Generalized Mehler Semigroup on White Noise Functionals and White Noise Evolution Equations

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Abstract: In this paper, we study a representation of generalized Mehler semigroup in terms of Fourier–Gauss transforms on white noise functionals and then we have an explicit form of the infinitesimal generator of the generalized Mehler semigroup in terms of the conservation operator and the generalized Gross Laplacian. Then we investigate a characterization of the unitarity of the generalized Mehler semigroup. As an application, we study an evolution equation for white noise distributions with n-th time-derivative of white noise as an additive singular noise.

Keywords: white noise theory; Gaussian space; generalized Fourier–Gauss transform; generalized Fourier–Mehler transform; generalized Mehler semigroup; evolution equation

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1. Introduction

Since the white noise theory initiated by Hida [1] as an infinite dimensional distribution theory, it has been extensively studied by many authors [2–8] (and references cited therein) with many applications to wide research fields, stochastic analysis, quantum field theory, mathematical physics, mathematical finance and etc. The white noise theory is based on a Gelfand triple:

$$(E) \subset (L^2) \subset (E)^*.$$

On the other hand, recently, the generalized Mehler semigroup as the transition semigroup of the infinite dimensional (Hilbert or Banach space valued) Ornstein–Uhlenbeck process described by the Langevin equation:

$$dX_t = AX_t dt + C dM_t$$

has been studied successfully by many authors [9–12] (see also [13–16]) and references cited therein. Here $\{M_t\}_{t \geq 0}$ is an infinite dimensional noise process and A and C are certain operators on the infinite dimensional space. In fact, the authors [12] studied systematically the generalized Mehler semigroup for cylindrical Wiener process $\{M_t\}_{t \geq 0}$ and then in [17], the authors generalized to the case of Lévy process $\{M_t\}_{t \geq 0}$. Furthermore, in [13,14], the authors studied the generalized Mehler semigroups and Langevin type equations with different noise processes. Recently, in [18], the author studied covariant generalized Mehler semigroup, and in [19], the authors studied time inhomogeneous generalized Mehler semigroup. For more details of the theory of Ornstein–Uhlenbeck operators and semigroups,

we refer the reader to [20] (see also [21]) which includes major recent achievements and open questions, and in which the generalized Mehler semigroups are briefly discussed.

The objective of this paper is twofold: the first one is to study the generalized Mehler semigroup on the space (E) of the test white noise functionals with its explicit form in terms of the generalized Fourier–Gauss transform. From the representation of the generalized Mehler semigroup, we investigate a characterization of the unitarity of the generalized Mehler semigroup. The second objective is to study a white noise Langevin (type) equation:

$$d\Phi_t = \Xi^* \Phi_t dt + dW_t^{(n)},$$

where $\Xi^* \in \mathcal{L}((E)^*, (E)^*)$ is the infinitesimal generator of an equicontinuous semigroup and $W_t^{(n)}$ is the n-th time-derivative of Gaussian white noise which is considered as a highly singular noise process. Specially, we are interested in the case of Ξ^* which is the infinitesimal generator of the adjoint of the generalized Mehler semigroup (see the Equation (16)). Recently, in [22], the author studied an evolution equation associated with the integer power of the Gross Laplacian Δ_G. We note that the Gross Laplacian Δ_G is a special case of the generator of the generalized Mehler semigroup.

As main results of this paper, we provide a representation of the generalized Mehler semigroup in terms of the generalized Fourier–Gauss transform on the space of the test white noise functionals, and then by applying the properties of the generalized Fourier–Gauss transform, we have an explicit form of the infinitesimal generator of the generalized Mehler semigroup, which is a perturbation of the Ornstein–Uhlenbeck generator. By duality, we study the generalized Fourier–Mehler transform and its infinitesimal generator, which induce the dual semigroup of the generalized Mehler semigroup and its infinitesimal generator, and then as application we investigate the unique weak solution of the Langevin type stochastic evolution equations with very singular noise forcing terms (see Theorem 9).

This paper is organized as follows. In Section 2, we recall basic notions for Gaussian space and (Gaussian) white noise functionals. In Section 3, we invite the general theory for white noise operators which is necessary for our main study. In particular, we review the generalized Fourier–Gauss and Fourier–Mehler transforms on white noise functionals. In Section 4, we study the generalized Mehler semigroups on (test) white noise functionals with their representations in terms of generalized Fourier–Gauss transform, and explicit forms of the infinitesimal generators of the generalized Mehler semigroups in terms of the conservation operators and the generalized Gross Laplacian. As the last result of Section 4, we investigate a characterization of the unitarity of the generalized Mehler semigroup. In Section 5, we consider the white noise integrals of white noise operator processes as integrands against with the highly singular noise processes (n-th time-derivatives of white noise). In Section 6, we investigate the unique existence of a weak solution of Langevin (type) white noise evolution equation for white noise distribution whose explicit solution is represented by the adjoint of the generalized Mehler semigroup.

2. White Noise Functionals

Let H be a separable Hilbert space and let A be a positive, selfadjoint operator in H. Suppose that there exist a complete orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H and an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers with $\lambda_1 > 1$ such that

(A1)　for all $n \in \mathbb{N}$, $Ae_n = \lambda_n e_n$,

(A2)　A^{-1} is of Hilbert-Schmidt type, i.e.

$$\|A^{-1}\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

For each $p \in \mathbb{R}$, we define a norm $|\cdot|_p$ by

$$|\xi|_p := |A^p \xi|_0, \quad \xi \in H,$$

where $|\cdot|_0$ is the Hilbertian norm on H, and for each $p \geq 0$, put $E_p = \{\xi \in H : |\xi|_p < \infty\}$ and $E_{-p} = \overline{H}^{|\cdot|_{-p}}$ the completion of H with respect to the norm $|\cdot|_{-p}$. Then for each $p \in \mathbb{R}$, E_p becomes a Hilbert space with the Hilbertian norm $|\cdot|_p$, and by identifying H^* (strong dual space) with H, we have a chain of Hilbert spaces:

$$\cdots \subset E_q \subset E_p \subset H \cong H^* \subset E_{-p} \subset E_{-q} \subset \cdots$$

for any $0 \leq p \leq q$, where E_{-p} and E_p^* are topologically isomorphic. Then by taking the projective limit space of E_p and the inductive limit space of E_{-p}, we have a Gelfand triple:

$$\operatorname*{proj\,lim}_{p \to \infty} E_p =: E \subset H \subset E^* \cong \operatorname*{ind\,lim}_{p \to \infty} E_{-p}, \tag{1}$$

where from the condition **(A2)**, E becomes a countably Hilbert nuclear space.

Example 1. *As a prototype of the Gelfand triple given as in* (1), *we consider the Hilbert space $H = L^2(\mathbb{R}, dt)$ of square integrable complex-valued functions of $\mathbb{R}$ with respect to the Lebesgue measure and the harmonic oscillator $A = -\frac{d^2}{dt^2} + t^2 + 1$ (see [7,8]). Then the projective limit space E coincides with the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing C^∞-functions and E^* coincides with the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions and then we have*

$$\mathcal{S}(\mathbb{R}) = E \subset H = L^2(\mathbb{R}) \subset E^* = \mathcal{S}'(\mathbb{R}),$$

which will be used for the concrete construction of Brownian motion and its higher-order time derivatives.

By the Bochner–Minlos theorem, we see that there exists a probability measure μ, called the standard Gaussian measure, on $E_\mathbb{R}^*$ such that

$$\int_{E_\mathbb{R}^*} e^{i\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2}\langle \xi, \xi \rangle}, \quad \xi \in E,$$

where $E_\mathbb{R}^*$ is the real nuclear space such that $E = E_\mathbb{R}^* + iE_\mathbb{R}^*$ and $\langle \cdot, \cdot \rangle$ is the canonical complex-bilinear form on $E^* \times E$. Note that the inner product on the Hilbert space H is given by $\langle \bar{\cdot}, \cdot \rangle$. The probability space $(E_\mathbb{R}^*, \mu)$ is called a standard Gaussian space and we put $(L^2) := L^2(E_\mathbb{R}^*, \mu)$ the space of square integrable complex-valued (Gaussian) random variables.

By the celebrated Wiener–Itô–Segal isomorphism, (L^2) is unitarily isomorphic to the (Boson) Fock space $\Gamma(H)$ defined by

$$\Gamma(H) = \left\{ \phi = (f_n)_{n=0}^\infty : f_n \in H^{\widehat{\otimes} n}, \ \|\phi\|_0^2 := \sum_{n=0}^\infty n! |f_n|_0^2 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $|\cdot|_0$ is the Hilbertian norm on $H^{\widehat{\otimes} n}$ again. Note that the Wiener–Itô–Segal (unitary) isomorphism between (L^2) and $\Gamma(H)$ is determined by the following correspondence:

$$\Gamma(H) \ni \phi_\xi := \left(1, \xi, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots \right) \quad \leftrightarrow \quad \phi_\xi(x) := e^{\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle} \in (L^2), \quad \xi \in E, \tag{2}$$

where ϕ_ξ is called an exponential vector (or coherent vector) associated with $\xi \in E$. The second quantization $\Gamma(A)$ of A is defined in $\Gamma(H)$ by

$$\Gamma(A)\phi := \left(A^{\otimes n}f_n\right)_{n=0}^{\infty}, \quad \phi = (f_n)_{n=0}^{\infty} \in \Gamma(H), \tag{3}$$

see (5).

From the (Boson) Fock space $\Gamma(H)$ and the positive, selfadjoint operator $\Gamma(A)$ in $\Gamma(H)$, by using the arguments used to construct the Gelfand triple given as in (1), we construct a chain of (Boson) Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \Gamma(H) \cong \Gamma(H)^* \subset \Gamma(E_{-p}) \subset \cdots$$

for any $p \geq 0$, and by taking the projective and inductive limit spaces, we have the Gelfand triple:

$$\operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p) =: (E) \subset \Gamma(H) \subset (E)^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}), \tag{4}$$

and then (E) becomes again a countably Hilbert nuclear space (see [4,7,8]). Then from the Gelfand triple (4), by using the Wiener–Itô–Segal unitary isomorphism, we have the Gelfand triple of Gaussian white noise functionals:

$$(E) \subset (L^2) \subset (E)^*,$$

which is referred as to the Hida-Kubo-Takenaka space [6].

For each $p \in \mathbb{R}$, the Hilbertian norm on $\Gamma(E_p)$ is denoted by $\|\cdot\|_p$ and given by $\|\cdot\|_p = \|\Gamma(A^p)\cdot\|_0$, i.e., for each $\phi = (f_n)_{n=0}^{\infty} \in \Gamma(E_p)$,

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2.$$

The canonical complex-bilinear form on $(E)^* \times (E)$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and then for each $\Phi = (F_n) \in (E)^*$ and $\phi = (f_n) \in (E)$, we have

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle.$$

Note that $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E). Therefore, every $\Phi \in (E)^*$ is uniquely determined by the function $S\Phi : E \to \mathbb{C}$ defined as

$$S\Phi(\xi) = \langle\!\langle \Phi, \phi_\xi \rangle\!\rangle, \quad \xi \in E,$$

which is called the S-transform of Φ. In fact, for each $\Phi \in (E)^*$ and $F := S\Phi$, we can easily see that

(S1) for each $\xi, \eta \in E$, the map $\mathbb{C} \ni z \mapsto F(\xi + z\eta) \in \mathbb{C}$ is entire holomorphic,
(S2) there exist constants $K, c \geq 0$ and $p \geq 0$ such that

$$|F(\xi)| \leq K\exp\left(c\,|\xi|_p^2\right), \quad \xi \in E.$$

The converse is also true as given in the next theorem, which is called the analytic characterization theorem for S-transform.

Theorem 1 ([23])**.** *A complex-valued function F on E is the S-transform of an element in* $(E)^*$ *if and only if F satisfies the conditions* **(S1)** *and* **(S2)**.

Remark 1. *Theorem 1 is originally from [23] and the proof of Theorem 1 in [23] had an essential gap and then the gap has been corrected later (see [24]). For a corrected proof of Theorem 1, we refer to [7,8] (see also [24]).*

3. White Noise Operators

For locally convex spaces $\mathfrak{X}$ and $\mathfrak{Y}$, the space of all continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$ is denoted by $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$. A continuous linear operator $\Xi \in \mathcal{L}((E), (E)^*)$ is called a white noise operator (or a generalized operator).

As an operator version of the *S*-transform, the symbol $\widehat{\Xi} : E \times E \to \mathbb{C}$ of white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\!\langle \Xi \phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in E.$$

Then since the exponential vectors span a dense subspace of (E), every white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is uniquely determined by the operator symbol $\widehat{\Xi}$. In fact, for each $\Xi \in \mathcal{L}((E), (E)^*)$ and $\Theta = \widehat{\Xi}$, we can easily see that

(Θ1)　for each $\xi_i, \eta_i \in E$ $(i = 1, 2)$, the function

$$\mathbb{C} \times \mathbb{C} \ni (z, w) \mapsto \Theta(\xi_1 + z\xi_2, \eta_1 + w\eta_2) \in \mathbb{C}$$

is entire holomorphic,

(Θ2)　there exist $K, c \geq 0$ and $p \geq 0$ such that

$$|\Theta(\xi, \eta)| \leq K \exp c \left(|\xi|_p^2 + |\eta|_p^2 \right), \quad \xi, \eta \in E.$$

Furthermore, if $\Xi \in \mathcal{L}((E), (E))$, then $\Theta = \widehat{\Xi}$ satisfies the following condition:

(Θ2′)　for any $p \geq 0$ and $\epsilon > 0$, there exist constants $K \geq 0$ and $q \geq 0$ such that

$$|\Theta(\xi, \eta)| \leq K \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \quad \xi, \eta \in E.$$

The converse is also true as given in the next theorem, which is called the analytic characterization theorem for operator symbol.

Theorem 2 ([25,26])**.**　*A complex-valued function Θ on $E \times E$ is the symbol of an operator $\Xi \in \mathcal{L}((E), (E)^*)$ if and only if Θ satisfies the conditions **(Θ1)** and **(Θ2)**. Moreover, Θ is the symbol of an operator $\Xi \in \mathcal{L}((E), (E))$ if and only if Θ satisfies the conditions **(Θ1)** and **(Θ2′)**.*

Throughout this paper, for a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$, the adjoint operator of Ξ with respect to the canonical complex-bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is denoted by Ξ^*. Then for each $\Xi \in \mathcal{L}((E), (E)^*)$, we have $\Xi^* \in \mathcal{L}((E), (E)^*)$ and for any $\phi, \psi \in (E)$,

$$\langle\!\langle \Xi^* \phi, \psi \rangle\!\rangle = \langle\!\langle \Xi \psi, \phi \rangle\!\rangle.$$

Example 2.　*Let $S \in \mathcal{L}(E, E^*)$ be given.*
(1)　Consider a function $\Theta_1 : E \times E \to \mathbb{C}$ defined by

$$\Theta_1(\xi, \eta) = \langle S\xi, \xi \rangle \, e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

*Then we can easily check that Θ_1 satisfies the conditions **(Θ1)** and **(Θ2′)** and then by Theorem 2, there exists a unique white noise operator, denoted by $\Delta_{\mathrm{G}}(S)$ and called the generalized Gross Laplacian (see [27]), in $\mathcal{L}((E), (E))$ such that $\widehat{\Delta_{\mathrm{G}}(S)} = \Theta_1$. In fact, the generalized Gross Laplacian $\Delta_{\mathrm{G}}(S)$ is uniquely determined by the action on exponential vectors:*

$$\Delta_{\mathrm{G}}(S)\phi_{\xi} = \langle S\xi, \xi \rangle \, \phi_{\xi}, \quad \xi \in E.$$

In particular, for $S = I$ (the identity operator), $\Delta_{\mathrm{G}}(I)$ is called the Gross Laplacian and denoted by Δ_{G}. For the adjoint operator of $\Delta_{\mathrm{G}}(S)$, we write $\Delta_{\mathrm{G}}^{*}(S) := \Delta_{\mathrm{G}}(S)^{*}$.

(2) Consider a function $\Theta_2 : E \times E \to \mathbb{C}$ defined by

$$\Theta_2(\xi, \eta) = \langle S\xi, \eta \rangle \, e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

*Then we can easily check that Θ_2 satisfies the conditions **(Θ1)** and **(Θ2)** and then by Theorem 2, there exists a unique white noise operator, denoted by $\Lambda(S)$ and called the conservation operator (see [8,27]), in $\mathcal{L}((E), (E)^{*})$ such that $\widehat{\Lambda(S)} = \Theta_2$. Furthermore, if $S \in \mathcal{L}(E, E)$, then we can see that $\Lambda(S) \in \mathcal{L}((E), (E))$. In fact, if $S \in \mathcal{L}(E, E)$ is an equicontinuous generator (see Section 4 or [28]), then for the equicontinuous semigroup $\{e^{tS}\}_{t \in \mathbb{R}}$ generated by S, the conservation operator $\Lambda(S)$ is uniquely determined by the action on exponential vectors:*

$$\Lambda(S)\phi_{\xi} = \frac{d}{d\epsilon}\phi_{e^{\epsilon S}\xi}\bigg|_{\epsilon=0}, \quad \xi \in E,$$

see [29]. Then we have $\Lambda(S)^{*} = \Lambda(S^{*})$.

Example 3. *Let $S \in \mathcal{L}(E, E^{*})$ be given. Then the second quantization $\Gamma(S)$ of S (see (3)) is defined by*

$$\Gamma(S)\phi := \left(S^{\otimes n} f_n\right)_{n=0}^{\infty}, \quad \phi = (f_n)_{n=0}^{\infty} \in (E). \tag{5}$$

Then $\Gamma(S) \in \mathcal{L}((E), (E)^{*})$ and we have

$$\widehat{\Gamma(S)}(\xi, \eta) = e^{\langle S\xi, \eta \rangle}, \quad \xi, \eta \in E,$$

see [8,27]. Therefore, the second quantization $\Gamma(S)$ is uniquely determined by the action on exponential vectors:

$$\Gamma(S)\phi_{\xi} = \phi_{S\xi}, \quad \xi \in E.$$

From the definition, we see that $\Gamma(S)^{*} = \Gamma(S^{*})$. Furthermore, if $S \in \mathcal{L}(E, E)$, then we see that $\Gamma(S) \in \mathcal{L}((E), (E))$.

Example 4. *Let $K \in \mathcal{L}(E, E^{*})$ and $S \in \mathcal{L}(E, E)$ be given.*

(1) Consider a function $\Theta_3 : E \times E \to \mathbb{C}$ defined by

$$\Theta_3(\xi, \eta) = e^{\langle K\xi, \xi \rangle + \langle S\xi, \eta \rangle}, \quad \xi, \eta \in E.$$

*Then we can check that Θ_3 satisfies the conditions **(Θ1)** and **(Θ2')**. Therefore, by Theorem 2, there exists a unique white noise operator, denoted by $\mathcal{G}_{K,S}$ and called the generalized Fourier–Gauss transform (see [7,27]), in $\mathcal{L}((E), (E))$ such that $\widehat{\mathcal{G}_{K,S}} = \Theta_3$. In fact, the generalized Fourier–Gauss transform $\mathcal{G}_{K,S}$ is uniquely determined by the action on exponential vectors:*

$$\mathcal{G}_{K,S}\phi_{\xi} = e^{\langle K\xi, \xi \rangle}\phi_{S\xi}, \quad \xi \in E,$$

and so we have

$$\mathcal{G}_{K,S} = \Gamma(S)e^{\Delta_{\mathrm{G}}(K)} \in \mathcal{L}((E), (E)).$$

(2) The adjoint operator of $\mathcal{G}_{K,S}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, denoted by $\mathcal{F}_{K,S}$, i.e., $\mathcal{F}_{K,S} = \mathcal{G}_{K,S}^*$ and called the generalized Fourier–Mehler transform (see [7,27]), belongs to $\mathcal{L}((E)^*, (E)^*)$. Then we have

$$\widehat{\mathcal{F}_{K,S}}(\xi, \eta) = e^{\langle K\eta, \eta \rangle + \langle S^*\xi, \eta \rangle}, \quad \xi, \eta \in E,$$

and

$$\mathcal{F}_{K,S} = e^{\Delta_G^*(K)} \Gamma(S^*) \in \mathcal{L}((E)^*, (E)^*).$$

4. Generalized Mehler Semigroup

Let $\{S_t^*\}_{t \geq 0}$ and $\{T_t^*\}_{t \geq 0}$ be families of continuous linear operators on $E_{\mathbb{R}}^*$. For each $\xi \in E$, we define

$$P_t \phi_\xi(x) := \int_{E_{\mathbb{R}}^*} \phi_\xi \left(S_t^* x + T_t^* y \right) d\mu(y).$$

In fact, from (2), we obtain that

$$\phi_\xi \left(S_t^* x + T_t^* y \right) = e^{\langle S_t^* x + T_t^* y, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle} = e^{\langle T_t^* y, \xi \rangle + \frac{1}{2} \langle (S_t^* S_t - I)\xi, \xi \rangle} \phi_{S_t \xi}(x)$$

$$= e^{\langle y, T_t \xi \rangle} \Gamma(S_t) e^{\frac{1}{2}\Delta_G(S_t^* S_t - I)} \phi_\xi(x).$$

Therefore, since μ is a Gaussian measure, we obtain that

$$\begin{aligned} P_t \phi_\xi(x) &= \int_{E_{\mathbb{R}}^*} \phi_\xi \left(S_t^* x + T_t^* y \right) d\mu(y) \\ &= \left(\int_{E_{\mathbb{R}}^*} e^{\langle y, T_t \xi \rangle} d\mu(y) \right) \Gamma(S_t) e^{\frac{1}{2}\Delta_G(S_t^* S_t - I)} \phi_\xi(x) \\ &= \Gamma(S_t) e^{\frac{1}{2}\Delta_G(S_t^* S_t - I)} e^{\frac{1}{2} \langle T_t \xi, T_t \xi \rangle} \phi_\xi(x) \\ &= \Gamma(S_t) e^{\frac{1}{2}\Delta_G(T_t^* T_t + S_t^* S_t - I)} \phi_\xi(x), \end{aligned} \tag{6}$$

and so we have

$$P_t = \Gamma(S_t) e^{\frac{1}{2}\Delta_G(T_t^* T_t + S_t^* S_t - I)}, \tag{7}$$

which holds on the linear spans of exponential vectors. By applying the analytic characterization theorem (see Theorem 2) for operator symbol, we can easily see that the operator given in the right hand side of (7) is a continuous linear operator from (E) into itself, i.e., $\Gamma(S_t) e^{\frac{1}{2}\Delta_G(T_t^* T_t + S_t^* S_t - I)} \in \mathcal{L}((E), (E))$ (see Example 4). Hence motivated by the above discussion, we have the following definition.

Definition 1. *Let $\{S_t^*\}_{t \geq 0}$ and $\{T_t^*\}_{t \geq 0}$ be families of continuous linear operators on E^*. For each $t \geq 0$, put*

$$P_t = \Gamma(S_t) e^{\frac{1}{2}\Delta_G(T_t^* T_t + S_t^* S_t - I)} \tag{8}$$

as an element of $\mathcal{L}((E), (E))$.

On the other hand, from Examples 3 and 4, for any $\xi \in E$, we obtain that

$$e^{\Delta_G(T)} \Gamma(S) \phi_\xi = e^{\langle S^* T S \xi, \xi \rangle} \phi_{S\xi} = \Gamma(S) e^{\Delta_G(S^* T S)} \phi_\xi,$$

which implies that

$$e^{\Delta_G(T)} \Gamma(S) = \Gamma(S) e^{\Delta_G(S^* T S)}. \tag{9}$$

Therefore, from (9), we obtain that

$$
\begin{aligned}
P_t P_s &= \Gamma(S_t) e^{\frac{1}{2}\Delta_G(T_t^* T_t + S_t^* S_t - I)} \Gamma(S_s) e^{\frac{1}{2}\Delta_G(T_s^* T_s + S_s^* S_s - I)} \\
&= \Gamma(S_t)\Gamma(S_s) e^{\frac{1}{2}\Delta_G(S_s^*(T_t^* T_t + S_t^* S_t - I)S_s)} e^{\frac{1}{2}\Delta_G(T_s^* T_s + S_s^* S_s - I)} \\
&= \Gamma(S_t S_s) e^{\frac{1}{2}\Delta_G(S_s^*(T_t^* T_t + S_t^* S_t)S_s + T_s^* T_s - I)}.
\end{aligned}
$$

Hence we have the following characterization of semigroup property.

Proposition 1. *Let $\{S_t^*\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ be families of continuous linear operators on $E_{\mathbb{R}}^*$. Suppose that $\{S_t\}_{t\geq 0}$ is a one-parameter semigroup. Then $\{P_t\}_{t\geq 0}$ is a one-parameter semigroup if and only if the following property:*

$$
S_s^* T_t^* T_t S_s + T_s^* T_s = T_{t+s}^* T_{t+s} \tag{10}
$$

holds.

Remark 2. *A general result for a characterization of the semigroup property of $\{P_t\}_{t\geq 0}$ can be found in Proposition 2.2 of [12]. In particular, a characterization of the semigroup property of $\{P_t\}_{t\geq 0}$ for a general Gaussian case on Hilbert space can be found in Proposition 4.1 of [12]. Furthermore, a definition of a generalized Mehler semigroup can be found in Definition 2.4 of [12].*

In (6), we used the fact that for any $\xi \in E$,

$$
\int_{E_{\mathbb{R}}^*} e^{\langle y, T_t\xi \rangle} d\mu(y) = e^{\frac{1}{2}\langle T_t\xi, T_t\xi \rangle} = e^{\frac{1}{2}\langle T_t^* T_t\xi, \xi \rangle}, \qquad t \geq 0.
$$

On the other hand, by applying the Bochner–Minlos theorem, there exists a unique Gaussian measure μ_t with the covariance operator $T_t^* T_t$ such that

$$
\int_{E_{\mathbb{R}}^*} e^{i\langle y, \xi \rangle} d\mu_t(y) = e^{-\frac{1}{2}\langle T_t^* T_t\xi, \xi \rangle}, \quad t \geq 0.
$$

Since the Gaussian measure μ on $E_{\mathbb{R}}^*$ is symmetric, from (6) we have

$$
\begin{aligned}
P_t \phi_\xi(x) &= \int_{E_{\mathbb{R}}^*} \phi_\xi\left(S_t^* x + T_t^* y\right) d\mu(y) \\
&= \int_{E_{\mathbb{R}}^*} \phi_\xi\left(S_t^* x \pm y\right) d\mu_t(y), \tag{11}
\end{aligned}
$$

and so from the continuities of P_t and the integral transform given as in the right hand side of (11), we see that

$$
P_t \phi(x) = \int_{E_{\mathbb{R}}^*} \phi\left(S_t^* x - y\right) d\mu_t(y), \quad \phi \in (E),
$$

see §2 of [12].

Definition 2. *Let $\{S_t^*\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ be families of continuous linear operators on $E_{\mathbb{R}}^*$ such that $\{S_t\}_{t\geq 0}$ is a one-parameter semigroup. Then the one-parameter family $\{P_t\}_{t\geq 0}$ defined as in (8) is called a generalized Mehler semigroup (associated with $\{S_t^*\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$) if $\{S_t^*\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ satisfy the equality given as in (10).*

Remark 3. *Let $S^* \in \mathcal{L}(E^*, E^*)$ be given. Then S^* is said to be real if $S^* \left(E_{\mathbb{R}}^* \right) \subset E_{\mathbb{R}}^*$. Consider the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ given by*

$$\int_{E_{\mathbb{R}}^*} \phi_\xi \left(S_t^* x + T_t^* y \right) \, d\mu(y) = P_t \phi_\xi(x) = \left(\Gamma(S_t) e^{\frac{1}{2}\Delta_{\mathrm{G}}(T_t^* T_t + S_t^* S_t - I)} \phi_\xi \right) (x) \tag{12}$$

for $\xi \in E$. If we consider P_t as the left hand side of (12), then the exponential vector ϕ_ξ is defined on $E_{\mathbb{R}}^*$ and so we have to consider the families $\{S_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ and $\{T_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ which are real. However, if we consider P_t as the right hand side of (12), then we do not need such restriction. Throughout this paper, we consider the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ defined as the right hand side of (12).

Remark 4. *In (8), if S_t and T_t satisfy that $T_t^* T_t + S_t^* S_t - I = 0$, i.e., $T_t = (I - S_t^* S_t)^{1/2}$, then we have $P_t = \Gamma(S_t)$ and so in this case, $\{P_t\}_{t \geq 0}$ becomes the Ornstein–Uhlenbeck semigroup and then we have the Mehler's formula of the Ornstein–Uhlenbeck semigroup (or second quantization) in the infinite dimensional case:*

$$\Gamma(S_t)\phi_\xi(x) = P_t \phi_\xi(x) = \int_{E_{\mathbb{R}}^*} \phi_\xi \left(S_t^* x + \sqrt{I - S_t^* S_t}\, y \right) \, d\mu(y), \tag{13}$$

which can be found in Theorem 6.1.1 of [2] (see also Theorem 4.5 of [16]).

Lemma 1. *Let $\{S_t^*\}_{t \geq 0}$ and $\{T_t^*\}_{t \geq 0}$ be families of continuous linear operators on $E_{\mathbb{R}}^*$. Suppose that $\{S_t\}_{t \geq 0}$ is a strongly continuous semigroup and the map $t \mapsto T_t^* T_t \in \mathcal{L}(E, E^*)$ is differentiable at 0. Then $\{P_t\}_{t \geq 0}$ given as in (8) is a generalized Mehler semigroup if and only if*

$$T_t^* T_t = \int_0^t S_s^* V S_s \, ds, \tag{14}$$

where $V = \frac{d}{dt} T_t^* T_t \Big|_{t=0}$.

Proof. Suppose that $\{P_t\}_{t \geq 0}$ is a generalized Mehler semigroup. For notational convenience, put

$$V(t) = T_t^* T_t, \quad t \geq 0.$$

Then from (10), by taking $s = t = 0$, we have $2V(0) = V(0)$, and so $V(0) = 0$. Furthermore, we obtain that

$$\frac{dV(t)}{dt} = \lim_{h \to 0} \frac{S_t^* V_h S_t}{h} = S_t^* \left(V'(0) \right) S_t,$$

from which we have (14). The proof of the converse is straightforward. □

Remark 5. *A similar explicit form of $T_t^* T_t$ for the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ for a general Gaussian case on a Hilbert space can be found in Proposition 4.3 of [12].*

Example 5. *Let $b \in \mathbb{C}$ and $S_t = e^{bt}$ for $t \geq 0$. Then from (14), we have*

$$T_t^* T_t = \left(\int_0^t e^{2bs} \, ds \right) V = \begin{cases} \frac{1}{2b} \left(e^{2bt} - 1 \right) V, & b \neq 0, \\ V t, & b = 0, \end{cases}$$

where $V \in \mathcal{L}(E, E^*)$ is a given operator, and so we have

$$K_t := T_t^* T_t + S_t^* S_t - I = \left(\frac{V}{2b} + 1 \right) \left(e^{2bt} - 1 \right).$$

Therefore, we have

$$P_t = \Gamma(S_t)e^{\frac{1}{2}\Delta_{\mathrm{G}}(K_t)} =: \mathcal{G}_{K_t, S_t}, \quad t \geq 0,$$

where $\mathcal{G}_{C,D}$ be a generalized Fourier–Gauss transform (see [27,30], and also Example 4).

Let $\mathfrak{X}$ be a barrelled locally convex Hausdorff space whose topology is generated by a family of seminorms $\{|\cdot|_p\}_{p \in \mathfrak{N}_{\mathfrak{X}}}$. An operator $S \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ is called an equicontinuous generator if for any $r > 0$, the family $\{(rS)^n/n!\}_{n=0}^{\infty}$ is equicontinuous, i.e., for any $p \in \mathfrak{N}_{\mathfrak{X}}$, there exist $C \geq 0$ and $q \in \mathfrak{N}_{\mathfrak{X}}$ such that $|((rS)^n/n!)x|_p \leq C|x|_q$ for all $x \in \mathfrak{X}$ (see [8,28]). For such equicontinuous generator S, we can prove that the series

$$e^{rS} := \sum_{n=0}^{\infty} \frac{1}{n!}(rS)^n$$

converges strongly on $\mathfrak{X}$, and then $\{e^{zS}\}_{z \in \mathbb{C}}$ becomes a holomorphic one-parameter subgroup of $\mathrm{GL}(\mathfrak{X})$ (the general linear group on $\mathfrak{X}$).

Let $\{S_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ be a family of continuous linear operators such that $\{S_t\}_{t \geq 0} \subset \mathcal{L}(E, E)$ is an equicontinuous semigroup with the infinitesimal generator $S \in \mathcal{L}(E, E)$. Then we obtain that

$$S_t^* S_t - I = \int_0^t \frac{d}{ds}(S_s^* S_s)\,ds = \int_0^t S_s^*(S^* + S)S_s\,ds,$$

and furthermore, if the map $t \mapsto T_t^* T_t \in \mathcal{L}(E, E^*)$ is differentiable at 0, then by Lemma 1, we have

$$T_t^* T_t + S_t^* S_t - I = \int_0^t S_s^*(V + S^* + S)S_s\,ds,$$

and hence we have the following theorem for the explicit representation of the generalized Mehler semigroup.

Theorem 3. *Let $\{S_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ and $\{T_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ be families of continuous linear operators. Suppose that $\{S_t\}_{t \geq 0} \subset \mathcal{L}(E, E)$ is an equicontinuous semigroup with the infinitesimal generator $S \in \mathcal{L}(E, E)$ and the map $t \mapsto T_t^* T_t \in \mathcal{L}(E, E^*)$ is differentiable at 0. Then the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ given as in (8) has the following representation in terms of the generalized Fourier–Gauss transform:*

$$P_t = \Gamma(S_t)e^{\frac{1}{2}\Delta_{\mathrm{G}}\left(\int_0^t S_s^*(V+S^*+S)S_s\,ds\right)}, \quad t \geq 0,$$

where $V = \frac{d}{dt}T_t^* T_t\Big|_{t=0}$.

Theorem 4. *Under the assumptions given as in Theorem 3, the infinitesimal generator of the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ is given by $\Lambda(S) + \frac{1}{2}\Delta_{\mathrm{G}}(V + S^* + S)$.*

Proof. We now give a sketch of the proof. A detailed proof of this theorem is a simple modification of the proof of Theorem 5.3.11 of [8] (also, see the proof of Theorem 4.3 of [30]). Consider the symbol of P_t and then for any $\xi, \eta \in E$, we obtain that

$$\widehat{P_t}(\xi, \eta) = e^{\frac{1}{2}\left\langle\left(\int_0^t S_s^*(V+S^*+S)S_s\,ds\right)\xi, \xi\right\rangle + \langle S_t\xi, \eta\rangle},$$

which implies that

$$\frac{d}{dt}\widehat{P}_t(\xi,\eta)\bigg|_{t=0} = \left[\frac{1}{2}\left\langle\left(S_t^*\left(V+S^*+S\right)S_t\right)\xi,\xi\right\rangle + \left\langle SS_t\xi,\eta\right\rangle\right]$$

$$\times e^{\frac{1}{2}\left\langle\left(\int_0^t S_s^*(V+S^*+S)S_sds\right)\xi,\xi\right\rangle + \left\langle S_t\xi,\eta\right\rangle}\bigg|_{t=0}$$

$$= \left[\frac{1}{2}\left\langle\left(V+S^*+S\right)\xi,\xi\right\rangle + \left\langle S\xi,\eta\right\rangle\right]e^{\langle\xi,\eta\rangle}$$

$$= \left\langle\!\!\left\langle\left(\Lambda(S)+\frac{1}{2}\Delta_G(V+S^*+S)\right)\phi_\xi,\phi_\eta\right\rangle\!\!\right\rangle,$$

from which we have the desired assertion. $\square$

For a Gelfand triple $\mathcal{X}\subset\mathcal{K}\subset\mathcal{X}^*$, where $\mathcal{X}=\mathcal{X}_\mathbb{R}+i\mathcal{X}_\mathbb{R}$ with the real vector space $\mathcal{X}_\mathbb{R}$, and for $L\in\mathcal{L}(\mathcal{X},\mathcal{X}^*)$, the complex conjugation $\overline{L}$ of L is defined by

$$\overline{L}x = \overline{L\overline{x}},\quad x\in\mathcal{X},$$

and then for the Hermitian adjoint $L^\dagger$ of L, we have $L^\dagger = \overline{(L^*)} = \left(\overline{L}\right)^*$.

Theorem 5. *Let $\{S_t\}_{t\geq 0}\subset\mathcal{L}(E,E)$ be an equicontinuous semigroup with the infinitesimal generator $S\in\mathcal{L}(E,E)$ and $\{T_t^*\}_{t\geq 0}\subset\mathcal{L}(E^*,E^*)$ be a family of continuous linear operators such that the map $t\mapsto T_t^*T_t\in\mathcal{L}(E,E^*)$ is differentiable at 0. Suppose that for each $t\geq 0$, S_t can be extended to H such that $\{S_t\}_{t\geq 0}\subset\mathcal{L}(H,H)$ is a strongly continuous semigroup. Let $\{P_t\}_{t\geq 0}$ be the generalized Mehler semigroup defined as in (7). Then the followings are equivalent:*

   (i)   *For each $t\geq 0$, P_t can be extended to $\Gamma(H)$ as a unitary operator,*

   (ii) *S_t is unitary and $K_t := T_t^*T_t + S_t^*S_t - I = 0$ for all $t\geq 0$,*

  (iii) *$S^\dagger = -S$ and $V+S+S^* = 0$, where $V = \frac{d}{dt}T_t^*T_t\big|_{t=0}$.*

Proof. (i) $\Leftrightarrow$ (ii) We first observe that for any $\xi,\eta\in E$,

$$\left\langle\!\!\left\langle P_t\phi_\xi,\overline{P_t\phi_\eta}\right\rangle\!\!\right\rangle = e^{\frac{1}{2}\left(\langle K_t\xi,\xi\rangle + \langle\overline{K_t}\overline{\eta},\overline{\eta}\rangle\right)}\left\langle\!\!\left\langle\Gamma(S_t^\dagger S_t)\phi_\xi,\overline{\phi_\eta}\right\rangle\!\!\right\rangle.$$

Since the exponential vectors span a dense subspace of $\Gamma(H)$ and ξ, η are chosen arbitrarily, $P_t^\dagger P_t = I$ holds if and only if $K_t = 0$ and $S_t^\dagger S_t = I$. Therefore, we have $P_t = \Gamma(S_t)$ and

$$\Gamma(S_t S_t^\dagger) = \Gamma(S_t)\Gamma(S_t^\dagger) = P_t P_t^\dagger = I$$

implies $S_t S_t^\dagger = I$. Hence S_t is unitary and $K_t = 0$. Conversely, $K_t = 0$ implies $P_t = \Gamma(S_t)$ and so S_t is unitary implies that P_t is unitary.

(ii) $\Leftrightarrow$ (iii) Note that, by Stone's theorem (Theorem 10.8 in [31]), S_t is unitary if and only if iS is selfadjoint if and only if $S^\dagger = -S$. Moreover, we observe that

$$\frac{d}{dt}\left(T_t^*T_t + S_t^*S_t - I\right)\bigg|_{t=0} = V+S^*+S$$

and

$$T_t^*T_t + S_t^*S_t - I = \int_0^t S_s^*(V+S^*+S)S_s ds.$$

Hence (ii) and (iii) are equivalent. $\square$

Remark 6. *The unitarity of the generalized Fourier–Gauss transform has been studied in [32,33] as transform from the space of Gaussian functionals onto another space of Gaussian functionals with different covariance operator.*

5. White Noise Integrals

Let $T > 0$ be fixed and let $\Phi : [0, T] \to (E)^*$ be a function. If the map $[0, T] \ni t \to S(\Phi(t))(\xi)$ is measurable for all $\xi \in E$ and there exist nonnegative constants $K, c \geq 0$ and $p \geq 0$ such that

$$\int_0^T |S(\Phi(t))(\xi)|dt \leq Ke^{c|\xi|_p^2}$$

for all $\xi \in E$, then by applying Theorem 13.4 of [7], we see that Φ is Pettis integrable and for any $\xi \in E$, we have

$$S\left(\int_0^T \Phi(t)dt\right)(\xi) = \int_0^T S(\Phi(t))(\xi)dt$$

for all $\xi \in E$.

Let $U^* : [0, T] \to \mathcal{L}((E)^*, (E)^*)$ and $\Phi : [0, T] \to (E)^*$ be functions. Suppose that $\Phi(t)$ is differentiable, i.e., $\Phi'(t) := \frac{d\Phi(t)}{dt}$ exists and belongs to $(E)^*$. If there exist nonnegative constants $K, c \geq 0$ and $p \geq 0$ such that

$$\int_0^T |S(U(t)(\Phi'(t)))(\xi)|dt \leq Ke^{c|\xi|_p^2}$$

for all $\xi \in E$, then by the discussion above we see that $U(t)(\Phi'(t))$ is Pettis integrable and

$$S\left(\int_0^T U(t)(\Phi'(t))\,dt\right)(\xi) = \int_0^T S(U(t)(\Phi'(t)))(\xi)dt$$

for all $\xi \in E$. In such a case, we write

$$\int_0^T U(t)d\Phi(t) := \int_0^T U(t)(\Phi'(t))\,dt.$$

From now on, we consider the case $H := L^2(\mathbb{R}, dt)$ and $E := \mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$. Then we have $E^* = \mathcal{S}'(\mathbb{R})$ the space of tempered distributions (see Example 1).

For each $t \geq 0$, we define $B_t \in (L^2)$ by

$$B_t = \left\langle \cdot, \mathbf{1}_{[0,t]} \right\rangle,$$

where we used the approximation procedure to define B_t, i.e, since E is dense in H and $\mathbf{1}_{[0,t]} \in H$, there exists a sequence $\{\xi_n\}_{n=1}^\infty \subset E$ such that $\lim_{n\to\infty} \xi_n = \mathbf{1}_{[0,t]}$ in H and

$$\mathbf{E}\left[|\langle \cdot, \xi_m \rangle - \langle \cdot, \xi_n \rangle|^2\right] = |\xi_m - \xi_n|_0^2,$$

which implies that

$$\left\langle \cdot, \mathbf{1}_{[0,t]} \right\rangle := \lim_{n\to\infty} \langle \cdot, \xi_n \rangle$$

exists in $L^2(E_{\mathbb{R}}^*, \mu)$. Then we can easily see that

$$B_0 = 0, \quad \mathbf{E}[B_t] = 0, \quad \mathbf{E}[B_s B_t] = \min\{s, t\},$$

from which we see that $\{B_t\}$ is a Brownian motion and it is called a realization of a Brownian motion.

Theorem 6 ([34]). *For each $n = 0, 1, 2, \cdots$, the map $t \mapsto \delta_t^{(n)} \in E_{-p}$ is continuous for*

$$p > \frac{5}{12} + \frac{n}{2}.$$

By applying Theorem 6, we see that the map $t \mapsto B_t \in (E)^*$ is a C^∞-map and we have

$$\frac{d}{dt} B_t = \langle \cdot, \delta_t \rangle =: W_t, \quad W_t^{(n)} := \frac{d^n}{dt^n} W_t = \left\langle \cdot, \delta_t^{(n)} \right\rangle, \quad t \in \mathbb{R}. \tag{15}$$

Proposition 2. *Let $\{S_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ and $\{T_t^*\}_{t \geq 0} \subset \mathcal{L}(E^*, E^*)$ be families of continuous linear operators. Suppose that $\{S_t\}_{t \geq 0} \subset \mathcal{L}(E, E)$ is an equicontinuous semigroup with the infinitesimal generator $S \in \mathcal{L}(E, E)$ and the map $t \mapsto T_t^* T_t \in \mathcal{L}(E, E^*)$ is differentiable at 0. Then for the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ given as in (8) and any $n \in \mathbb{N}$, $P_{t-s}^* \left(W_s^{(n)} \right)$ is Pettis integrable over $[0, t]$ for all $t > 0$.*

Proof. From Theorem 3, the explicit form of the generalized Mehler semigroup $\{P_t\}_{t \geq 0}$ is given by

$$P_t = \Gamma(S_t) e^{\frac{1}{2}\Delta_G \left(\int_0^t S_s^* (V + S^* + S) S_s ds \right)}, \quad t \geq 0,$$

where $V = \frac{d}{dt} T_t^* T_t \big|_{t=0} \in \mathcal{L}(E, E^*)$. For notational convenience, we put $Q := V + S^* + S \in \mathcal{L}(E, E^*)$. Since $\{S_t\}_{t \geq 0}$ is an equicontinuous semigroup with the infinitesimal generator $S \in \mathcal{L}(E, E)$, we can write $S_t = e^{tS}$ and then we have

$$P_t = \Gamma\left(e^{tS}\right) e^{\frac{1}{2}\Delta_G \left(\int_0^t e^{sS^*} Q e^{sS} ds \right)}, \quad t \geq 0.$$

Therefore, for any $\xi \in E$, we obtain that

$$\begin{aligned}
S\left(P_{t-s}^*\left(W_s^{(n)}\right)\right)(\xi) &= \left\langle\!\left\langle W_s^{(n)}, P_{t-s}\phi_\xi \right\rangle\!\right\rangle \\
&= e^{\frac{1}{2}\left\langle \left(\int_0^{t-s} e^{vS^*} Q e^{vS} dv \right)\xi, \xi \right\rangle} \left\langle\!\left\langle W_s^{(n)}, \phi_{e^{(t-s)S}\xi} \right\rangle\!\right\rangle \\
&= e^{\frac{1}{2}\left\langle \left(\int_0^{t-s} e^{vS^*} Q e^{vS} dv \right)\xi, \xi \right\rangle} \left\langle \delta_s^{(n)}, e^{(t-s)S}\xi \right\rangle \\
&= e^{\frac{1}{2}\int_0^{t-s} \langle Q e^{vS}\xi, e^{vS}\xi \rangle dv} (-1)^n \left(e^{(t-s)S}\xi \right)^{(n)}(s).
\end{aligned}$$

On the other hand, since $\{S_t\}_{t \geq 0}$ is an equicontinuous semigroup, by applying Lemma 2.1 of [28], we see that for any $p \geq 0$, there exist a constant $C \geq 0$ and $q \geq p$ such that

$$\left| e^{vS}\xi \right|_p \leq C|\xi|_q, \quad 0 \leq v \leq t,$$

where the constants C and p are depending on $t > 0$. Hence by applying Theorem 6 and the continuity of the differential operator on $E = \mathcal{S}(\mathbb{R})$, we see that there exist nonnegative constants $K, c \geq 0$ and $p \geq 0$ such that

$$\left| S\left(P_{t-s}^*\left(W_s^{(n)}\right)\right)(\xi) \right| \leq K e^{c|\xi|_p^2}$$

for all $\xi \in E$, then by the discussion above we see that $P_{t-s}^*\left(W_s^{(n)}\right)$ is Pettis integrable. $\quad\square$

6. Stochastic Evolution Equations

In this section, motivated by the results obtain in Section 4 (see Theorem 4), we study the following stochastic evolution equation (for white noise distribution):

$$d\Phi_t = \left(\Lambda(S^*) + \frac{1}{2}\Delta_G^*(V + S^* + S) \right) \Phi_t dt + dW_t^{(n)}, \quad 0 \le t \le T, \tag{16}$$
$$\Phi_0 = \Psi,$$

where $\Psi \in (E)^*$ is given and $W_t^{(n)}$ is the n-th distributional derivative of the white noise process $\{W_t\}_{t \ge 0}$ (see (15)). As an abstract extension of (16), we study the stochastic evolution equation:

$$d\Phi_t = \Xi^* \Phi_t dt + dW_t^{(n)}, \quad 0 \le t \le T, \tag{17}$$
$$\Phi_0 = \Psi,$$

where $\Xi^* \in \mathcal{L}((E)^*, (E)^*)$ is a given operator.

Definition 3. *A (generalized) stochastic process $\{\Phi_t\}_{t \ge 0} \subset (E)^*$ is called a **weak solution** of the stochastic evolution Equation (17) if Φ_t ($0 \le t \le T$) satisfies*

$$\Phi_t = \Psi + \int_0^t \Xi^* \Phi_s ds + W_t^{(n)} \tag{18}$$

in the weak sense in $(E)^*$ (see Definition 13.42 of [7]).

In the white noise theory, the integral equation given as in (18) can be represented by a white noise integral equation as following:

$$\Phi_t = \Psi + \int_0^t f(s, \Phi_s) ds,$$

where $f : [0, T] \times (E)^* \to (E)^*$ is a function defined by

$$f(s, \Psi) = \Xi^* \Psi + W_s^{(n+1)}, \quad \Psi \in (E)^*.$$

In fact, we have

$$\int_0^t \Xi^* \Phi_s ds + W_t^{(n)} = \int_0^t \Xi^* \Phi_s ds + \int_0^t dW_s^{(n)}$$
$$= \int_0^t \left(\Xi^* \Phi_s ds + \frac{dW_s^{(n)}}{ds} \right) ds.$$

Hence as a general case, we consider the following white noise integral equation:

$$X(t) = X(0) + \int_0^T f(s, X(s)) ds, \quad 0 \le t \le T, \tag{19}$$

where f is a function from $[0, T] \times (E)^*$ into $(E)^*$ (see (13.74) in [7]). Then as a special case of Theorem 13.43 of [7], i.e., by taking $\beta = 0$ in Theorem 13.43 of [7], we have the following theorem.

Theorem 7. *Suppose that the function $f : [0, T] \times (E)^* \to (E)^*$ satisfies the following conditions:*

(i)   *for any weak measurable function $X : [0, T] \to (E)^*$, the function $[0, T] \ni t \mapsto f(t, X(t)) \in (E)^*$ is weak measurable,*

(ii)　*there exist two nonnegative functions $L, \rho : [0, T] \times E \to [0, \infty)$ satisfying that*

$$\int_0^T L(t, \xi)dt \leq Ke^{c|\xi|_p^2}, \quad \int_0^T \rho(t, \xi)dt \leq Ke^{c|\xi|_p^2}, \quad \xi \in E$$

for some constants $K, c \geq 0$ and $p \geq 0$, such that for almost all $t \in [0, T]$,

$$|Sf(t, \Phi)(\xi) - Sf(t, \Psi)(\xi)| \leq L(t, \xi)|S\Phi(\xi) - S\Psi(\xi)|,$$
$$|Sf(t, \Phi)(\xi)| \leq \rho(t, \xi) \left(1 + |S\Phi(\xi)|\right) \tag{20}$$

for any $\Phi, \Psi \in (E)^*$ and $\xi \in E$.

Then for each $X(0) \in (E)^*$, the Equation (19) has a unique weak solution X.

Proposition 3. *Let $V \in \mathcal{L}(E, E^*)$. Suppose that there exists a family $\{T_t\}_{t \geq 0} \subset \mathcal{L}(E, E)$ such that the map $t \mapsto T_t^* T_t$ is differentiable and $V = \frac{d}{dt} T_t^* T_t \big|_{t=0}$. Then the stochastic evolution equation:*

$$d\Phi_t = \frac{1}{2}\Delta_G^*(V)\Phi_t dt + dW_t^{(n)}, \quad 0 \leq t \leq T, \tag{21}$$
$$\Phi_0 = \Psi,$$

has a unique weak solution $\Phi_t \in (E)^*$ given by

$$\Phi_t = P_t^*\Psi + \int_0^t P_{t-s}^* dW_s^{(n)}, \quad 0 \leq t \leq T, \tag{22}$$

where $\{P_t\}_{t \geq 0}$ is the generalized Mehler semigroup given by

$$P_t = e^{\frac{t}{2}\Delta_G(V)}, \quad 0 \leq t \leq T.$$

Moreover, Φ_t is given by

$$\Phi_t = W_t^{(n)} + P_t^*\Psi + \frac{1}{2}\int_0^t P_{t-s}^*\Delta_G^*(V)W_s^{(n)} ds. \tag{23}$$

Proof. Consider the function $f : [0, T] \times (E)^* \to (E)^*$ defined by

$$f(s, \Psi) = \frac{1}{2}\Delta_G^*(V)\Psi + W_s^{(n+1)}, \quad \Psi \in (E)^*.$$

　　Then we can check that the function f satisfies the conditions (i) and (ii) given as in Theorem 7. Then by applying Theorem 7, we see the unique existence of a weak solution of (21). The explicit form of Φ_t given as in (22) can be proved by direct computation (see Theorem 8). The representation of Φ_t given as in (23) is from (22) by applying integration by parts formula. □

Remark 7. *For the unique existence of weak solution of stochastic evolution equation given as in (16), it may be not easy to apply Theorem 7. In fact, it may be not easy to check the Lipschitz and growth conditions given as in (20) with the function $f : [0, T] \times (E)^* \to (E)^*$ defined by*

$$f(s, \Psi) = \left(\Lambda(S^*) + \frac{1}{2}\Delta_G^*(V + S^* + S)\right)\Psi + W_s^{(n+1)}, \quad \Psi \in (E)^*.$$

Then to overcome this difficulty, we have the following general theorem for stochastic evolution equation.

Theorem 8. *Let $\Xi^* \in \mathcal{L}((E)^*, (E)^*)$ be a given such that $\Xi := (\Xi^*)^* \in \mathcal{L}((E), (E))$ is an equicontinuous generator with the corresponding equicontinuous, holomorphic group $\{e^{z\Xi^*}\}_{z \in \mathbb{C}}$. Suppose that for given $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, there exist nonnegative constants $K, c \geq 0$ and $p \geq 0$ such that*

$$\left| S\left(e^{(t-s)\Xi^*} \left(W_s^{(n)} \right) \right)(\xi) \right| \leq K e^{c|\xi|_p^2} \tag{24}$$

for all $\xi \in E$ and $0 \leq s \leq t$ with $0 \leq t \leq T$. Then the stochastic evolution equation of (17) has a unique weak solution $\Phi_t \in (E)^*$ given by

$$\Phi_t = e^{t\Xi^*}\Psi + \int_0^t e^{(t-s)\Xi^*} dW_s^{(n)}, \quad 0 \leq t \leq T. \tag{25}$$

Moreover, Φ_t is given by

$$\Phi_t = W_t^{(n)} + e^{t\Xi^*}\Psi + \int_0^t e^{(t-s)\Xi^*}\Xi^* W_s^{(n)} ds. \tag{26}$$

Proof. Uniqueness. If $U(t)$ and $V(t)$ are weak solutions of (18), then $Y(t) = U(t) - V(t)$ is a solution of the equation:

$$Y(t) = \int_0^t \Xi^* Y(s) ds, \quad 0 \leq t \leq T,$$

which is equivalent to

$$\frac{d}{dt} Y(t) = \Xi^* Y(t), \quad Y(0) = 0, \ 0 \leq t \leq T,$$

from which, since Ξ is an equicontinuous generator, we have $Y(t) = 0$ and so $U(t) = V(t)$ for all $0 \leq t \leq T$.

Existence. From the condition given as in (24), for each $t > 0$, we can see that the white noise integral:

$$\int_0^t e^{(t-s)\Xi^*} dW_s^{(n)} = \int_0^t e^{(t-s)\Xi^*} \left(W_s^{(n+1)} \right) ds$$

is well-defined (see the proof of Proposition 2). We now prove that Φ_t given as in (25) is a weak solution of the stochastic evolution equation of (17). For any $\varphi \in (E)$, we have

$$\langle\!\langle \Phi_t, \varphi \rangle\!\rangle = \left\langle\!\left\langle \Psi, e^{t\Xi}\varphi \right\rangle\!\right\rangle + \int_0^t \left\langle\!\left\langle W_s^{(n+1)}, e^{(t-s)\Xi}\varphi \right\rangle\!\right\rangle ds$$

for all $t \geq 0$, and so we obtain that

$$\frac{d}{dt} \langle\!\langle \Phi_t, \varphi \rangle\!\rangle = \left\langle\!\left\langle \Psi, e^{t\Xi}\Xi\varphi \right\rangle\!\right\rangle + \int_0^t \left\langle\!\left\langle W_s^{(n+1)}, e^{(t-s)\Xi}\Xi\varphi \right\rangle\!\right\rangle ds + \int_0^t \left\langle\!\left\langle W_s^{(n+1)}, \varphi \right\rangle\!\right\rangle ds$$

$$= \langle\!\langle \Phi_t, \Xi\varphi \rangle\!\rangle + \int_0^t \left\langle\!\left\langle W_s^{(n+1)}, \varphi \right\rangle\!\right\rangle ds$$

$$= \left\langle\!\left\langle \Xi^*\Phi_t + W_t^{(n)}, \varphi \right\rangle\!\right\rangle,$$

which proves that Φ_t is a weak solution of (25). The representation of Φ_t given as in (26) can be obtained by applying the integration by parts formula from (25). □

Theorem 9. *Let $S \in \mathcal{L}(E, E)$ be an equicontinuous generator and $V \in \mathcal{L}(E, E^*)$. Suppose that there exists a family $\{T_t\}_{t \geq 0} \subset \mathcal{L}(E, E)$ such that the map $t \mapsto T_t^* T_t$ is differentiable and $V = \frac{d}{dt} T_t^* T_t \big|_{t=0}$. Then the stochastic evolution equation of (16) has a unique weak solution $\Phi_t \in (E)^*$ given by*

$$\Phi_t = P_t^* \Psi + \int_0^t P_{t-s}^* dW_s^{(n)}, \quad 0 \leq t \leq T,$$

where $\{P_t\}_{t \geq 0}$ is the generalized Mehler semigroup associated with $\{S_t^* = e^{tS^*}\}_{t \geq 0}$ and $\{T_t^*\}_{t \geq 0}$. Moreover, Φ_t is given by

$$\Phi_t = W_t^{(n)} + P_t^* \Psi + \int_0^t P_{t-s}^* \left[\Lambda(S^*) + \frac{1}{2} \Delta_G^*(V + S^* + S) \right] W_s^{(n)} ds.$$

Proof. The proof is immediate by applying Theorem 8, Proposition 2 and Theorem 4. □

Example 6. *As given in Remark 4, if S_t and T_t satisfy that $T_t^* T_t + S_t^* S_t - I = 0$, i.e., $T_t = (I - S_t^* S_t)^{1/2}$, then $\{P_t\}_{t \geq 0}$ becomes the Ornstein–Uhlenbeck semigroup, i.e., $P_t = \Gamma(S_t)$, and then from Theorem 9, the unique weak solution of the following Langevin type white noise evolution equation:*

$$d\Phi_t = \Lambda(S^*)\Phi_t dt + dW_t^{(n)}, \quad 0 \leq t \leq T, \tag{27}$$
$$\Phi_0 = \Psi \in (E)^*,$$

where $W_t^{(n)}$ is the n-th distributional derivative of the white noise process $\{W_t\}_{t \geq 0}$, is given by

$$\Phi_t = \Gamma(S_t^*)\Psi + \int_0^t \Gamma(S_{t-s}^*) dW_s^{(n)}$$
$$= W_t^{(n)} + \Gamma(S_t^*)\Psi + \int_0^t \Gamma(S_{t-s}^*)\Lambda(S^*) W_s^{(n)} ds, \quad 0 \leq t \leq T,$$

where $S_t^* = e^{tS^*}$ for $t \geq 0$.

7. Conclusions

The Mehler's formula given as in (13) provides the integral representation of the Ornstein–Uhlenbeck semigroup (or second quantization, i.e., roughly speaking, an exponential of a conservation operator), see Remark 4 (see also Theorem 6.1.1 of [2] and Theorem 4.5 of [16]). Then the Mehler semigroup has been generalized within an integral form in [12] (see Definition 2.4) which is called a generalized Mehler semigroup. As one of main results of this paper, in the converse direction of the Mehler's formula, we have considered the generalized Fourier–Gauss transform as an exponential form of the generalized Mehler semigroup. From the representation of the generalized Mehler semigroup, we investigated a characterization of the unitarity of the generalized Mehler semigroup.

The generalized Fourier–Gauss transform induces a one-parameter semigroup (group) with the infinitesimal generator which is a perturbation of the generator of the Ornstein–Uhlenbeck semigroup by the generalized Gross Laplacian from which we have obtained the following result that the infinitesimal generator of the generalized Mehler semigroup is given explicitly by the perturbation of the conservation operator by the generalized Gross Laplacian. The generalized Fourier–Mehler transform is defined as the adjoint operator of the generalized Fourier–Gauss transform and then, by the duality, which is acting on the space of the generalized white noise functionals, and hence the generalized Fourier–Mehler transform induces a one-parameter semigroup (group) with the infinitesimal generator which is a perturbation of the conservation operator $\Lambda(S)$ (the generator of the Ornstein–Uhlenbeck semigroup) by the adjoint $\Delta_G^*(K)$ of the generalized Gross Laplacian $\Delta_G(K)$. Hence it is very natural to consider a Langevin type stochastic evolution equation given as in (16) with

a very singular noise forcing term of which the unique weak solution is given as in Theorem 9 which is one of main results of this paper.

We should emphasis that our approach provides a useful tool to study the generalized Mehler semigroup and associated Langevin type stochastic evolution equations with singular noise forcing terms, and can be applied to study more general Langevin type equations associated with the perturbations of Ornstein–Uhlenbeck generators.

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