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The Relationship between the Core and the Modified Cores of a Dynamic Game

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Abstract: The core as a solution to a cooperative game has the advantage that any imputation from it is undominated. In cooperative dynamic games, there is a known transformation that demonstrates another advantage of the core—time consistency—keeping players adhering to it during the course of the game. Such a transformation may change the solution, so it is essential that the new core share common imputations with the original one. In this paper, we will establish the relationship between the original core of a dynamic game and the core after the transformation, and demonstrate that the latter can be a subset of the former.

Keywords: discrete-time games; cooperation; the core; linear transformation

MSC: 91A12; 91A50

1. Introduction

The theory of cooperative dynamic games is useful for modeling and analyzing real world problems. Examples include advertising, public goods provision, resource extraction, environmental management, and others which are extensively discussed in [1,2]. The core as a solution to a cooperative game has the advantage that any imputation from it is undominated. This solution is quite popular in the literature on the application of dynamic game theory, not only because of the aforementioned property, but also because of its flexibility, allowing allocating the cooperative outcome in several ways, for instance, in lot sizing [3–5], pollution control [6–8], or non-renewable resource extraction [9]. In cooperative dynamic games, there is a known transformation of a characteristic function, which is a key component of any cooperative game measuring the claims of any group of players [10,11]. The core determined by the modified characteristic function possesses another advantage—time consistency—keeping players adhering to it and being non-negotiable during the course of the game [1,12]. Such a transformation, however, may change the core, so it is essential that the modified core share common imputations with the original one. This allows players to expect if not all of the imputations from the original one, but a part of them. For this reason, we will establish the relationship between the original core of a dynamic game and the core after the transformation, and demonstrate that in some instances the latter can be a subset of the former. It was proven in [11] that the proposed transformation rule applied an infinite number of times converges when the total players' payoffs along the agreed upon behavior are positive. Here we will relax this assumption and refine the conditions that ensure the convergence of the transformation rule.

The structure of the paper is as follows. Section 2 introduces necessary definitions and concepts. The main results of the paper are formulated in Section 3. We then study the relationship between the original core of a dynamic game and the modified ones and establish conditions for the limiting core to be a subset of the original core also when considering classes of linear symmetric games, linear-state games, and two-stage network games. Section 4 concludes.

2. Background

We consider a standard formulation of a dynamic game with complete information. Let $N = \{1, \ldots, n\}$, $|N| = n \geqslant 2$, be a finite set of players. The set of game stages (periods) is described by a finite set $\mathcal{T} = \{1, \ldots, T\}$. We denote a state variable at stage $t \in \mathcal{T}$ by $x(t)$ which belongs to a state space X. Let $x(1) = x_1$. Next, we denote an action of player $i \in N$ at stage $t \in \mathcal{T}$ by $u_i(t)$ which belongs to her action space $U_i(t)$. (Since the sets of actions $U_1(t), \ldots, U_n(t)$, $t \in \mathcal{T}$, and the state space X have not been precisely defined, we suppose that they are not empty, and the values of all optimization problems below exist and are finite). We suppose that the state dynamics is governed by the difference equation

$$x(t+1) = f_t(x(t), u(t)) \in X \tag{1}$$

for any $t \in \mathcal{T}$ from the initial state x_1. It is supposed that $x(t+1)$ is uniquely defined. At each of T game stages, players simultaneously choose actions and thus form an action profile $u = (u_1, \ldots, u_n)$ with $u_i = (u_i(1), \ldots, u_i(T))$ for $i \in N$. The payoff to player $i \in N$ in the game is defined by the real-valued function

$$J_i(x_1, u) = \sum_{t=1}^{T} h_{it}(x(t), u(t)), \quad i \in N,$$

and amounts to the sum of her stage payoffs. (As with [11], players are not rewarded at a terminal state $x(T+1)$, yet the setting can easily be generalized to this case as well).

A player chooses an action according to her *strategy* which accounts for the current information about the game available to her at the time of decision: this can be the information about the game stage, the value of the state variable, the actions that players have taken at previous stages, etc. We denote the information available to player $i \in N$ at stage $t \in \mathcal{T}$ by $\eta_i(t)$. A strategy u_i of player i is a rule that maps the player's information space to her action space, i.e., at stage t player i with information $\eta_i(t)$ chooses the action $u_i(t) = \mathrm{u}_i(\eta_i(t)) \in U_i(t)$. (See [13] for more details). A collection $\mathrm{u} = (\mathrm{u}_1, \ldots, \mathrm{u}_n)$ is a strategy profile. Each strategy profile generates a *trajectory* which is a profile $x = (x(1), \ldots, x(T))$ whose entries are determined by (1). One can introduce the *payoff function* J_i of player $i \in N$ defined on the set of players' strategy profiles as follows: $\mathrm{J}_i(\mathrm{u}) = J_i(x_1, u)$ where u is an action profile corresponding to u.

In the cooperative formulation of the game, players choose their strategies jointly to maximize the payoff they generate, that is to maximize the sum $\sum_{i \in N} \mathrm{J}_i(\mathrm{u})$. Let a strategy profile $\mathrm{u}^* = (\mathrm{u}_1^*, \ldots, \mathrm{u}_n^*)$ maximize the latter sum. This profile is called a cooperative strategy profile and the associated trajectory $x^* = (x^*(1), \ldots, x^*(T))$ with $x^*(1) = x_1$ is called a cooperative trajectory.

We now define a cooperative transferable utility game, or a TU game, (N, v) which is determined by the same player set N and a characteristic function v. This function is defined on 2^N, that is the set of all subsets of set N, and for a subset $S \subseteq N$, called a coalition, its value (a real number) $v(S)$ measures the worth, or claims, of this coalition in the game. Additionally, $v(\varnothing) = 0$. We will not specify how this function is determined as it is not relevant to the analysis we will perform; we only note that $v(N) = \sum_{i \in N} \mathrm{J}_i(\mathrm{u}^*)$, i.e., the *grand coalition* claims the maximum payoff it generates. (See different concepts for determining the characteristic function in dynamic games in [12]). Once the value of $v(N)$ is obtained, players allocate it among them as an *imputation* which is a profile $\xi(v) = (\xi_1(v), \ldots, \xi_n(v))$ satisfying efficiency, i.e., $\sum_{i \in N} \xi_i(v) = v(N)$, and individual rationality, i.e., $\xi_i(v) \geqslant v(\{i\})$, $i \in N$. The set of all imputations, or the imputation set, will be denoted by $\mathcal{I}(v)$. A *cooperative solution*, or simply a *solution*, to the cooperative dynamic game (N, v) is a rule that assigns a subset $\mathcal{M}(v) \subseteq \mathcal{I}(v)$ to this game. In this paper, we suppose that the solution is the core, that is the subset of the imputation set given by $\mathcal{C}(v) = \{\xi(v) \in \mathcal{I}(v) : \sum_{i \in S} \xi_i(v) \geqslant v(S),\ S \subset N\}$. Having chosen the agreed upon cooperative solution $\mathcal{C}(v)$, players jointly implement cooperative strategy profile u^* moving along cooperative trajectory x^*, and after obtaining the value $v(N)$ as their payoff, the players allocate it among them as an imputation from the chosen solution $\mathcal{C}(v)$.

In cooperative dynamic games, it is important that players adhere to the same solution chosen at the initial stage as the game develops along the agreed upon cooperative trajectory x^*. A *time-consistent* solution is stable to its revision during the course of the game, and implementing certain mechanisms one can make cooperation sustainable. When the solution is time inconsistent, there are effective mechanisms of game *regularization*, that is a change in players' stage payoffs along the cooperative trajectory, so that the solution becomes time consistent in the regularized game. (See [1,12] for a comprehensive analysis of sustainable cooperation and the associated time consistency property of a cooperative solution). In the vast majority of cases, such mechanisms are designed on a special redistribution of players' stage payoffs determined by an imputation distribution procedure and they require consideration of proper subgames of the original game along the cooperative trajectory. Each subgame is a dynamic game of $T-t+1$ stages starting from the initial state $x^*(t)$, $t \in \mathcal{T} \setminus \{1\}$. In a similar way, one can define a cooperative subgame (N, v_t), the imputation set $\mathcal{I}(v_t)$, and the cooperative solution (the core $\mathcal{C}(v_t)$) to each subgame, $t \in \mathcal{T} \setminus \{1\}$. (From now on, the original cooperative game (N, v) will be denoted by (N, v_1) for consistency in notation). We suppose that the solution is not empty along the cooperative trajectory x^*. In other words, for each state $x^*(t)$, $t \in \mathcal{T}$, the core $\mathcal{C}(v_t)$ is not empty. If it is not the case, then from the first game stage when this assumption is violated, the players are unable to follow the agreed upon solution.

Petrosyan et al. [11] examine the time consistency of the core based on a *transformation* of the characteristic functions and reveal that the core of the transformed game becomes strong time consistent. (Strong time consistency is a stricter property of a cooperative solution to a dynamic game; it is applicable to set solutions and it coincides with the property of time consistency for point solutions (see [10,14–16] for details)). The strong time consistency of the core was established with the use of a *modified* characteristic function $\hat{v}_t$, $t \in \mathcal{T}$, which for each coalition $S \subseteq N$ accounts for values $v_\tau(S)$ and $v_\tau(N)$, $\tau \geqslant t$, along the cooperative trajectory x^* and is given by:

$$\hat{v}_t(S) = \sum_{\tau=t}^{T} \frac{v_\tau(S) \sum_{i \in N} h_{i\tau}(x^*(\tau), u^*(\tau))}{v_\tau(N)}, \quad S \subseteq N. \tag{2}$$

We call the sets $\mathcal{I}(\hat{v}_t)$ and $\mathcal{C}(\hat{v}_t)$ the modified imputation set and the modified core: these sets are the imputation set and the core in the modified game $(N, \hat{v}_t)$, $t \in \mathcal{T}$.

Since the transformation rule changes the solution, a player or a group of players may want to apply the rule again (or several times subsequently) to change the characteristic function of the game and therefore their payoffs prescribed by the solution which is based on the characteristic function. For a given cooperative trajectory x^* and a coalition $S \subseteq N$, let $\mathbf{v}(S) = (v_1(S), \ldots, v_T(S))'$ and $\hat{\mathbf{v}}(S) = (\hat{v}_1(S), \ldots, \hat{v}_T(S))'$. Using this notation, transformation rule (2) can be written in matrix form:

$$\hat{\mathbf{v}}(S) = \Theta\, \mathbf{v}(S), \tag{3}$$

where Θ is the upper-triangular matrix

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_{T-1} & \theta_T \\ 0 & \theta_2 & \cdots & \theta_{T-1} & \theta_T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \theta_{T-1} & \theta_T \\ 0 & 0 & \cdots & 0 & \theta_T \end{pmatrix},$$

whose entries are given by

$$\theta_t = \frac{\sum_{i\in N} h_{it}(x^*(t), u^*(t))}{\sum_{\tau=t}^{T}\sum_{i\in N} h_{i\tau}(x^*(\tau), u^*(\tau))}, \quad t \in \mathcal{T}.$$

Since $\theta_T = 1$, the last column of Θ consists of ones. Using relation (3), for each coalition $S \subseteq N$ we construct an iterative process $\mathbf{v}^{(m)}(S) = \Theta\,\mathbf{v}^{(m-1)}(S)$, $m = 1, 2, \ldots$ with the initial condition $\mathbf{v}^{(0)}(S) = \mathbf{v}(S)$ where $\mathbf{v}^{(m)}(S) = (v_1^{(m)}(S), \ldots, v_T^{(m)}(S))'$ and $\mathbf{v}^{(1)}(S) = \hat{\mathbf{v}}(S)$. The iterative process can be rewritten as:

$$\mathbf{v}^{(m)}(S) = \Theta^m\,\mathbf{v}(S), \quad m = 1, 2, \ldots \tag{4}$$

It was established in [11] that under the assumption of the non-negativity of players' stage payoffs along the cooperative trajectory, the sequence of modified characteristic functions defined by (4) converges. A *limiting* characteristic function $\bar{v}_t$, $t \in \mathcal{T}$, is given by:

$$\bar{v}_t(S) = \frac{v_t(N)}{v_T(N)} \cdot v_T(S), \quad S \subseteq N. \tag{5}$$

We call the sets $\mathcal{I}(\bar{v}_t)$ and $\mathcal{C}(\bar{v}_t)$ the limiting imputation set and the limiting core: these sets are the imputation set and the core in the limiting game $(N, \bar{v}_t)$, $t \in \mathcal{T}$. In [11], it was shown that when the iterative process (4) converges and (i) when the core $\mathcal{C}(v_t) \neq \varnothing$ for any $t \in \mathcal{T}$, then the modified core $\mathcal{C}(\hat{v}_t) \neq \varnothing$ for any $t \in \mathcal{T}$, (ii) when the core at the terminal stage $\mathcal{C}(v_T) \neq \varnothing$, the limiting core $\mathcal{C}(\bar{v}_t) \neq \varnothing$ for any $t \in \mathcal{T}$. Since we suppose that the original cores are non-empty along the cooperative trajectory, all modified and limiting cores $\mathcal{C}(v_t^{(m)})$ and $\mathcal{C}(\bar{v}_t)$ will be non-empty as well for all $t \in \mathcal{T}$, $m = 1, 2, \ldots$, provided that (4) converges.

3. The Results

3.1. General Results

The convergence of transformation rule (4) was only established for non-negative payoffs along the cooperative trajectory. We now relax the non-negativity condition, yet still assume that the total payoff $\sum_{i\in N} h_{it}(x^*(t), u^*(t))$ is non-zero at each game stage, i.e., at least one player contributes into the grand coalition's payoff.

Proposition 1. *The limiting characteristic function $\bar{v}_1$ exists if and only if $v_t(N)v_{t+1}(N) > 0$ and $|v_{t+1}(N)| \leqslant 2|v_t(N)|$ for $t \in \mathcal{T} \setminus \{T\}$.*

Proof. We suppose that matrix Θ can be decomposed as $\Theta = P\Lambda P^{-1}$ where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of Θ, and P is a matrix whose columns are the corresponding eigenvectors. When the limiting characteristic function exists, it holds that

$$\bar{\mathbf{v}}(S) = \lim_{m\to\infty} \mathbf{v}^{(m)}(S) = \lim_{m\to\infty} \Theta^m\,\mathbf{v}(S) = \lim_{m\to\infty} P\Lambda^m P^{-1}\,\mathbf{v}(S). \tag{6}$$

Since the transformation matrix Θ is upper triangular, we have that $\Lambda = \mathrm{diag}\{\theta_1, \ldots, \theta_T\}$ and $\Lambda^m = \mathrm{diag}\{\theta_1^m, \ldots, \theta_T^m\}$ for $m = 1, 2, \ldots$ According to (6), the limiting characteristic function exists if and only if the limit $\lim_{m\to\infty} \Lambda^m$ exists. This is the case when the absolute values of the eigenvalues of matrix Θ do not exceed 1: $|\theta_t| \in [0, 1]$ for $t \in \mathcal{T}$. Recall that $\theta_T = 1$.

If $\theta_t = (v_t(N) - v_{t+1}(N))\,/v_t(N) = 1$ for some $t \in \mathcal{T} \setminus \{T\}$, then $v_{t+1}(N) = 0$. However, the linear transformation requires that $v_t(N) \neq 0$ for all $t \in \mathcal{T}$. Therefore, it must hold that $\theta_t \in [-1, 1)$ for $t \in \mathcal{T} \setminus \{T\}$ which is equivalent to

$$-1 \leqslant \frac{v_t(N) - v_{t+1}(N)}{v_t(N)} < 1, \quad t \in \mathcal{T} \setminus \{T\}. \tag{7}$$

For $t \in \mathcal{T} \setminus \{T\}$ if $v_t(N) > 0$, then (7) is equivalent to $0 < v_{t+1}(N) \leqslant 2v_t(N)$, whereas if $v_t(N) < 0$, (7) is equivalent to $2v_t(N) \leqslant v_{t+1}(N) < 0$. Summarizing the above, (7) is equivalent to the conditions mentioned in the statement of the proposition for every $t \in \mathcal{T} \setminus \{T\}$. □

Remark 1. *When the limiting characteristic function $\bar{v}_1$ exists, then the limiting characteristic functions $\bar{v}_t$, $t \in \mathcal{T} \setminus \{1\}$, exist as well for any subgame along the cooperative trajectory.*

The conditions that ensure the convergence of the iterative process (4) and, therefore, the existence of the limiting characteristic function, have the following meaning. First, the grand coalition's payoff in the original game and its proper subgames along the cooperative trajectory are of same sign. Second, the grand coalition's payoff in any subgame must be at most twice its payoff in the preceding subgame in absolute values.

Now we study the relationship between the core of the cooperative dynamic game and the modified (limiting) core. As we noted, the transformation rule (4) changes the solution. Therefore, players having agreed on the core $\mathcal{C}(v_1)$ as a solution to game (N, v_1) have to be sure that they will be able to realize an imputation from it even after game transformation. Since for the grand coalition N it holds that $v_t(N) = v_t^{(1)}(N) = \cdots = v_t^{(m)}(N) = \cdots = \bar{v}_t(N)$ for every $t \in \mathcal{T}$, the value to be allocated is invariant to the transformation rule. Our main goal is to establish the relationship between the original core $\mathcal{C}(v_1)$, modified cores $\mathcal{C}(v_1^{(m)})$, $m = 1, 2, \ldots$, and the limiting one $\mathcal{C}(\bar{v}_1)$. When the latter cores intersect with $\mathcal{C}(v_1)$, players are able to realize an imputation from the original core after one-time or even multifaceted transformation of the characteristic function. We will need the following definitions. A set-function $v : 2^N \mapsto \mathbb{R}$ is *monotone* if for every $R \subset S \subseteq N$ we have that either $v(R) \leqslant v(S)$ or $v(R) \geqslant v(S)$. A set-function v is called *supermodular* if for any subsets (coalitions) $S, R \subseteq N$ the following holds: $v(S \cup R) + v(S \cap R) \geqslant v(S) + v(R)$. When the opposite inequality holds for every pair of coalitions, the function v is called *submodular*. It is well known that the core of a *convex* cooperative game, i.e., the game whose characteristic function is supermodular, is not empty [17]. Therefore, when the characteristic functions $\max\{v_1, v_1^{(m)}\}$, $m = 1, 2 \ldots$, and $\max\{v_1, \bar{v}_1\}$ are supermodular, we will have non-empty intersections $\mathcal{C}(v_1) \cap \mathcal{C}(v_1^{(m)}) \neq \varnothing$ and $\mathcal{C}(v_1) \cap \mathcal{C}(\bar{v}_1) \neq \varnothing$ respectively. As it is pointed out in [18], neither the minimum nor the maximum of two submodular set-functions is in general submodular. However, the following result is useful:

Proposition 2 ([18]). *Let v and w be real-valued submodular set-functions on 2^N such that $v - w$ is either monotone increasing or decreasing. Then $\min\{v, w\}$ is also submodular.*

The case when the modified cores $\mathcal{C}(v_1^{(m)})$, $m = 1, 2, \ldots$, and the limiting core $\mathcal{C}(\bar{v}_1)$ are the subsets of $\mathcal{C}(v_1)$ is even more desirable. It ensures that players can realize an imputation from the original core after the transformation(s) of the characteristic function. We would like to establish the conditions providing a nested structure for the cores. Obviously, the inclusions $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(m)})$, $m = 1, 2, \ldots$, and $\mathcal{C}(v_t) \subseteq \mathcal{C}(\bar{v}_t)$ with $t \in \mathcal{T}$ hold if and only if $v_t(S) \geqslant v_t^{(m)}(S)$ and $v_t(S) \geqslant \bar{v}_t(S)$ for every coalition $S \subset N$. Recall that in the subgame which starts at the terminal stage, the original, the modified, and the limiting cores coincide. Similarly, $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(m)})$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ if and only if $v_t(S) \leqslant v_t^{(m)}(S)$ and $v_t(S) \leqslant \bar{v}_t(S)$ for every $S \subset N$. As the above inequalities require the comparison of the original and the modified characteristic functions, we would prefer to establish relationship that require only the definition of the original characteristic function. The following proposition addresses this issue.

Proposition 3. *Let $v_t(N)$ be non-increasing in t and positive. It holds that*

1. *If $\frac{v_1(S)}{v_1(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ for any coalition $S \subseteq N$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ for every game stage $t \in \mathcal{T}$.*

2. *If* $\frac{v_1(S)}{v_1(N)} \geqslant \cdots \geqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$, *then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{v_1(S)}{v_1(N)} = \cdots = \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$, *then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

Proof. Prove the first claim. We suppose that $\frac{v_1(S)}{v_1(N)} \leqslant \frac{v_2(S)}{v_2(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ for any coalition $S \subseteq N$. Then for any $t \in \mathcal{T}$ and S it holds that the modified characteristic function

$$\begin{aligned} v_t^{(1)}(S) &= \sum_{\tau=t}^{T-1} \frac{v_\tau(S)}{v_\tau(N)}[v_\tau(N) - v_{\tau+1}(N)] + v_T(S) \\ &\geqslant \frac{v_t(S)}{v_t(N)} \left(\sum_{\tau=t}^{T-1} [v_\tau(N) - v_{\tau+1}(N)] + v_T(N) \right) = v_t(S). \end{aligned} \tag{8}$$

For the modified characteristic function, we next prove that $\frac{v_1^{(1)}(S)}{v_1^{(1)}(N)} \leqslant \frac{v_2^{(1)}(S)}{v_2^{(1)}(N)} \leqslant \cdots \leqslant \frac{v_T^{(1)}(S)}{v_T^{(1)}(N)}$ for every coalition $S \subseteq N$. Given a coalition S and a game stage $t \in \mathcal{T} \setminus \{T\}$, we obtain

$$\begin{aligned} \frac{v_{t+1}^{(1)}(S)}{v_{t+1}^{(1)}(N)} - \frac{v_t^{(1)}(S)}{v_t^{(1)}(N)} &= \frac{v_{t+1}^{(1)}(S)}{v_{t+1}(N)} - \frac{v_{t+1}^{(1)}(S) + \frac{v_t(S)}{v_t(N)}(v_t(N) - v_{t+1}(N))}{v_t(N)} \\ &= \frac{v_t(N)v_{t+1}^{(1)}(S) - v_{t+1}(N)\left(v_{t+1}^{(1)}(S) + \frac{v_t(S)}{v_t(N)}(v_t(N) - v_{t+1}(N))\right)}{v_t(N)v_{t+1}(N)} \\ &= \frac{(v_t(N) - v_{t+1}(N))\left(v_{t+1}^{(1)}(S) - v_{t+1}(N)\frac{v_t(S)}{v_t(N)}\right)}{v_t(N)v_{t+1}(N)} \\ &\geqslant \frac{(v_t(N) - v_{t+1}(N))}{v_t(N)v_{t+1}(N)}\left(v_{t+1}(S) - v_{t+1}(N)\frac{v_t(S)}{v_t(N)}\right) \\ &= \frac{(v_t(N) - v_{t+1}(N))}{v_t(N)}\left(\frac{v_{t+1}(S)}{v_{t+1}(N)} - \frac{v_t(S)}{v_t(N)}\right) \geqslant 0. \end{aligned}$$

The latter inequality holds true because $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_{t+1}(S)}{v_{t+1}(N)}$ for any stage $t \in \mathcal{T} \setminus \{T\}$ and $v_t(N)$ is non-increasing in t and positive. Therefore, $\frac{v_1^{(1)}(S)}{v_1^{(1)}(N)} \leqslant \frac{v_2^{(1)}(S)}{v_2^{(1)}(N)} \leqslant \cdots \leqslant \frac{v_T^{(1)}(S)}{v_T^{(1)}(N)}$ for every coalition $S \subseteq N$. Similar to (8), we conclude with $v_t^{(2)}(S) \geqslant v_t^{(1)}(S)$ for every S and $t \in \mathcal{T}$.

By induction, we get the following relation $v_t(S) \leqslant v_t^{(1)}(S) \leqslant \cdots \leqslant v_t^{(m)}(S) \leqslant v_t^{(m+1)}(S) \leqslant \ldots$ for all $S \subseteq N$ and $t \in \mathcal{T}$. It immediately implies that $\mathcal{C}(v_t^{(m)}) \supseteq \mathcal{C}(v_t^{(m+1)})$ for $m = 0, 1, \ldots$ with understanding $v_t^{(0)}(S) = v_t(S)$. Since $v_t(N)$ is positive and non-increasing in t, then by Proposition 1, the limiting characteristic function exists. Thus, $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

The second claim is proved in a similar way with the third one being a special case. □

We note that the conditions in the above proposition require the monotonicity of the *relative* worth of all coalitions along the cooperative trajectory. This proposition can be extended for the case when $v_t(N)$ is non-decreasing in t and negative. We formulate additional instances in the next corollary. As we already showed in Proposition 1, the case when $v_t(N)$ changes its sign in t does not lead to the convergence of the iterative process and, as a result, to the existence of the limiting core.

Corollary 1. *Let* $v_t(N)$ *be non-decreasing in* t *and negative. It holds that*

1. *If* $\frac{v_1(S)}{v_1(N)} \leqslant \cdots \leqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
2. *If* $\frac{v_1(S)}{v_1(N)} \geqslant \cdots \geqslant \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{v_1(S)}{v_1(N)} = \cdots = \frac{v_T(S)}{v_T(N)}$ *for any coalition* $S \subseteq N$*, then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

3.2. Linear Symmetric Games

As a special class of cooperative dynamic games, we consider a class of linear symmetric games with the characteristic function depending only upon the number of players in a coalition, that is, $v_t(S) = A_t|S| + B_t$ for all coalitions $S \subseteq N$ and game stages $t \in \mathcal{T}$. Following [19], cooperative game (N, v_t) has a non-empty core $\mathcal{C}(v_t)$ if and only if $\frac{v_t(S)}{|S|} \leqslant \frac{v_t(N)}{|N|}$ for any non-empty coalition $S \subseteq N$. For the characteristic function under consideration, the latter inequality transforms into $\frac{B_t}{|S|} \leqslant \frac{B_t}{|N|}$, $S \subseteq N$, $t \in \mathcal{T}$, which holds true for non-positive B_t. Since players consider the core to be the solution to the cooperative dynamic game, the solution must prescribe a non-empty subset of the imputation set. For this reason, we introduce the assumption $B_t \leqslant 0$ for each $t \in \mathcal{T}$. In practical situations, it is reasonable to assume that the worth of grand coalition $v_1(N)$ is positive, i.e., players generate a positive gain in the game under cooperation. At the same time in view of Proposition 1, the iterative process (4) converges when the grand coalition's payoff does not change its sign along the cooperative trajectory. This implies that $v_t(N) \geqslant 0$ and, therefore, $A_t \geqslant 0$ as well for all $t \in \mathcal{T}$. The next results summarize the relationship between the cores for the class of games under consideration. We let $s = |S|$.

Corollary 2. *Let* $v_t(S) = A_t s + B_t$, $A_t \geqslant 0$, $B_t \leqslant 0$, $t \in \mathcal{T}$. *If the limiting characteristic function* $\bar{v}_t$ *exists, then* $\mathcal{C}(v_t) \bigcap \mathcal{C}(\bar{v}_t) \neq \varnothing$ *for every game stage* $t \in \mathcal{T}$.

Proof. By the definition of the limiting characteristic function (5), we note that v_t and $\bar{v}_t$ are monotone. Taking into account their difference, it holds that $v_t - \bar{v}_t$ is monotone as well. Using Proposition 2, we prove the result. $\square$

Corollary 3. *Let* $v_t(N) = A_t n + B_t$ *be non-increasing in* t *and positive with* $A_t \geqslant 0$, $B_t < 0$ *for all* $t \in \mathcal{T}$. *It holds that*

1. *If* $\frac{A_1}{B_1} \geqslant \cdots \geqslant \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
2. *If* $\frac{A_1}{B_1} \leqslant \cdots \leqslant \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) \subseteq \mathcal{C}(v_t^{(1)}) \subseteq \mathcal{C}(v_t^{(2)}) \subseteq \cdots \subseteq \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.
3. *If* $\frac{A_1}{B_1} = \cdots = \frac{A_T}{B_T}$*, then* $\mathcal{C}(v_t) = \mathcal{C}(v_t^{(1)}) = \mathcal{C}(v_t^{(2)}) = \cdots = \mathcal{C}(\bar{v}_t)$ *for every game stage* $t \in \mathcal{T}$.

Proof. Prove the first claim. We suppose that $\frac{A_t}{B_t} \geqslant \frac{A_{t+1}}{B_{t+1}}$ for any $t \in \mathcal{T} \setminus \{T\}$. Then the following sequence of equivalent relations holds:

$$
\begin{aligned}
\frac{A_t}{B_t} \geqslant \frac{A_{t+1}}{B_{t+1}} &\Leftrightarrow A_t B_{t+1} \geqslant A_{t+1} B_t \\
&\Leftrightarrow A_t B_{t+1}(s-n) \leqslant A_{t+1} B_t (s-n) \\
&\Leftrightarrow A_t B_{t+1} s + A_{t+1} B_t n \leqslant A_t B_{t+1} n + A_{t+1} B_t s \\
&\Leftrightarrow A_t A_{t+1} sn + A_t B_{t+1} s + A_{t+1} B_t n + B_t B_{t+1} \leqslant A_t A_{t+1} sn + A_t B_{t+1} n + A_{t+1} B_t s + B_t B_{t+1} \\
&\Leftrightarrow (A_t s + B_t)(A_{t+1} n + B_{t+1}) \leqslant (A_{t+1} s + B_{t+1})(A_t n + B_t) \\
&\Leftrightarrow v_t(S) v_{t+1}(N) \leqslant v_{t+1}(S) v_t(N).
\end{aligned}
$$

Since $v_t(N)$ is positive for all $t \in \mathcal{T} \setminus \{T\}$, the latter inequality is equivalent to $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_{t+1}(S)}{v_{t+1}(N)}$, $t \in \mathcal{T} \setminus \{T\}$. By Proposition 3, we get the inclusions $\mathcal{C}(v_t) \supseteq \mathcal{C}(v_t^{(1)}) \supseteq \mathcal{C}(v_t^{(2)}) \supseteq \cdots \supseteq \mathcal{C}(\bar{v}_t)$, $t \in \mathcal{T}$.

The second claim is proved in a similar way with the third one being a special case. □

To establish the relationship between the core $\mathcal{C}(v_t)$ and the limiting core $\mathcal{C}(\bar{v}_t)$, we can relax the monotonicity of the ratio A_t/B_t.

Corollary 4. *Let $v_t(N) = A_t n + B_t$ be positive with $A_t \geqslant 0$, $B_t < 0$ for all $t \in \mathcal{T}$. If the limiting characteristic function $\bar{v}_t$ exists, then for any game stage it holds that*

1. *If $\frac{A_t}{B_t} \geqslant \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$.*
2. *If $\frac{A_t}{B_t} \leqslant \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) \subseteq \mathcal{C}(\bar{v}_t)$.*
3. *If $\frac{A_t}{B_t} = \frac{A_T}{B_T}$, then $\mathcal{C}(v_t) = \mathcal{C}(\bar{v}_t)$.*

Proof. We prove the first statement. As with the proof of Corollary 3, it is easy to verify that $\frac{v_t(S)}{v_t(N)} \leqslant \frac{v_T(S)}{v_T(N)}$, $t \in \mathcal{T}$, $S \subseteq N$, and provided that the limiting characteristic function $\bar{v}_t$ exists, we obtain the inclusion $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$.

The second and the third statements are proved similarly. □

Remark 2. *It is worth noting that Corollaries 3 and 4 can be extended to the non-decreasing in game stage and negative values of the grand coalition's payoffs along the cooperative trajectory (recall that by Proposition 1 for convergence, these values must be of same sign). If it is the case, then relaxing the assumption $A_t \geqslant 0$ for all $t \in \mathcal{T}$, one can easily show that the core inclusions become opposite.*

3.3. Two-Stage Network Games

In this section, we establish the relationship between the cores for a class of cooperative two-stage network games studied in [20,21] for a general model and in [22] for their applications in public goods provision and market competition. We will define the characteristic functions in the two-stage cooperative network game according to transformation rule (2) when implementing the cooperative agreement. Taking into account that players receive their payoffs only at the second stage of the game, that is $v_1(N) = v_2(N)$, then it holds that $\hat{v}_1(S) = \hat{v}_2(S) = v_2(S)$ for any coalition $S \subseteq N$. Next, the transformation matrix Θ takes the form

$$\Theta = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Although the players' payoffs at the first game stage are zero, the iterative process (4) converges. From (5), we conclude that $\bar{v}_1(S) = \bar{v}_2(S) = v_2(S)$ for any coalition $S \subseteq N$ as well. Since characteristic functions $\hat{v}_1$, $\bar{v}_1$ in the cooperative two-stage network game and characteristic functions $\hat{v}_2$, $\bar{v}_2$ in its cooperative one-stage subgame coincide, we get the equality $\mathcal{C}(\hat{v}_1) = \mathcal{C}(\hat{v}_2) = \mathcal{C}(\bar{v}_1) = \mathcal{C}(\bar{v}_2) = \mathcal{C}(v_2)$ for the cores.

3.4. A Class of Linear-State Games

Now we examine a class of linear-state games. For the model under consideration, we take one studied in [23] with the purpose to establish the relationship between the cores in this class of games. For convenience, we change the set of game stages $\mathcal{T} = \{0, 1, \ldots, T\}$ and start indexing stages from zero. In the model, the state dynamics is governed by the state equation

$$x(t+1) = b_0 x(t) + b_1 \sum_{i \in N} u_i(t) \in X, \quad t \in \mathcal{T} \setminus \{T\},$$

with the initial condition $x(0) = x_0 \in X$. Here $u_i(t) \in \operatorname{comp} U_i \subset \mathbb{R}_+$ for each player $i \in N$ and $X = \mathbb{R}_+$. The player i's stage payoffs are defined by the functions

$$\begin{aligned} h_{it}(x(t), u(t)) &= a_{i0} u_i(t) + \frac{a_{i1}}{2} u_i^2(t) + a_{i2} x(t), \quad t \in \mathcal{T} \setminus \{T\}, \\ h_{iT}(x(T)) &= a_{i2} x(T). \end{aligned}$$

Additionally, we assume that $a_{i1} < 0$, and $a_{i2} \neq 0$ are of same sign for each $i \in N$, and $b_0, b_1 \neq 0$.

When the game is played cooperatively, players jointly maximize the sum $\sum_{i \in N} J_i(x_0, u) = \sum_{i \in N} (\sum_{t=0}^{T-1} h_{it}(x(t), u(t)) + h_{iT}(x(T)))$.

First, we introduce the following functions of stage number $t \in \mathcal{T}$:

$$\chi_1(t) = \sum_{\tau=1}^{T-t} b_0^{\tau}, \qquad \chi_2(t) = \sum_{\tau=1}^{T-t} \sum_{m=0}^{\tau-1} b_0^{\tau-m}, \qquad \chi_3(t) = \sum_{\tau=0}^{T-t-1} \left(\sum_{m=1}^{T-t-\tau} b_0^m \right)^2, \qquad t \in \mathcal{T} \setminus \{T\},$$

with $\chi_1(T) = \chi_2(T) = \chi_3(T) = 0$. In [23], it was established that the cooperative trajectory is given by

$$x^*(t) = \begin{cases} x_0, & t = 0, \\ b_0^t x_0 - \dfrac{b_1}{b_0} \displaystyle\sum_{i \in N} \frac{1}{a_{i1}} \sum_{\tau=0}^{t-1} b_0^{t-\tau} \left(a_{i0} + \frac{b_1}{b_0} \chi_1(\tau) \sum_{j \in N} a_{j2} \right), & t \in \mathcal{T} \setminus \{0\}, \end{cases}$$

and the characteristic functions in the game and its cooperative proper subgames along this trajectory equal

$$\begin{aligned} v_t(N) &= \sum_{i \in N} \left(-\frac{a_{i0}^2}{2a_{i1}} (T-t) + a_{i2} x^*(t)(1 + \chi_1(t)) - \frac{a_{i2} b_1}{b_0} \chi_2(t) \sum_{j \in N} \frac{a_{j0}}{a_{j1}} \right) \\ &\quad - \left(\frac{b_1}{b_0} \sum_{j \in N} a_{j2} \right)^2 \chi_3(t) \sum_{j \in N} \frac{1}{2a_{j1}}, \\ v_t(S) &= \sum_{i \in S} \left(-\frac{a_{i0}^2}{2a_{i1}} (T-t) + a_{i2} x^*(t)(1 + \chi_1(t)) - \frac{a_{i2} b_1}{b_0} \chi_2(t) \sum_{j \in N} \frac{a_{j0}}{a_{j1}} \right) \\ &\quad - \frac{b_1^2}{b_0^2} \left(\left(\sum_{j \in S} a_{j2} \right)^2 \sum_{j \in S} \frac{1}{2a_{j1}} + \sum_{j \in S} a_{j2} \sum_{j \in N \setminus S} \frac{a_{j2}}{a_{j1}} \right) \chi_3(t), \quad S \subset N, \end{aligned}$$

while for $t = T$ and any $S \subseteq N$, we have $v_T(S) = \sum_{i \in S} a_{i2} x^*(T)$. Please note that characteristic function v_T is additive, therefore, the core $\mathcal{C}(v_T)$ is non-empty and consists of a single imputation $\xi(v_T) = (a_{12} x^*(T), \ldots, a_{n2} x^*(T))$. Moreover, if there exists the core $\mathcal{C}(\bar{v}_T)$, it consists of the same imputation as $\mathcal{C}(v_T) = \mathcal{C}(\bar{v}_T)$.

Before studying the relationship between the cores, we consider the following example. It demonstrates that for the class of games under consideration (i) the modified core and the limiting core can be subsets of the original one, (ii) they can share no common imputation with the original core, and (iii) the original core can intersect with the modified core, but does not intersect with the limiting one.

Example 1. *We consider a 3-person game with* $T = 3$ *and perform simulation with the following parameters:* $x_0 = 15$, $b_0 = 1$, $b_1 = -1$, $a_{11} = a_{21} = a_{31} = -2$, $a_{12} = a_{22} = a_{32} = 0.05$ *whereas parameters* a_{10}, a_{20}, a_{30} *vary. Figure 1 demonstrates the situation when the original core* $\mathcal{C}(v_0)$ *intersects with the modified core* $\mathcal{C}(v_0^{(1)})$*, but does not intersect with the limiting core* $\mathcal{C}(\bar{v}_0)$*. Next, the instance when the modified core and the limiting core are subsets of the core* $\mathcal{C}(v_0)$ *is depicted in Figure 2. Finally, in Figure 3, the original core intersects neither with the modified core nor with the limiting one.*

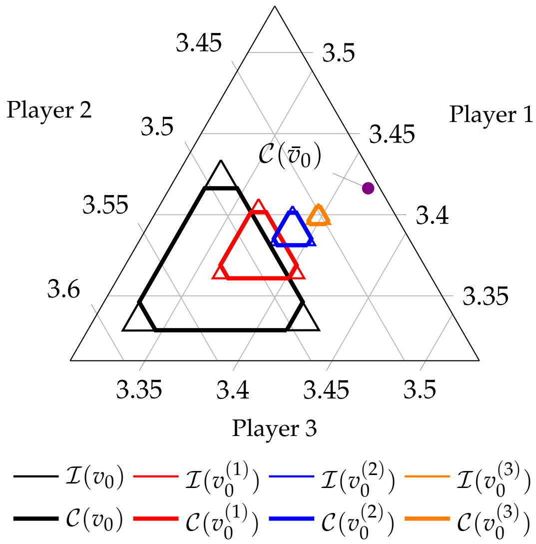

Figure 1. A non-empty intersection of the original core and the modified core ($a_{10} = 1$, $a_{20} = 1.1$, $a_{30} = 1$).

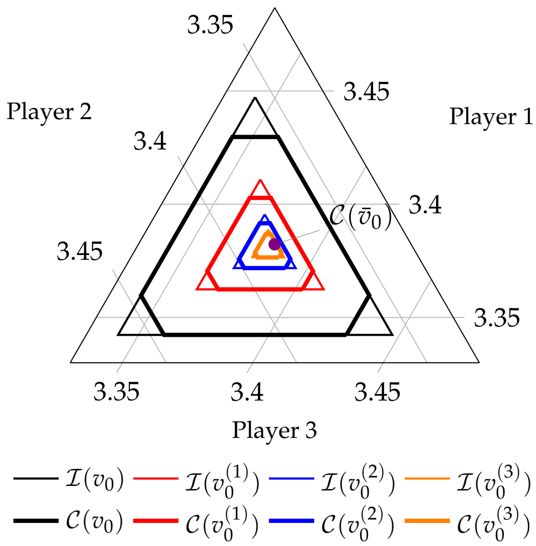

Figure 2. A nested cores pattern ($a_{10} = 1$, $a_{20} = 1.01$, $a_{30} = 1$).

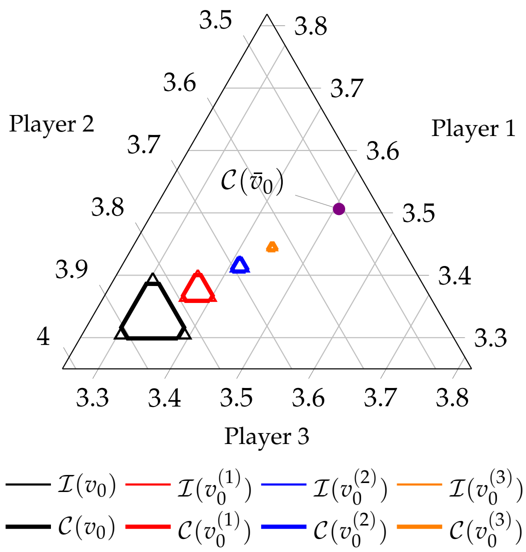

Figure 3. All cores are pairwise disjoint ($a_{10} = 1$, $a_{20} = 1.3$, $a_{30} = 1$).

The next proposition provides conditions under which the limiting core is a subset of the original core.

Proposition 4. *Let the limiting characteristic function $\bar{v}_0$ exist. If the inequality $(\sum_{j\in S}\frac{a_{j0}^2}{a_{j1}})/(\sum_{j\in N}\frac{a_{j0}^2}{a_{j1}}) \leqslant (\sum_{j\in S} a_{j2})/(\sum_{j\in N} a_{j2})$ holds for every coalition $S \subseteq N$ and $\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}} \geqslant 1$ for any $i, j \in N$, then $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for every $t \in \mathcal{T}$. Moreover, when players are symmetric, $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for every $t \in \mathcal{T}$.*

Proof. First, we prove the statement assuming that players are asymmetric. Having the required conditions satisfied, for any $a_{i2}, i \in N$, of same sign we have:

$$\begin{aligned}
\bar{v}_t(S) - v_t(S) &= \frac{v_t(N)}{v_T(N)} v_T(S) - v_t(S) \\
&\geqslant -\frac{b_1^2}{b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in N} a_{i2}\sum_{j\in N}\frac{1}{2a_{j1}} - \sum_{i\in S} a_{i2}\sum_{j\in S}\frac{1}{2a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \\
&= -\frac{b_1^2}{2b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} - 2\sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \\
&\geqslant -\frac{b_1^2}{2b_0^2}\chi_3(t)\sum_{i\in S} a_{i2}\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right).
\end{aligned}$$

When a_{i2} is positive for all $i \in N$, the following sequence of relations holds true:

$$\begin{aligned}
\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}} \geq 1,\ \forall i, j \in N &\Rightarrow \sum_{i\in S}\left(\frac{a_{i2}}{a_{j2}} + \frac{a_{j1}}{a_{i1}}\right) \geqslant 1,\ \forall S \subseteq N, \forall j \in N \\
&\Leftrightarrow \frac{1}{a_{j2}}\sum_{i\in S} a_{i2} + a_{j1}\sum_{i\in S}\frac{1}{a_{i1}} \geqslant 1,\ \forall S \subseteq N, \forall j \in N \\
&\Leftrightarrow \frac{1}{a_{j1}}\sum_{i\in S} a_{i2} + a_{j2}\sum_{i\in S}\frac{1}{a_{i1}} \leqslant \frac{a_{j2}}{a_{j1}},\ \forall S \subseteq N, \forall j \in N \\
&\Rightarrow \sum_{j\in N\setminus S}\left(\frac{1}{a_{j1}}\sum_{i\in S} a_{i2} + a_{j2}\sum_{i\in S}\frac{1}{a_{i1}} - \frac{a_{j2}}{a_{j1}}\right) \leqslant 0,\ \forall S \subseteq N \\
&\Leftrightarrow \left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \leqslant 0,\ \forall S \subseteq N.
\end{aligned}$$

Thus, $\bar{v}_t(S) - v_t(S) \geqslant 0$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

When a_{i2} is negative for all $i \in N$, we obtain:

$$\left(\sum_{i\in S} a_{i2}\sum_{j\in N\setminus S}\frac{1}{a_{j1}} + \sum_{i\in N\setminus S} a_{i2}\sum_{j\in S}\frac{1}{a_{j1}} - \sum_{j\in N\setminus S}\frac{a_{j2}}{a_{j1}}\right) \geqslant 0,\ \forall S \subseteq N.$$

Then $\bar{v}_t(S) - v_t(S) \geqslant 0$ and $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$. Therefore, when $a_{i2} \neq 0$, $i \in N$, are of same sign, $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ for $t \in \mathcal{T}$.

Now suppose that players are symmetric. We note that in this case, the required conditions from the first part are always met. Therefore, the inclusion $\mathcal{C}(v_t) \supseteq \mathcal{C}(\bar{v}_t)$ holds as well for $t \in \mathcal{T}$. □

4. Conclusions

In this paper, we studied the relationship between the core of the original game and the cores of modified games determined by a transformation rule because these cores may not intersect for a dynamic game of a general structure. First, we extended the conditions known in the literature which lead to the convergence of an iterative process based on this transformation rule. Second, we found conditions under which one core is a subset of the other: these conditions require the monotonicity of the relative worth of coalitions along the cooperative trajectory. Finally, for several classes of dynamic games, we characterized the relationship between the cores.

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