenishment by the duration time to derive the average cost per unit of time, denoted as $EAC^{u}(Q,k,L)$.

They proved that $EAC^{u}(Q,k,L)$ is concave up with respect to L and then they implied that the objective function will reduce to $L = L_i$ or $L = L_{i-1}$. To simplify the expression, this study still uses L to represent for L_i or L_{i-1}.

The system of the first partial derivative is derived by Wu and Ouyang [2] as follows

$$Q=\sqrt{\frac{2D}{h\delta}\left\{A+\frac{\sigma}{2}\sqrt{L}\left[\pi+\pi_0(1-\beta)\right]\left(\sqrt{1+k^2}-k\right)+R(L)\right\}} \tag{2}$$

with $\delta=1-2fE(p)+f^2E(p^2)$, and

$$\frac{2\sqrt{1+k^2}}{\sqrt{1+k^2}-k}=(1-\beta)+\frac{D[\pi+\pi_0(1-\beta)]}{hQ[1-fE(p)]} \tag{3}$$

Wu and Ouyang [2] mentioned that explicit general solutions for Q and k are not possible because the evaluation of Equations (2) and (3) requires knowledge of the value of the other. Thus, an iterative algorithm is established to find the optimal value of Q and k. Hence, Wu and Ouyang [2] assumed that $k_1=0$ and inserted it into Equation (2) to find a value of Q, denoted as Q_1, the value of Q_1 is then used in Equation (3) to obtain a value of k, denoted as k_2. Iteratively, k_i is plugged into Equation (2) to derive Q_i, Q_i is then plugged into Equation (3) to find the next one, k_{i+1}. They derive two sequences: (k_i) and (Q_i), until no change occurs in the values of Q_i and k_i.

The procedure of Wu and Ouyang [2] contains three questionable results. First, why will the two sequences (k_i) and (Q_i) converge? Second, given two different initial values of k_1 as k_{11} and k_{12}, if the results of the two sequences, (a) based on k_{11}, (k_{i1}) and (Q_{i1}), and (b) based on k_{12}, (k_{i2}) and (Q_{i2}), do indeed converge then they may have different limit points. That is, the limits may be dependent on the initial point. Third, if the two sequences converge, why are the limit points the optimal solution? Sometimes the iterative sequence approach may not derive the desired result, an example is provided in the following to illustrate this point. To solve $x^2-12x+20=0$ for positive solutions, there are two solutions $x=2$ and $x=10$. Our goal is to illustrate that by using the iterative method, it is almost impossible to construct a sequence that converges to 10. If the researchers try to find x by the iterative method, implicitly motivated by $x=\frac{x^2+20}{12}$, the researchers use the following iterative formula

$$x_{n+1}=\frac{x_n^2+20}{12} \tag{4}$$

From Equation (4), the researchers know that $x_{n+1}>x_n$ if and only if $x_n<2$ or $x_n>10$, which means that for the starting point, say x_0, there are the following five cases:

(a) $0\le x_0<2$, the generated sequence (x_n) will increase and converge to its limit, 2. For example, $x_0=0$, $x_1=1.667$, $x_2=1.899$, $x_3=1.967$, $x_4=1.989$, and $x_5=1.996$, which indicates that $\lim_{n\to\infty}x_n=2$;

(b) $x_0=2$, the researchers have $(x_n)=(2)$, a constant sequence. It converges to 2;

(c) $2<x_0<10$, owing to $x_{n+1}<x_n$, the generated sequence will decrease and converge to its limit, 2. For example, if researchers select $x_0=9$, then $x_1=8.417$, $x_2=7.570$, $x_4=5.125$, $x_6=2.905$, $x_8=2.135$, $x_{10}=2.016$, and $x_{12}=2.002$, which indicates that the derived sequence will converge to its limit, $\lim_{n\to\infty}x_n=2$;

(d) $x_0=10$, the researchers have $(x_n)=(10)$, a constant sequence. It converges to 10;

(e) $x_0 > 10$, according to $x_{n+1} > x_n$, the generated sequence will increase and diverge to ∞. For example, if the researchers select $x_0 = 11$, then $x_1 = 11.75$, $x_2 = 13.172$, $x_3 = 16.125$, $x_4 = 23.334$, $x_5 = 47.041$, $x_6 = 186.068$, and $x_7 = 2886.783$, which indicates that the derived sequence will diverge to ∞.

Based on the above example, the starting point will influence the result. Moreover, most of the time $x_0 = 10$ will not be selected. Therefore, there is little chance to construct a convergent sequence which will converge to our goal, with its limit of $\lim_{n\to\infty} x_n = 10$. Our example demonstrates that sometimes the iterative approach may not be as robust as previously thought. The result of the above discussion provides us with enough motivation to apply an analytical approach to prove the existence and uniqueness of the interior optimal solution and find the conditions for the boundary minimum.

In the next section, this study provides an analytical approach to prove that with three reasonable conditions, there exists a unique pair of Q and k that is the optimal solution. This study will show that the three reasonable conditions for the optimal order quantity are as one lower bound, Equation (10) and two upper bounds, and Equations (6) and (14). The first partial derivative system of Wu and Ouyang [2] consists of Equations (2) and (3). This study combines them into one equation by deleting the variable k such that in our findings, Equation (15) only contains one variable, Q. Owing to that, the order quantity must be positive and the safety factor must be non-negative, so during our derivation (Equations (5)–(14)), this proposed model derived that there are three necessary conditions to ensure the existence of the optimal solution.

4. Improved Mathematical Analysis

This study rewrites Equation (3) as

$$1 - \frac{2}{\Delta} = \frac{k}{\sqrt{1+k^2}} \tag{5}$$

where $\Delta = (1-\beta) + \frac{a}{Q}$ with $a = \frac{D[\pi + \pi_0(1-\beta)]}{h[1 - fE(p)]}$. Owing to $0 \le k < \sqrt{1+k^2}$, this study finds $0 < \frac{2}{\Delta} \le 1$ so that $2 \le 1 - \beta + \frac{a}{Q}$ to derive an upper bound of Q as

$$Q(1+\beta) \le a \tag{6}$$

This study rewrites Equation (5) as

$$\frac{k}{\sqrt{1+k^2}} = \frac{a - (1+\beta)Q}{a + (1-\beta)Q} \tag{7}$$

This study squares Equation (7) and cross multiplies to derive k in a function of Q, then

$$k = \frac{a - (1+\beta)Q}{2\sqrt{Q(a - \beta Q)}} \tag{8}$$

This study rewrites Equation (2) as

$$\frac{Q^2 - b}{c} = \sqrt{1+k^2} - k \tag{9}$$

where $b = \left[\frac{2D}{h\delta}(A + R(L))\right]$, and $c = \left[\frac{D\sigma}{h\delta}\sqrt{L}[\pi + \pi_0(1-\beta)]\right]$.

From Equation (9), owing to $\sqrt{1+k^2} > k$, it yields a lower bound for Q such that

$$Q > \sqrt{b} \tag{10}$$

From Equation (9), it yields that

$$\frac{c}{Q^2-b} = \frac{1}{\sqrt{1+k^2}-k} = \sqrt{1+k^2}+k \tag{11}$$

If the researchers take the sum and the difference of Equations (9) and (11), then it implies that

$$\sqrt{1+k^2} = \frac{Q^2-b}{2c} + \frac{c}{2\left(Q^2-b\right)}, \tag{12}$$

and

$$k = \frac{c}{2\left(Q^2-b\right)} - \frac{Q^2-b}{2c}. \tag{13}$$

Based on Equation (13) with $k \geq 0$ and Equation (10), this study obtains an upper of Q as

$$Q \leq \sqrt{b+c}. \tag{14}$$

If this study plugs Equations (12) and (13) into Equation (7), then it yields a fifth-degree polynomial as

$$\left(a-\beta Q\right)\left(Q^2-b\right)^2 - Qc^2 = 0. \tag{15}$$

The goal is to find the condition that guarantees Equation (15) will have a unique positive solution. From Equation (6), it follows that

$$a > \beta Q \tag{16}$$

to assure the well-defined of Equation (15).

However, the restriction in Equation (16) is unnecessary, owing to Equation (6), as this study already has $Q(1+\beta) \leq a$. Based on Equations (6) and(14), this study knows that

$$Q < \min\left\{\sqrt{b+c}, a/(1+\beta)\right\}. \tag{17}$$

In the following, this study will prove that under the conditions of Equations (10) and (17), $\sqrt{b} < Q \leq \min\left\{\sqrt{b+c}, a/(1+\beta)\right\}$, there is a unique solution for Equation (15). This study assumes an auxiliary function, say $F(Q)$, where

$$F(Q) = (a-\beta Q)\left(Q^2-b\right)^2 - c^2 Q. \tag{18}$$

This study knows that

$$F''(Q) = 4(a-\beta Q)\left(3Q^2-b\right) - 8\beta Q\left(Q^2-b\right) \tag{19}$$

From Equation (6), it yields that $a - \beta Q > Q$. On the other hand, owing to $0 \leq \beta \leq 1$, this study knows that

$$F''(Q) > 4Q\left[\left(3Q^2-b\right) - 2\left(Q^2-b\right)\right] = 4Q\left(Q^2+b\right) > 0 \tag{20}$$

such that $F(Q)$ is a convex function. According to $F\left(\sqrt{b}\right) = -\sqrt{b}c^2 < 0$,

$$F(a/(1+\beta))=(a/(1+\beta))\left[(a/(1+\beta))^2-b-c\right]\left[(a/(1+\beta))^2-b+c\right]. \tag{21}$$

and

$$F\left(\sqrt{b+c}\right)=\left(a-(1+\beta)\sqrt{b+c}\right)c^2>0. \tag{22}$$

From Equation (21), depending on the sign of $F(a/(1+\beta))$, this study will divide the solution procedure into the following three cases: (i) $(a/(1+\beta))^2>b+c$, (ii) $(a/(1+\beta))^2\leq b+c$ and $c+(a/(1+\beta))^2<b$, and (iii) $(a/(1+\beta))^2\leq b+c$ and $c+(a/(1+\beta))^2\geq b$.

For case (i), this study knows that $F(a/(1+\beta))>0$ and then

$$F\left(\min\left\{\sqrt{b+c},a/(1+\beta)\right\}\right)>0. \tag{23}$$

Due to the convexity of $F(Q)$, there is therefore a unique point, say Q^*, with $\sqrt{b}<Q^*<\min\left\{\sqrt{b+c},a/(1+\beta)\right\}$ that satisfies Equation (15). Using Equation (13), it implies that

$$k^*=\left(c/2\left(Q^{*2}-b\right)\right)-\left(\left(Q^{*2}-b\right)/2c\right) \tag{24}$$

such that Q^* and k^* are the interior optimal solution for the distribution-free inventory model.

For case (ii), owing to $(a/(1+\beta))^2\leq b+c$ and $c+(a/(1+\beta))^2<b$, this study still has $F(a/(1+\beta))>0$. Hence, the rest of the derivation is the same for case (i).

For case (iii), owing to $(a/(1+\beta))^2\leq b+c$ and $c+(a/(1+\beta))^2\geq b$, this study implies that $F(a/(1+\beta))\leq 0$. From the convexity property of $F(Q)$, this study derives that there is no point with $\sqrt{b}<Q^*<\min\left\{\sqrt{b+c},a/(1+\beta)\right\}$ which satisfies Equation (15). Consequently, the interior minimum solution does not exist. Though subsequently, by using our approach, this study finds the conditions

$$b-c\leq(a/(1+\beta))^2\leq b+c \tag{25}$$

to locate the boundary minimum.

Next, this study considers the minimums that are located on the boundary with $Q=0$ or $k=0$. Though in the paper of Wu and Ouyang [2], there is no discussion about the domain for their distribution free inventory model. This study knows that k is the safety factor with $k\geq 0$ and Q is the order quantity with $Q>0$.

During their derivation, Wu and Ouyang [2] adopted $Q/(Q-1)\approx 1$ to simplify their inventory model. That means that they assumed that Q would be a very large number. The assumption of $Q/(Q-1)\approx 1$ is supported from their numerical examples, the range of Q is derived to be from 145 to 184. Based on the above discussion, this study will first prove that the local minimum will not happen along the line with $Q=0$. This study derives that

$$\begin{aligned}\frac{\partial}{\partial Q}EAC^u(Q,k,L)=&\frac{-AD}{Q^2(1-fE(p))}+\frac{h}{2}(1-fE(p))+\frac{f^2h\left(E\left(p^2\right)-E^2(p)\right)}{2[1-fE(p)]}\\&-\frac{D\sigma\sqrt{L}}{2Q^2[1-fE(p)]}\left(\pi+\pi_0(1-\beta)\right)\left(\sqrt{1+k^2}-k\right)-\frac{R(L)D}{Q^2[1-fE(p)]}\end{aligned} \tag{26}$$

and

$$\frac{\partial^2}{\partial Q^2}EAC^u(Q,k,L)=\frac{D\sigma\sqrt{L}}{Q^3[1-fE(p)]}(\pi+\pi_0(1-\beta))\left(\sqrt{1+k^2}-k\right)+\frac{2AD}{Q^3(1-fE(p))}+\frac{2R(L)D}{Q^3[1-fE(p)]}>0 \tag{27}$$

From Equation (27), $\frac{\partial^2}{\partial Q^2}EAC^u(Q,k,L)>0$, this study knows that $EAC^u(Q,k,L)$ is convex in Q when k and L are fixed, where $L=L_i$ or $L=L_{i-1}$.

Moreover, $\lim_{Q\to 0}\frac{\partial}{\partial Q}EAC^u(Q,k,L)=-\infty<0$ and

$$\lim_{Q\to\infty}\frac{\partial}{\partial Q}EAC^u(Q,k,L)=\frac{h}{2}(1-fE(p))+\frac{f^2h\left(E(p^2)-E^2(p)\right)}{2[1-fE(p)]}>0 \tag{28}$$

there is a point, say $Q(k,L)$ satisfying $\frac{\partial}{\partial Q}EAC^u(Q(k,L),k,L)=0$, such that $Q(k,L)$ is the minimum along the ray $\{(Q,k,L):0<Q<\infty\}$. Hence, there is no need to discuss the boundary along the line of $Q=0$.

On the other hand, along the boundary of $k=0$, the objective function of Equation (1) is reduced to

$$\begin{aligned}EAC^u(Q,k=0,L)=&\frac{AD}{Q(1-fE(p))}+\frac{h}{2}Q(1-fE(p))+\frac{Qf^2h\left(E(p^2)-E^2(p)\right)}{2[1-fE(p)]}\\&+\frac{f(1-f)hE(p(1-p))}{2[1-fE(p)]}+h\sigma\sqrt{L}\left\{\frac{1-\beta}{2}\right\}+\frac{Dvf}{1-fE(p)}\\&+\frac{D\sigma\sqrt{L}}{2Q[1-fE(p)]}(\pi+\pi_0(1-\beta))+\frac{Dw(1-f)E(p)}{1-fE(p)}+\frac{R(L)D}{Q[1-fE(p)]}\end{aligned} \tag{29}$$

Based on Equation (29), this study assumes that.

$$\Delta_1=\frac{AD}{(1-fE(p))}+\frac{D\sigma\sqrt{L}}{2[1-fE(p)]}(\pi+\pi_0(1-\beta))+\frac{R(L)D}{[1-fE(p)]}, \tag{30}$$

$$\Delta_2=\frac{h}{2}(1-fE(p))+\frac{f^2h\left(E(p^2)-E^2(p)\right)}{2[1-fE(p)]}, \tag{31}$$

and

$$\Delta_3=\frac{f(1-f)hE(p(1-p))}{2[1-fE(p)]}+h\sigma\sqrt{L}\left\{\frac{1-\beta}{2}\right\}+\frac{Dvf}{1-fE(p)}+\frac{Dw(1-f)E(p)}{1-fE(p)}, \tag{32}$$

with Δ_1, Δ_2, and Δ_3 being positive.

Hence, based on the expression of Equations (30)–(32), this study can simplify Equation (29) as

$$EAC^u(Q,k=0,L)=(\Delta_1/Q)+\Delta_2Q+\Delta_3. \tag{33}$$

From Equation (33), this study derives that

$$EAC^{u}(Q,k=0,L)=\left(\sqrt{\Delta_1/Q}-\sqrt{\Delta_2 Q}\right)^2+2\sqrt{\Delta_1\Delta_2}+\Delta_3 \tag{34}$$

to the boundary minimum point $Q=\sqrt{\Delta_1/\Delta_2}$ and minimum value $2\sqrt{\Delta_1\Delta_2}+\Delta_3$.

In the next theorem, the results will be summarized as follows.

Theorem 1. *If* $(a/(1+\beta))^2 > b+c$ *or* $(a/(1+\beta))^2 < b-c$, *there is an interior* Q^* *that satisfies Equation (15) and* k^* *which satisfies Equation (13).*

If $b-c \le (a/(1+\beta))^2 \le b+c$, *then the boundary minimum will happen at* $Q^*=\sqrt{\Delta_1/\Delta_2}$ *and* $k^*=0$.

Lead time is an important factor for inventory models. This study presents a detailed examination between the crashable lead time and annual total cost. Hence, the decision-makers can decide how many investments should be provided to balance the cost and efficiency.

5. Numerical Examples

In order to demonstrate our approach, this study considers the same numerical examples that are proposed by Wu and Ouyang [2], with the following data: $D=600$ units/year, $A=\$200$ per order, $h=\$20$, $w=\$12$, $v=\$1.6$, $\pi=\$50$, $\pi_0=\$150$, $f=0.1$, $\sigma=7$ units/week, the lead time has three components such that $L_0=8$, $L_1=6$, $L_2=4$, and $L_3=3$ with the crashing cost, $R(L_0)=0$, $R(L_1)=5.6$, $R(L_2)=22.4$, and $R(L_3)=57.4$, the defective rate p has a Beta distribution with parameters $s=1$ and $t=4$ to imply the probability density function is $g(p)=4(1-p)^3$ for $0<p<1$ to imply that $E(p)=\frac{s}{s+t}=\frac{1}{5}$ and $E(p^2)=\frac{s(s+1)}{(s+t)(s+t+1)}=\frac{1}{15}$, the backordered rate, β, is assumed as 0, 0.5, 0.8, and 1. This study derives Q from Equation (15) and then uses Equation (13) to find k, to plug them into Equation (1) to find the minimum value. This study lists our findings in Table 2.

Table 2. Our solution for the example in Wu and Ouyang [2].

β	L_i	Q_i^*	k_i^*	$EAC(Q_i^*,k_i^*,L_i)$	β	L_i	Q_i^*	k_i^*	$EAC(Q_i^*,k_i^*,L_i)$
0.0	8	184.234	2.796	6173.397	0.8	8	159.560	1.776	5262.014
	6	176.074	2.864	5883.168		6	154.546	1.811	5079.805
	4	168.514	2.931	5567.743		4	150.988	1.836	4901.551
	3	169.320	2.923	5468.756		3	154.548	1.811	4895.407
0.5	8	170.225	2.237	5664.529	1.0	8	150.689	1.348	4908.181
	6	163.823	2.286	5433.926		6	146.842	1.372	4769.548
	4	158.527	2.328	5194.376		4	144.760	1.385	4646.160
	3	160.887	2.309	5146.848		3	149.335	1.356	4376.801

In Table 3 shown below, this study compares our findings with that of Wu and Ouyang [2].

Table 3. Comparison of ours and that of Wu and Ouyang [2].

	Wu and Ouyang				**Our Findings**			
β	L^*	Q^*	r^*	EAC^*	L^*	Q^*	k^*	EAC^*
0.0	3	169	70	5469.20	3	169.320	2.923	5468.756
0.5	3	161	62	5147.27	3	160.887	2.309	5146.848
0.8	3	155	56	4895.82	3	154.548	1.811	4895.407
1.0	3	149	51	4377.20	3	149.335	1.356	4376.801

Based on the comparison of the minimum solution and the minimum value, both approaches are comparatively very similar, under the parameters set in Wu and Ouyang [2]. In the following, this study constructed a counter-example to demonstrate that sometimes the iterative method cannot work. This study assumes that $D=60$, $h=200$ with $\beta=1$, $L_0=8$ and $R(L_0)=0$ to imply that

$$b-c=-184.231<(a/(1+\beta))^2=58.569<b+c=434.058 \tag{35}$$

which satisfies the condition of Equation (25). By our approach, according to the derivation of Equation (34), this study finds that $Q^*=20.834$ and $k^*=0$ with $EAC^u(Q^*,k^*,L_0)=44226.894$. On the other hand, by using the iterative method proposed by Wu and Ouyang [2] with $k_1=0$, by Equation (2), it yields that $Q_1=20.834$. If the researchers follow the suggestion of Wu and Ouyang [2], this study then faces the following problem

$$\frac{k}{\sqrt{1+k^2}}=1-\frac{1}{0.367335}=-1.722 \tag{36}$$

such that this study cannot derive a non-negative k_2 from Equation (36). The above counter-example demonstrates that in our Theorem 1, if the condition $b-c\le(a/(1+\beta))^2\le b+c$ is satisfied, then the iterative approach proposed by Wu and Ouyang [2] can not generate two sequences that converge to the optimal solution.

6. Managerial Insights and Real-Life Implication

In real life situations, the distribution of demand sometimes is unknown. The decision-makers can observe the historical data to find the mean and variance, and then based on the distribution-free approach, this study shows how to derive the optimal solutions under crashable lead time environments. Based on this study, investments in reducing the lead time are considered for any concerned decision-makers to minimize the total annual cost. On the other hand, this study provides information for extra costs that will be spent to accelerate the delivery and then compress the lead time to its minimum. Therefore, the decision-makers can decide his best policy under practical implications. There are so many real-life examples of this field for practical applications. In the following, a practical example of this study is presented below. After an earthquake, a company decides to order a necessary pipeline for future construction from a nearby country. The transport contains three components: sea, railroad, and bus. The transportation period for the pipeline can be crashed if extra investments are spent. The decision-maker can decide his best policy depending on the trade-off between cost and the delivery time. For example, if there is much competition, then the decision-maker should cut the delivery time to the minimum to earn the market such that the profit owing to the annual cost is increased. Hence, this study can provide help to the decision-maker to handle a real-life issue.

7. Conclusions

Retailers incur a variety of additional costs when dealing with defective products. Thus manufacturers must constantly seek to improve product quality and minimize inspection errors. This study identifies the criteria by which to determine whether the optimal solution to a production inventory with defective items is an interior minimum or a boundary minimum. This study also constructed a counter-example to illustrate the fact that iterative methods are sometimes unable to derive the boundary minimum. This study provides an analytical foundation for the optimal solution to the inventory problem with defective units and sub-lot inspection. Moreover, this model can be extended to a fuzzy environment under various conditions by considering the fuzzy setup and inventory holding cost. This model can also be extendable to consider quality-dependent demand and imperfect production environments.

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