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A Note on Symmetry of Birkhoff-James Orthogonality in Positive Cones of Locally C^* -algebras

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Abstract: In the present note some results of Kimuro, Saito, and Tanaka on symmetry of Birkhoff-James orthogonality in positive cones of C^* -algebras are extended to locally C^* -algebras.

Keywords: C^* -algebras; locally C^* -algebras; projective limit of projective family of C^* -algebras; locally Hilbert spaces; Birkhoff-James orthogonality; A^+ -local left (right) symmetric point for Birkhoff-James orthogonality

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1. Introduction

In 1935 Birkhoff introduced in [1], and in 1947 James studied in [2,3] a notion of orthogonality in Banach spaces in an attempt to generalize the standard notion of orthogonality in Hilbert spaces, which is important for study of Geometry of Banach spaces. More precisely, given a complex Banach space X and two elements $x, y \in X$, one says that x is *Birkhoff-James orthogonal* to y , denoted

$$x \perp_B y,$$

if

$$\|x + \lambda y\| \geq \|x\|,$$

holds for all $\lambda \in \mathbb{C}$. This relation generalizes the standard notion of orthogonality in Hilbert spaces. However, for most Banach spaces it is not a symmetric relation, that is,

$$x \perp_B y \text{ does not imply } y \perp_B x$$

in general.

A good overview of Birkhoff-James orthogonality for Banach spaces is given in 2012 by Alonso, Martini and Wu in their paper [4].

A special type of Banach spaces, and more specifically Banach algebras- the so called C^* -algebras, were introduced in 1943 by Gelfand and Naimark in their seminal paper [5]. At the present time the theory of C^* -algebras is well developed. For the development of the subject, see for example [6–8]. In [9,10] one can find very important applications of C^* -algebras to Quantum Physics.

Recently, in 2019, Komuro, Saito and Tanaka in [11] studied elements in C^* -algebras on which Birkhoff-James orthogonality is symmetric in a suitable sense. More precisely, denoting by A a C^* -algebra and by A^+ its positive cone, the authors say that an element $a \in A^+$ is an *A^+ -local left symmetric point for Birkhoff-James orthogonality* if for all $b \in A^+$ the relation

$$a \perp_B b \text{ implies } b \perp_B a.$$

Analogously, $a \in A^+$ is an A^+ -local right symmetric point for Birkhoff-James orthogonality if for all $b \in A^+$ the relation

$$b \perp_B a \text{ implies } a \perp_B b.$$

Let A be a C^* -algebra. The two main results of the aforementioned paper of Komuro, Saito and Tanaka characterize A^+ -local left symmetric points and A^+ -local right symmetric points in the following way:

Theorem 1 (Komuro-Saito-Tanaka). *Let A be a C^* -algebra and $a \in A^+$ a positive element of norm one. Then a is an A^+ -local left symmetric point for Birkhoff-James orthogonality if and only if a is a projection satisfying for each $b \in A^+$ the property that there exists $\lambda \in \mathbb{C}$, such that*

$$aba = \lambda a.$$

Proof. See [11] for details. \square

Theorem 2 (Komuro-Saito-Tanaka). *Let A be a unital C^* -algebra and $a \in A^+$ a non-zero positive element. Then a is an A^+ -local right symmetric point for Birkhoff-James orthogonality if and only if a is invertible.*

Proof. See [11] for details. \square

The Hausdorff projective limits of projective families of Banach algebras as natural locally-convex generalizations of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [12] and Michael [13]. The Hausdorff projective limits of projective families of C^* -algebras were first mentioned by Arens [12]. They have since been studied under various names by many authors. Development of the subject is reflected in the monograph of Fragoulopoulou [14]. We will follow Inoue [15] in the usage of the name **locally C^* -algebras** for these algebras. A lot of research has been done in the attempts to generalize to locally C^* -algebras the known results for C^* -algebras (see for example [15–19] to name a few). In particular, in [20] it has been shown how important it is in Physics to go out of category of C^* -algebras to the category of locally C^* -algebras- the reason is that not all operators of Quantum Physics, say for example the Energy operator, are bounded.

In the present notes we extend Theorems 1 and 2 of Komuro, Saito and Tanaka to locally C^* -algebras. This is the first attempt to study Geometry of Birkhoff-James orthogonality in locally convex spaces and algebras.

2. Preliminaries

First, let us recall some basic notions on topological $*$ -algebras. A $*$ -algebra (or involutory algebra) is a complex algebra A with an involution

$$* : A \rightarrow A,$$

such that

$$(a + \lambda b)^* = a^* + \bar{\lambda}b^*,$$

and

$$(ab)^* = b^*a^*,$$

for every $a, b \in A$ and every $\lambda \in \mathbb{C}$.

A seminorm $\|\cdot\|$ on A is a C^* -seminorm if it is submultiplicative, i.e.,

$$\|ab\| \leq \|a\| \|b\|,$$

and satisfies the C^* -condition, i.e.,

$$\|a^*a\| = \|a\|^2,$$

for every $a, b \in A$. Note that the C^* -condition alone implies that $\|\cdot\|$ is submultiplicative, and in particular

$$\|a^*\| = \|a\|,$$

for every $a \in A$ (cf. for example [14]).

In the case when a seminorm $\|\cdot\|$ on a $*$ -algebra A is a C^* -norm, and A is complete in the topology generated by this norm, A is called a **C^* -algebra**.

A topological $*$ -algebra is a $*$ -algebra A equipped with a topology making the operations (addition, multiplication, additive inverse, involution) separately continuous. For a topological $*$ -algebra A , we denote by $N(A)$ the set of continuous C^* -seminorms on A . If topology τ_A is generated by a separating and saturated family $N(A)$, A is called a **locally C^* -algebra**. One can see that in this case $N(A)$ is a directed set with respect to pointwise ordering, because

$$\max\{\|\cdot\|_\alpha, \|\cdot\|_\beta\} \in N(A)$$

for every $\|\cdot\|_\alpha, \|\cdot\|_\beta \in N(A)$, where $\alpha, \beta \in \Lambda$, with Λ being a certain directed set.

For a topological $*$ -algebra A , and $\|\cdot\|_\alpha \in N(A)$, $\alpha \in \Lambda$,

$$\ker \|\cdot\|_\alpha = \{a \in A : \|a\|_\alpha = 0\}$$

is a $*$ -ideal in A , and $\|\cdot\|_\alpha$ induces a C^* -norm (we as well denote it by $\|\cdot\|_\alpha$) on the quotient algebra

$$A_\alpha = A / \ker \|\cdot\|_\alpha,$$

and A_α is automatically complete in the topology generated by the norm $\|\cdot\|_\alpha$, thus is a C^* -algebra (see [14] for details). Each pair $\|\cdot\|_\alpha, \|\cdot\|_\beta \in N(A)$, such that

$$\beta \succeq \alpha,$$

$\alpha, \beta \in \Lambda$, induces a natural (continuous) surjective $*$ -homomorphism

$$g_\alpha^\beta : A_\beta \rightarrow A_\alpha,$$

such that for any $\alpha, \beta, \gamma \in \Lambda$, such that

$$\gamma \succeq \beta \succeq \alpha,$$

$$g_\alpha^\gamma = g_\alpha^\beta \circ g_\beta^\gamma.$$

Let now again, Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) " \preceq ". Let

$$\{A_\alpha, \alpha \in \Lambda\}$$

be a family of C^* -algebras, and g_α^β be, for

$$\alpha \preceq \beta,$$

the continuous linear $*$ -mappings

$$g_\alpha^\beta : A_\beta \longrightarrow A_\alpha,$$

so that

$$g_\alpha^\alpha(x_\alpha) = x_\alpha,$$

for all $\alpha \in \Lambda$, and

$$g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma,$$

whenever

$$\alpha \preceq \beta \preceq \gamma.$$

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let A be a $*$ -subalgebra of the direct product algebra

$$\prod_{\alpha \in \Lambda} A_\alpha,$$

so that for its elements

$$x_\alpha = g_\alpha^\beta(x_\beta),$$

for all

$$\alpha \preceq \beta,$$

where

$$x_\alpha \in A_\alpha,$$

and

$$x_\beta \in A_\beta.$$

Definition 1. The $*$ -algebra A constructed above is called a **Hausdorff projective limit** of the projective family

$$\{A_\alpha, \alpha \in \Lambda\},$$

of C^* -algebras relatively to the collection

$$\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda : \alpha \preceq \beta\},$$

and is denoted by

$$\varprojlim_{\alpha \in \Lambda} A_\alpha,$$

and is called the *Arens-Michael decomposition* of A .

It is well known (see, for example [21] or [22]) that for each $x \in A$, and each pair $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$, there is a natural projection

$$\pi_\beta : A \longrightarrow A_\beta,$$

with the basic property

$$\pi_\alpha(x) = g_\alpha^\beta(\pi_\beta(x)),$$

and each projection π_α for all $\alpha \in \Lambda$ is continuous.

Theorem 3. A topological $*$ -algebra (A, τ_A) over \mathbb{C} is a locally C^* -algebra iff A is a complete Hausdorff topological $*$ -algebra in which the topology τ_A is generated by a saturated separating family $N(A)$ of C^* -seminorms.

Proof. See for example [14] for details. \square

Example 1. Every C^* -algebra is a locally C^* -algebra.

Example 2. A closed $*$ -subalgebra of a locally C^* -algebra is a locally C^* -algebra.

Example 3. The product $\prod_{\alpha \in \Lambda} A_\alpha$ of C^* -algebras A_α , with the product topology, is a locally C^* -algebra.

Example 4. Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$). These spaces are called k -spaces in [23]. Then the algebra $C(X)$ of all continuous, not necessarily bounded complex-valued functions on X , with the topology of uniform convergence on compact subsets, is a locally C^* -algebra. It is well known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see [24] or [23] for details).

Let A be a locally C^* -algebra. Then an element $a \in A$ is called **bounded**, if $\|a\|_\infty < \infty$, where

$$\|a\|_\infty = \sup\{\|a\|_\alpha : \alpha \in \Lambda, \|\cdot\|_\alpha \in N(A)\}.$$

The set of all bounded elements of A is denoted by $b(A)$.

It is well-known that for each locally C^* -algebra A , its set $b(A)$ of bounded elements of A is a locally C^* -subalgebra, which is a C^* -algebra in the norm $\|\cdot\|_\infty$, such that it is dense in A in its topology τ_A (see for example [14]).

Let

$$U_\alpha = \{a \in A : \|a\|_\alpha \leq 1\},$$

$\alpha \in \Lambda$. Then $\{U_\alpha\}_{\alpha \in \Lambda}$ is a directed base with the associated family $\{\|\cdot\|_\alpha\}_{\alpha \in \Lambda}$ of C^* -seminorms on A generating the topology τ_A of A .

Let A^* as usual denote the conjugate space of A . For a given $\alpha \in \Lambda$,

$$A^*(\alpha) = \{f \in A^* : f \text{ is bounded on } U_\alpha\},$$

denotes all functionals in A^* that are bounded on U_α . Let

$$\Lambda(f) = \{\alpha \in \Lambda : f \text{ is bounded on } U_\alpha\}$$

denote all indices α in Λ so that a given $f \in A^*$ is bounded on U_α .

A linear functional on a locally convex space is continuous if and only if it is bounded on some neighborhood of zero. It means in our notation that $f \in A^*$ if and only if $\Lambda(f) \neq \emptyset$.

From that it follows that if $f \in A^*(\alpha)$, then there exists a linear functional f_α on A_α , such that

$$f(a) = f_\alpha(\pi_\alpha(a)),$$

for all $a \in A$. Thus,

$$A^* = \bigcup_{\alpha \in \Lambda} A^*(\alpha).$$

Let now

$$A^*(+) = \{f \in A^* : f(a^*a) \geq 0 \text{ for all } a \in A\},$$

$$A^*(\alpha, +) = \{f \in A^*(\alpha) : f(a^*a) \geq 0 \text{ for all } a \in A\},$$

$\alpha \in \Lambda$, and

$$A_\alpha^*(+) = \{f_\alpha \in A_\alpha^* : f_\alpha(a_\alpha^*a_\alpha) \geq 0 \text{ for all } a_\alpha \in A_\alpha\}.$$

$\alpha \in \Lambda$.

One can see that:

(i). the mapping

$$f \rightarrow f_\alpha,$$

$\alpha \in \Lambda$, is an algebraic isomorphism of $A^*(\alpha)$ onto A_α^* ;

(ii).

$$\|f_\alpha\| = \sup_{a \in U_\alpha} |f(a)|,$$

$a \in A, \alpha \in \Lambda;$

(iii). a functional $f \in A^*(\alpha)$ is positive (Hermitian) if and only iff $f_\alpha \in A_\alpha^*$ is positive (Hermitian);

(iv). if A is unital, then the mapping

$$f \rightarrow f_\alpha,$$

$\alpha \in \Lambda$, from $A^*(\alpha, +)$ onto $A_\alpha^*(+)$ is bi-continuous with respect to the relative w^* - topologies $\sigma(A^*, A)$ and $\sigma(A_\alpha^*, A_\alpha)$ of those spaces. Thus,

$$A^*(+) = \bigcup_{\alpha \in \Lambda} A^*(\alpha, +).$$

(see for example [15] for details.)

A non-zero positive functional f on A is called *pure*, it cannot be represented as a linear combination of two other positive functionals with non-negative coefficients. It is equivalent to a statement that from

$$g \leq f,$$

it follows that there exists $\lambda \in [0, 1]$, so that

$$g = \lambda f.$$

One can see that for $f \in A^*(\alpha, +)$ to be pure it is necessary and sufficient for f_α to be pure.

A family of Hilbert spaces $\{H_\alpha\}_{\alpha \in \Lambda}$ called *inductive* if for $\alpha, \beta \in \Lambda$, and

$$\alpha \preceq \beta,$$

the following conditions hold:

(i).

$$H_\alpha \subset H_\beta;$$

(ii).

$$(\cdot, \cdot)_\alpha = (\cdot, \cdot)_\beta$$

on H_α .

Let

$$H = \bigcup_{\alpha \in \Lambda} H_\alpha$$

be a union of an inductive family of Hilbert spaces $\{H_\alpha\}_{\alpha \in \Lambda}$.

We define a topology τ_H on H as follows:

(i).

$$X \subset H$$

is a closed set in H ;

if and only if

(ii).

$$X = H,$$

or

(iii). for some $\alpha \in \Lambda$,

$$X \subset H_\alpha,$$

and H is closed in H_α .

With that topology τ_H , H is called a *locally Hilbert space*. It is shown in [15] that (H, τ_H) is a T_1 topological space.

Let for each $\alpha \in \Lambda$,

$$P_\alpha : H \rightarrow H_\alpha,$$

be the orthogonal projection from H onto H_α , and for each pair of indices $\alpha, \beta \in \Lambda$, such that

$$\beta \succeq \alpha,$$

$$P_\alpha^\beta : H_\beta \rightarrow H_\alpha,$$

be the orthogonal projection from H_β onto H_α . One can see that

$$P_\alpha = P_\alpha^\beta \circ P_\beta,$$

and for each $\alpha \in \Lambda$, and each vector $\xi \in H$, there exists a decomposition

$$\xi = \xi' + \eta,$$

such that

$$(\xi', \eta) = 0,$$

$\xi' \in H_\alpha$, and η belongs to the complement of H_α in H , i.e.,

$$P_\alpha(\xi) = \xi',$$

and

$$P_\alpha(\eta) = 0_H.$$

(see [15] for details.)

Each locally Hilbert space (H, τ_H) is a pre-Hilbert space in a canonical way, and each pre-Hilbert space can be endowed with structure of a locally Hilbert space, and

$$H \cong \varinjlim_{\alpha \in \Lambda} H_\alpha,$$

where Λ can be taken of the same cardinality as the set of all finite-dimensional Hilbert subspaces of H ordered by inclusion. (see [25] for details).

Let (A, τ_A) be a locally C^* -algebra in which the topology τ_A generated by separating and saturated family $N(A)$ of C^* -seminorms $\{\|\cdot\|_\alpha\}_{\alpha \in \Lambda}$. Thus,

$$A \cong \varprojlim_{\alpha \in \Lambda} A_\alpha,$$

i.e., A is topologically $*$ -isomorphic to the projective limit of the projective family of C^* -algebras $\{A_\alpha\}_{\alpha \in \Lambda}$, where for each $\alpha \in \Lambda$, A_α is isometrically $*$ -isomorphic to the factor algebra

$$A / \ker \|\cdot\|_\alpha$$

endowed with the factor norm. (see for example [14] for details). In this case we call A a *locally C^* -algebra of type Λ* .

Remark 1. Generally speaking, there may be locally C^* -algebras of different types when Λ is more than countable.

We say that a *locally Hilbert space* (H, τ_H) is of *type Λ* , if there exists an inductive family of its Hilbert subspaces generating it, i.e.,

$$H \cong \varinjlim_{\alpha \in \Lambda} H_\alpha.$$

Let $B(H_\alpha)$ for each $\alpha \in \Lambda$, be the C^* -algebra of all bounded linear operators on H_α . From the fact that the family of Hilbert spaces $\{H_\alpha\}_{\alpha \in \Lambda}$, generates the locally Hilbert spaces H of type Λ , it follows

that it is inductive, and that the family of C^* -algebras $B(H_\alpha)$ is projective. We denote by $L(H)$ the projective limit

$$L(H) \cong \varprojlim_{\alpha \in \Lambda} B(H_\alpha),$$

of that family $\{B(H_\alpha)\}_{\alpha \in \Lambda}$.

Let

$$T : H \rightarrow H,$$

be a linear operator on locally Hilbert space H of type Λ . For each $\alpha \in \Lambda$, let

$$T_\alpha(\xi) = T(\xi) \text{ when } \xi \in H_\alpha.$$

The operator T is called *coherent* if and only if for every $\alpha, \beta \in \Lambda$, such that $\alpha \preceq \beta$,

$$P_\alpha^\beta \circ T = T \circ P_\alpha^\beta.$$

Lemma 1 (Inoue). *A coherent linear operator T on H is continuous if and only if for each $\alpha \in \Lambda$, $T_\alpha \in B(H_\alpha)$.*

Proof. See [15] for details. \square

As a corollary one gets that each coherent continuous linear operator on H leaves H_α for each $\alpha \in \Lambda$ invariant, i.e., for each $\alpha \in \Lambda$,

$$T(H_\alpha) \subset H_\alpha.$$

For that reason a linear coherent continuous linear operator T on H is called a **locally bounded operator** (see [26]). If we define

$$\pi_\alpha : \varprojlim_{\alpha \in \Lambda} B(H_\alpha) \rightarrow B(H_\alpha)$$

as

$$\pi_\alpha : T \rightarrow T_\alpha$$

for each $\alpha \in \Lambda$, then $L(H)$ can be identified with the locally C^* -algebra of type Λ of all locally bounded linear operators on the locally Hilbert space H of type Λ .

Let A and B be locally C^* -algebras of the same type Λ with respective Arens-Michael decompositions

$$A \cong \varprojlim_{\alpha \in \Lambda} A_\alpha \text{ and } B \cong \varprojlim_{\alpha \in \Lambda} B_\alpha.$$

We call a $*$ -homomorphism

$$\varphi : A \rightarrow B$$

a *locally $*$ -homomorphism*, if and only if for each $\alpha \in \Lambda$, there exists a $*$ -homomorphism of C^* -algebras A_α and B_α ,

$$\varphi_\alpha : A_\alpha \rightarrow B_\alpha,$$

so that

$$\pi_\alpha^B \circ \varphi = \varphi_\alpha \circ \pi_\alpha^A,$$

where π_α^A (resp. π_α^B) is the natural projection from A onto A_α (resp. the natural projection from B onto B_α)

By a *locally $*$ -representation* $\{A, \mu, H\}$ of a locally C^* -algebra A of type Λ on a locally Hilbert space H of type Λ we understand a locally $*$ -homomorphism

$$\mu : A \rightarrow L(H),$$

from A into the C^* -algebra $B(H)$ of bounded linear operators on some Hilbert space H .

Let

$$A \cong \varprojlim_{\alpha \in \Lambda} A_\alpha,$$

be a locally C^* -algebra of type Λ ,

$$\mu : A \rightarrow L(H)$$

be a $*$ -homomorphism from A into $L(H)$, where

$$H \cong \varinjlim_{\alpha \in \Lambda} H_\alpha,$$

is a locally Hilbert space of type Λ . Then one can see that $\{A, \mu, H\}$ is a locally $*$ -representation of the locally C^* -algebra A of type Λ if and only if for each $\alpha \in \Lambda$, $\{A_\alpha, \mu_\alpha, H_\alpha\}$ is a $*$ -representation of the C^* -algebra A_α on the Hilbert space H_α .

3. Symmetry of Birkhoff-James Orthogonality in Positive Cones of Locally C^* -algebras

Let A be a locally C^* -algebra, and $a, b \in A$. We say that a is *Birkhoff-James orthogonal* to b , denoted by

$$a \perp_B b,$$

if for each $\alpha \in \Lambda$, the inequality

$$\|a + \lambda b\|_\alpha \geq \|a\|_\alpha,$$

holds for all $\lambda \in \mathbb{C}$. We say that if A^+ is the positive cone of A , the element $a \in A^+$ is an A^+ -local left symmetric point for Birkhoff-James orthogonality if for all $b \in A^+$ the relation

$$a \perp_B b \text{ implies } b \perp_B a.$$

Analogously, $a \in A^+$ is an A^+ -local right symmetric point for Birkhoff-James orthogonality if for all $b \in A^+$ the relation

$$b \perp_B a \text{ implies } a \perp_B b.$$

The following three lemmata are valid:

Lemma 2. *Let*

$$A \cong \varprojlim_{\alpha \in \Lambda} A_\alpha,$$

be a locally C^ -algebra of type Λ , and $a, b \in A$. Then*

$$a \perp_B b$$

in A if and only if for each $\alpha \in \Lambda$,

$$a_\alpha \perp_B b_\alpha$$

in A_α .

Proof. First assume that

$$a \perp_B b.$$

It implies that for all $\alpha \in \Lambda$,

$$\|a + \lambda b\|_\alpha \geq \|a\|_\alpha,$$

holds for all $\lambda \in \mathbb{C}$. For a given arbitrary $\alpha \in \Lambda$, we conclude that

$$\|a_\alpha + \lambda b_\alpha\|_{A_\alpha} = \|a + \lambda b\|_\alpha \geq \|a\|_\alpha = \|a_\alpha\|_{A_\alpha},$$

thus, due to the arbitrariness of α , in each A_α ,

$$a_\alpha \perp_B b_\alpha.$$

Conversely, let us assume that for each $\alpha \in \Lambda$,

$$a_\alpha \perp_B b_\alpha$$

in A_α . It means that for each $\alpha \in \Lambda$,

$$\|a_\alpha + \lambda b_\alpha\|_{A_\alpha} \geq \|a_\alpha\|_{A_\alpha}.$$

From the last inequality we get that for each $\alpha \in \Lambda$,

$$\|a + \lambda b\|_\alpha = \|a_\alpha + \lambda b_\alpha\|_{A_\alpha} \geq \|a_\alpha\|_{A_\alpha} = \|a\|_\alpha,$$

thus

$$a \perp_B b,$$

which was required. \square

Lemma 3. *Let*

$$A \cong \varprojlim_{\alpha \in \Lambda} A_\alpha,$$

be a locally C^* -algebra of type Λ , and $a, b \in A$. Then for each $\alpha \in \Lambda$,

$$\pi_\alpha^A(A^+) = A_\alpha^+.$$

Proof. See [15] for details. \square

Lemma 4. *Let A be a locally C^* -algebra. Then $a \in A^+$ is a projection if and only if for all $\alpha \in \Lambda$, a_α is a projection in A_α^+ .*

Proof. Let $a \in A^+$ be a projection, i.e., $a^2 = a$. Applying, for all $\alpha \in \Lambda$, the natural projection π_α^A to the last identity, we get

$$a_\alpha^2 = \pi_\alpha^A(a)^2 = \pi_\alpha^A(a^2) = \pi_\alpha^A(a) = a_\alpha,$$

thus $a_\alpha^2 = a_\alpha$, i.e., a_α is a projection in A_α^+ .

Conversely, let for all $\alpha \in \Lambda$, $a_\alpha \in A_\alpha^+$ be such that $a_\alpha^2 = a_\alpha$. Let us consider a family $\{a_\alpha\}_{\alpha \in \Lambda}$. This family is a projective family, i.e., there exists a unique $a \in A$, such that for each $\alpha \in \Lambda$,

$$\pi_\alpha^A(a) = a_\alpha.$$

Let us consider $a^2 \in A^+$. For each $\alpha \in \Lambda$,

$$\pi_\alpha^A(a^2) = \pi_\alpha^A(a)^2 = a_\alpha^2 = a_\alpha,$$

which implies that $a^2 = a$. Indeed, let us on the contrary assume that $a^2 \neq a$. It would mean that there exists at least one $\alpha_0 \in \Lambda$, such that $a_{\alpha_0}^2 \neq a_{\alpha_0}$. This contradiction completes the proof. \square

Now we are ready to prove a version of Theorem 1 for locally C^* -algebras.

Theorem 4. Let A be a locally C^* -algebra of type Λ and $a \in A^+$ a positive element of A , such that for each $\alpha \in \Lambda$,

$$\|a\|_\alpha = 1.$$

Then a is an A^+ -local left symmetric point for Birkhoff-James orthogonality if and only if a is a projection satisfying for each $b \in A^+$, the property that there exists $\lambda \in \mathbb{C}$, such that

$$aba = \lambda a.$$

Proof. First note, that since $b \in A^+$, if a is a projection and $\lambda \in \mathbb{C}$, then

$$(aba)(\lambda a) = (\lambda a)(aba) = \lambda aba,$$

thus the elements λa and aba always commute, and therefore both belong to the maximal commutative locally C^* -subalgebra of A that contains a .

Let $a \in A^+$ be an A^+ -local left symmetric point for Birkhoff-James orthogonality such that for each $\alpha \in \Lambda$,

$$\|a\|_\alpha = 1.$$

From Lemmas 2 and 3 it follows that for each $\alpha \in \Lambda$, $a_\alpha \in A_\alpha^+$ is an A_α^+ -local left symmetric point for Birkhoff-James orthogonality in A_α , and

$$\|a_\alpha\|_{A_\alpha} = \|a\|_\alpha = 1.$$

From Theorem 1 applicable to A_α it follows that a_α is a projection and for each $b_\alpha \in A_\alpha^+$, there exists $\lambda \in \mathbb{C}$, such that

$$a_\alpha b_\alpha a_\alpha = \lambda a_\alpha,$$

where b_α without loss of generality can be taken in be arbitrary projective family $\{b_\beta\}_{\beta \in \Lambda}$. Indeed, let

$$\alpha \preccurlyeq \beta,$$

$\alpha, \beta \in \Lambda$, and

$$a_\alpha b_\alpha a_\alpha = \lambda_1 \cdot a_\alpha,$$

but

$$a_\beta b_\beta a_\beta = \lambda_2 \cdot a_\beta.$$

We get that

$$\lambda_1 \cdot a_\alpha = a_\alpha b_\alpha a_\alpha = g_\alpha^\beta(a_\beta b_\beta a_\beta) = g_\alpha^\beta(\lambda_2 \cdot a_\beta) = \lambda_2 \cdot a_\alpha,$$

and

$$\lambda_1 = \lambda_2,$$

thus λ does not depend of $\alpha \in \Lambda$, as long as b_α is taken from a projective family $\{b_\beta\}_{\beta \in \Lambda}$.

Therefore, there exists a unique $b \in A$, such that for each $\alpha \in \Lambda$,

$$\pi_\alpha^A(b) = b_\alpha \in A.$$

We show that

$$aba = \lambda a.$$

Indeed, if we assume that

$$aba \neq \lambda a,$$

for any $\lambda \in \Lambda$, there will exist $\alpha_0 \in \Lambda$, such that

$$\pi_{\alpha_0}^A(aba) = \pi_{\alpha_0}^A(a)\pi_{\alpha_0}^A(b)\pi_{\alpha_0}^A(a) = a_{\alpha_0}b_{\alpha_0}a_{\alpha_0} \neq \lambda a_{\alpha_0} = \pi_{\alpha_0}^A(\lambda a).$$

Contradiction proves what required.

Conversely, let $a \in A^+$ be a projection satisfying for each $\alpha \in \Lambda$,

$$\|a\|_\alpha = 1,$$

such that for each $b \in A^+$, there exists $\lambda \in \mathbb{C}$, such that

$$aba = \lambda a.$$

The later means that for each $\alpha \in \Lambda$,

$$\pi_\alpha^A(aba) = \pi_\alpha^A(a)\pi_\alpha^A(b)\pi_\alpha^A(a) = a_\alpha b_\alpha a_\alpha = \pi_\alpha^A(\lambda a) = \lambda a_\alpha,$$

i.e.,

$$a_\alpha b_\alpha a_\alpha = \lambda a_\alpha,$$

for each $b_\alpha \in A_\alpha^+$, and a_α is a projection in A_α due to Lemma 4.

Applying Theorem 1 to algebra A_α we get that $a_\alpha \in A_\alpha^+$ is an A_α^+ -local left symmetric point for Birkhoff-James orthogonality in A_α , for each $\alpha \in \Lambda$. Applying Lemma 2 we get that a is an A^+ -local left symmetric point for Birkhoff-James orthogonality, which was required. \square

In order to establish a version of Theorem 2 for locally C^* -algebras, we would need the following:

Lemma 5. *Let A be a unital locally C^* -algebra of type Λ . Then $a \in A$ is invertible if and only if for all $\alpha \in \Lambda$, a_α is invertible.*

Proof. Let $a \in A$ be invertible, i.e., there exists a unique $b \in A$, such that

$$ab = \mathbf{1}_A.$$

Applying for each $\alpha \in \Lambda$, the natural projection π_α^A , we get that

$$\pi_\alpha^A(ab) = \pi_\alpha^A(a)\pi_\alpha^A(b) = a_\alpha b_\alpha = \pi_\alpha^A(\mathbf{1}_A) = \mathbf{1}_{A_\alpha},$$

and a_α is invertible in A_α .

Conversely, let $a \in A$ be such that for each $\alpha \in \Lambda$, a_α is invertible in A_α , i.e., there exists a unique $b_\alpha \in A_\alpha$ for each $\alpha \in \Lambda$, such that

$$a_\alpha b_\alpha = \mathbf{1}_{A_\alpha}.$$

Let B be maximal commutative locally C^* -subalgebra of A , such that $a \in B$. It is obvious that B is of the same type Λ , and $a_\alpha, b_\alpha \in B_\alpha$ for all $\alpha \in \Lambda$, and

$$B \cong \varprojlim_{\alpha \in \Lambda} B_\alpha,$$

where each B_α is a commutative unital C^* -subalgebra of A_α . Using a Gelfand-Naimark type theorem for commutative locally C^* -algebras (see [27]) we get a functional locally C^* -algebra $C(X)$ of type Λ of all continuous complex-valued functions on compactly generated completely regular topological space X with generating family of Hausdorff compacts X_α , such that

$$B \cong C(X),$$

for each $\alpha \in \Lambda$, and

$$B_\alpha \cong C(X_\alpha),$$

where $C(X_\alpha)$ is a commutative unital C^* -algebra under supremum norm. Let $f_a \in C(X)$, be the function corresponding to the element $a \in A$, and let $f_{a_\alpha} \in C(X_\alpha)$ be the function corresponding to a_α . Since each a_α is invertible in A_α , f_{a_α} is never equal to 0 for all $x_\alpha \in X_\alpha$, for each $\alpha \in \Lambda$. Thus f_a is never equal to 0 for all $x \in X$, thus, there exists a function

$$\frac{1}{f_a} \in C(X).$$

Let b be an element in B corresponding to the function $\frac{1}{f_a}$. We get that

$$ab = \mathbf{1}_B = \mathbf{1}_A,$$

thus, a is invertible, and for each $\alpha \in \Lambda$,

$$\pi_\alpha^A(b) = b_\alpha$$

due to uniqueness of b_α . \square

Now we are ready to prove a version of Theorem 2 for locally C^* -algebras.

Theorem 5. *Let A be a unital locally C^* -algebra and $a \in A^+$ a non-zero positive element. Then a is an A^+ -local right symmetric point for Birkhoff-James orthogonality if and only if a is invertible.*

Proof. Let a be an A^+ -local right symmetric point for Birkhoff-James orthogonality in A . Due to Lemma 2, for each $\alpha \in \Lambda$, each

$$a_\alpha = \pi_\alpha^A(a)$$

is an A_α^+ -local right symmetric point for Birkhoff-James orthogonality in A_α . Applying Theorem 2 to the algebra A_α , we get that each a_α is invertible, and from Lemma 5 it follows that a is invertible in A .

Conversely, let $a \in A^+$ be non-zero positive invertible element. From Lemma 3 and Lemma 5 it follows that each a_α is a non-zero positive invertible element in A_α^+ . Applying Theorem 2 to the algebra A_α , we get that each a_α is an A_α^+ -local right symmetric point for Birkhoff-James orthogonality in A_α . From Lemma 2 it now follows that a is an A^+ -local right symmetric point for Birkhoff-James orthogonality in A . \square

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