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Kuelbs–Steadman Spaces for Banach Space-Valued Measures

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Abstract: We introduce Kuelbs–Steadman-type spaces (KS^p spaces) for real-valued functions, with respect to countably additive measures, taking values in Banach spaces. We investigate the main properties and embeddings of L^q -type spaces into KS^p spaces, considering both the norm associated with the norm convergence of the involved integrals and that related to the weak convergence of the integrals.

Keywords: Kuelbs–Steadman space; Henstock–Kurzweil integrable function; vector measure; dense embedding; completely continuous embedding; Köthe space; Banach lattice

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1. Introduction

Kuelbs–Steadman spaces have been the subject of many recent studies (see, e.g., [1–3] and the references therein). The investigation of such spaces arises from the idea to consider the L^1 spaces as embedded in a larger Hilbert space with a smaller norm and containing in a certain sense the Henstock–Kurzweil integrable functions. This allows giving several applications to functional analysis and other branches of mathematics, for instance Gaussian measures (see also [4]), convolution operators, Fourier transforms, Feynman integrals, quantum mechanics, differential equations, and Markov chains (see also [1–3]). This approach allows also developing a theory of functional analysis that includes Sobolev-type spaces, in connection with Kuelbs–Steadman spaces rather than with classical L^p spaces.

Moreover, in recent studies about integration theory, multifunctions have played an important role in applications to several branches of science, like for instance control theory, differential inclusions, game theory, aggregation functions, economics, problems of finding equilibria, and optimization. Since neither the Riemann integral, nor the Lebesgue integral are completely satisfactory concerning the problem of the existence of primitives, different types of integrals extending the previous ones have been introduced and investigated, like Henstock–Kurzweil, McShane, and Pettis integrals. These topics have many connections with measures taking values in abstract spaces, and in particular, the extension of the concept of integrability to set-valued functions can be used in order to obtain a larger number of selections for multifunctions, through their estimates and properties, in several applications (see, e.g., [5–13]).

In this paper, we extend the theory of Kuelbs–Steadman spaces to measures μ defined on a σ -algebra and with values in a Banach space X . We consider an integral for real-valued functions f

with respect to X -valued countably additive measures. In this setting, a fundamental role is played by the separability of μ . This condition is satisfied, for instance, when T is a metrizable separable space, not necessarily with a Schauder basis (such spaces exist; see, for instance, [1]), and μ is a Radon measure. In the literature, some deeply investigated particular cases are when $X = \mathbb{R}^n$ and μ is the Lebesgue measure, and when X is a Banach space with a Schauder basis (see also [1–3]). Since the integral of f with respect to μ is an element of X , in general, it is not natural to define an inner product, when it is dealt with by norm convergence of the involved integrals. Moreover, when μ is a vector measure, the spaces $L^p[\mu]$ do not satisfy all classical properties as the spaces L^p with respect to a scalar measure (see also [14–16]). However, it is always possible to define Kuelbs–Steadman spaces as Banach spaces, which are completions of suitable L^p spaces. We introduce them and prove that they are normed spaces and that the embeddings of $L^q[\mu]$ into $KS^p[\mu]$ are completely continuous and dense. We show that the norm of KS^p spaces is smaller than that related to the space of all Henstock–Kurzweil integrable functions (the Alexiewicz norm). We prove that KS^p spaces are Köthe function spaces and Banach lattices, extending to the setting of $KS^p[\mu]$ -spaces some results proven in [16] for spaces of type $L^p[\mu]$. Furthermore, when X' is separable, it is possible to consider a topology associated with the weak convergence of integrals and to define a corresponding norm and an inner product. We introduce the Kuelbs–Steadman spaces related to this norm and prove the analogous properties investigated for KS^p spaces related to norm convergence of the integrals. In this case, since we deal with a separable Hilbert space, it is possible to consider operators like convolution and Fourier transform and to extend the theory developed in [1–3] to the context of Banach space-valued measures.

2. Vector Measures, (HKL)- and (KL)-Integrals

Let $T \neq \emptyset$ be an abstract set, $\mathcal{P}(T)$ be the class of all subsets of T , $\Sigma \subset \mathcal{P}(T)$ be a σ -algebra, X be a Banach space, and X' be its topological dual. For each $A \in \Sigma$, let us denote by χ_A the characteristic function of A , defined by:

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \in T \setminus A. \end{cases}$$

A vector measure is a σ -additive set function $\mu : \Sigma \rightarrow X$. By the Orlicz–Pettis theorem (see also [17] (Corollary 1.4)), the σ -additivity of μ is equivalent to the σ -additivity of the scalar-valued set function $x'\mu : A \mapsto x'(\mu(A))$ on Σ for every $x' \in X'$. For the literature on vector measures, see also [14,15,17–21] and the references therein.

The variation $|\mu|$ of μ is defined by setting:

$$|\mu|(A) = \sup \left\{ \sum_{i=1}^r \|\mu(A_i)\| : A_i \in \Sigma, i = 1, 2, \dots, r; A_i \cap A_j = \emptyset \text{ for } i \neq j; \bigcup_{i=1}^r A_i \subset A \right\}.$$

We define the semivariation $\|\mu\|$ of μ by:

$$\|\mu\|(A) = \sup_{x' \in X', \|x'\| \leq 1} |x'\mu|(A). \quad (1)$$

Remark 1. Observe that $\|\mu\|(A) < +\infty$ for all $A \in \Sigma$ (see also [17] (Corollary 1.19), [15] (§1)).

The completion of Σ with respect to $\|\mu\|$ is defined by:

$$\tilde{\Sigma} = \{A = B \cup N : B \in \Sigma, N \subset M \in \Sigma \text{ with } \|\mu\|(M) = 0\}. \quad (2)$$

A function $f : T \rightarrow \mathbb{R}$ is said to be μ -measurable if:

$$f^{-1}(B) \cap \{t \in T : f(t) \neq 0\} \in \tilde{\Sigma}$$

for each Borel subset $B \subset \mathbb{R}$.

Observe that from (1) and (2), it follows that every μ -measurable real-valued function is also $x'\mu$ -measurable for every $x' \in X'$. Moreover, it is readily seen that every Σ -measurable real-valued function is also μ -measurable.

We say that μ is Σ -separable (or separable) if there is a countable family $\mathbb{B} = (B_k)_k$ in Σ such that, for each $A \in \Sigma$ and $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that:

$$\|\mu\|(A \Delta B_{k_0}) = \sup_{x' \in X', \|x'\| \leq 1} [|x'\mu|(A \Delta B_{k_0})] \leq \varepsilon \quad (3)$$

(see also [22]). Such a family $\mathbb{B}$ is said to be μ -dense.

Observe that μ is separable if and only if Σ is μ -essentially countably generated, namely there is a countably generated σ -algebra $\Sigma_0 \subset \Sigma$ such that for each $A \in \Sigma$, there is $B \in \Sigma_0$ with $\mu(A \Delta B) = 0$. The separability of μ is satisfied, for instance, when T is a separable metrizable space, Σ is the Borel σ -algebra of the Borel subsets of T , and μ is a Radon measure (see also [23] (Theorem 4.13), [24] (Theorem 1.0), [19] (§1.3 and §2.6), and [22] (Propositions 1A and 3)).

From now on, we assume that μ is separable, and $\mathbb{B} = (B_k)_k$ is a μ -dense family in Σ with:

$$\|\mu\|(B_k) \leq M = \|\mu\|(T) + 1 \quad \text{for all } k \in \mathbb{N}. \quad (4)$$

Now, we recall the Henstock–Kurzweil (HK) integral for real-valued functions, defined on abstract sets, with respect to (possibly infinite) non-negative measures. For the related literature, see also [5–13,25–33] and the references therein. When we deal with the (HK)-integral, we assume that T is a compact topological space and Σ is the σ -algebra of all Borel subsets of T . We will not use these assumptions to prove the results, which do not involve the (HK)-integral.

Let $\nu : \Sigma \rightarrow \mathbb{R} \cup \{+\infty\}$ be a σ -additive non-negative measure. A decomposition of a set $A \in \Sigma$ is a finite collection $\{(A_1, \xi_1), (A_2, \xi_2), \dots, (A_N, \xi_N)\}$ such that $A_j \in \Sigma$ and $\xi_j \in A_j$ for every $j \in \{1, 2, \dots, N\}$, and $\nu(A_i \cap A_j) = 0$ whenever $i \neq j$. A decomposition of subsets of $A \in \Sigma$ is called a partition of A when $\bigcup_{j=1}^N A_j = A$. A gauge on a set $A \in \Sigma$ is a map δ assigning to each point $x \in A$ a neighborhood $\delta(x)$ of x . If $\mathcal{D} = \{(A_1, \xi_1), (A_2, \xi_2), \dots, (A_N, \xi_N)\}$ is a decomposition of A and δ is a gauge on A , then we say that $\mathcal{D}$ is δ -fine if $A_j \subset \delta(\xi_j)$ for any $j \in \{1, 2, \dots, N\}$.

An example is when T_0 is a locally compact and Hausdorff topological space and $T = T_0 \cup \{x_0\}$ is the one-point compactification of T_0 . In this case, we will suppose that all involved functions f vanish on x_0 . For instance, this is the case when $T_0 = \mathbb{R}^n$ is endowed with the usual topology and x_0 is a point “at the infinity”, or when T is the unbounded interval $[a, +\infty] = [a, +\infty) \cup \{+\infty\}$ of the extended real line, considered as the one-point compactification of the locally compact space $[a, +\infty)$. In this last case, the base of open sets consists of the open subsets of $[a, +\infty)$ and the sets of the type $(b, +\infty]$, where $a < b < +\infty$. Any gauge in $[a, +\infty]$ has the form $\delta(x) = (x - d(x), x + d(x))$, if $x \in [a, +\infty) \cap \mathbb{R}$, and $\delta(+\infty) = (b, +\infty] = (b, +\infty) \cup \{+\infty\}$, where d denotes a positive real-valued function defined on $[a, +\infty)$. Now, we define the Riemann sums by: $S(f, \mathcal{D}) = \sum_{j=1}^N f(\xi_j) \nu(A_j)$ if the sum exists in $\mathbb{R}$, with the convention $0 \cdot (+\infty) = 0$. Note that for any gauge δ , there exists at least one δ -fine partition $\mathcal{D}$ such that $S(f, \mathcal{D})$ is well defined.

A function $f : T \rightarrow \mathbb{R}$ is said to be Henstock–Kurzweil integrable ((HK)-integrable) on a set $A \in \Sigma$ if there is an element $I_A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge δ on A with $|S(f, \mathcal{D}) - I_A| \leq \varepsilon$ whenever $\mathcal{D}$ is a δ -fine partition of A such that $S(f, \mathcal{D})$ exists in $\mathbb{R}$, and we write:

$$(HK) \int_A f \, dv = I_A.$$

Observe that, if $A, B \in \Sigma$, $B \subset A$, and $f : T \rightarrow \mathbb{R}$ is (HK)-integrable on A , then f is also (HK)-integrable on B and on $A \setminus B$, and:

$$(HK) \int_A f(t) \, dv = (HK) \int_B f(t) \, dv + (HK) \int_{A \setminus B} f(t) \, dv \quad (5)$$

(see also [25] (Propositions 5.14 and 5.15), [33] (Lemma 1.10 and Proposition 1.11)). From (5) used with $A = T$ and $\chi_B f$ instead of f , it follows that, if f is (HK)-integrable on T and $B \in \Sigma$, then:

$$(HK) \int_T \chi_B(t) f(t) \, dv = (HK) \int_B f(t) \, dv.$$

We say that a Σ -measurable function $f : T \rightarrow \mathbb{R}$ is Kluváněk–Lewis–Lebesgue μ -integrable, (KL) μ -integrable (resp. Kluváněk–Lewis–Henstock–Kurzweil μ -integrable, or (HKL) μ -integrable) if the following properties hold:

$$f \text{ is } |x' \mu| \text{-Lebesgue (resp. } |x' \mu| \text{-Henstock–Kurzweil) integrable for each } x' \in X', \quad (6)$$

and for every $A \in \Sigma$, there is $x_A^{(L)}$ (resp. $x_A^{(HK)}$) $\in X$ with:

$$x'(x_A^{(L)}) = (L) \int_A f \, d|x' \mu| \quad (\text{resp. } x'(x_A^{(HK)}) = (HK) \int_A f \, d|x' \mu|) \text{ for all } x' \in X', \quad (7)$$

where the symbols (L) and (HK) in (7) denote the usual Lebesgue (resp. Henstock–Kurzweil) integral of a real-valued function with respect to an (extended) real-valued measure. A Σ -measurable function $f : T \rightarrow \mathbb{R}$ is said to be weakly (KL) (resp. weakly (HKL)) μ -integrable if it satisfies only condition (6) (see also [18,21,34]). We recall the following facts about the (KL)-integral.

Proposition 1. (See also [21] (Theorem 2.1.5 (i))) If $s : T \rightarrow \mathbb{R}$, $s = \sum_{i=1}^r \alpha_i \chi_{A_i}$ is Σ -simple, with $\alpha_i \in \mathbb{R}$, $A_i \in \Sigma$, $i = 1, 2, \dots, r$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then s is (KL) μ -integrable on T , and:

$$(KL) \int_A s \, d\mu = \sum_{i=1}^r \alpha_i \mu(A \cap A_i) \text{ for all } A \in \Sigma.$$

Proposition 2. (See also [21] (Theorem 2.1.5 (vi))) If $f : T \rightarrow \mathbb{R}$ is (KL) μ -integrable on T and $A \in \Sigma$, then $\chi_A f$ is (KL) μ -integrable on T and:

$$(KL) \int_A f \, d\mu = (KL) \int_T \chi_A f \, d\mu.$$

The space $L^1[\mu]$ (resp. $L_w^1[\mu]$) is the space of all (equivalence classes of) (KL) μ -integrable functions (resp. weakly (KL) μ -integrable functions) up to the complement of μ almost everywhere sets. For $p > 1$, the space $L^p[\mu]$ (resp. $L_w^p[\mu]$) is the space of all (equivalence classes of) Σ -measurable

functions f such that $|f|^p$ belongs to $L^1[\mu]$ (resp. $L^1_w[\mu]$). The space $L^\infty[\mu]$ is the space of all (equivalence classes of) μ -essentially bounded functions. The norms are defined by:

$$\begin{cases} \|f\|_{L^p[\mu]} = \|f\|_{L^p_w[\mu]} = \sup_{x' \in X', \|x'\| \leq 1} \left((L) \int_T |f(t)|^p d|x'\mu| \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \|f\|_{L^\infty[\mu]} = \sup_{x' \in X', \|x'\| \leq 1} (|x'\mu|-\text{ess sup } |f|) \end{cases}$$

(see also [35–37]).

If $f : T \rightarrow \mathbb{R}$ is an (HKL) -integrable function, then the Alexiewicz norm of f is defined by:

$$\|f\|_{HKL} = \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{A \in \Sigma} \left| (HK) \int_A f(t) d|x'\mu| \right| \right)$$

(see also [38,39]). Observe that, by arguing analogously as in [30] (Theorem 9.5) and [40] (Example 3.1.1), for each $x' \in X'$, we get that $f = 0$ $|x'\mu|$, almost everywhere if and only if $(HK) \int_A f(t) d|x'\mu| = 0$ for every $A \in \Sigma$. Thus, it is not difficult to see that $\|\cdot\|_{HKL}$ is a norm. In general, the space of the real-valued Henstock–Kurzweil integrable functions endowed with the Alexiewicz norm is not complete (see also [39] (Example 7.1)).

3. Construction of the Kuelbs–Steadman Spaces and Main Properties

We begin with giving the following technical results, which will be useful later.

Proposition 3. Let $(a_k)_k$ and $(\eta_k)_k$ be two sequences of non-negative real numbers, such that $a = \sup_k a_k < +\infty$, and

$$\sum_{k=1}^{\infty} \eta_k = 1, \quad (8)$$

and $p > 0$ be a fixed real number. Then,

$$\left(\sum_{k=1}^{\infty} \eta_k a_k^p \right)^{1/p} \leq a. \quad (9)$$

Proof. We have $\eta_k a_k^p \leq a^p \eta_k$ for all $k \in \mathbb{N}$, and hence:

$$\sum_{k=1}^{\infty} \eta_k a_k^p \leq a^p \sum_{k=1}^{\infty} \eta_k = a^p,$$

getting (9). $\square$

Proposition 4. Let $(b_k)_k, (c_k)_k$ be two sequences of real numbers, $(\eta_k)_k$ be a sequence of positive real numbers, satisfying (8), and $p \geq 1$ be a fixed real number. Then,

$$\left(\sum_{k=1}^{\infty} \eta_k |b_k + c_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} \eta_k (|b_k| + |c_k|)^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} \eta_k |b_k|^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} \eta_k |c_k|^p \right)^{1/p}. \quad (10)$$

Proof. It is a consequence of Minkowski's inequality (see also [41] (Theorem 2.11.24)). $\square$

Let $\mathbb{B} = (B_k)_k$ be as in (4), and set $\mathcal{E}_k = \chi_{B_k}, k \in \mathbb{N}$.

For $1 \leq p \leq \infty$, let us define a norm on $L^1[\mu]$ by setting:

$$\|f\|_{KSP[\mu]} = \begin{cases} \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right|^p \right]^{1/p} \right\} & \text{if } 1 \leq p < \infty, \\ \sup_{x' \in X', \|x'\| \leq 1} \left[\sup_{k \in \mathbb{N}} \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right| \right] & \text{if } p = \infty. \end{cases} \quad (11)$$

The following inequality holds.

Proposition 5. For any $f \in L^1[\mu]$ and $p \geq 1$, it is:

$$\|f\|_{KSP[\mu]} \leq \|f\|_{KS^\infty[\mu]}. \quad (12)$$

Proof. By (9) used with:

$$a_k = \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu|(t) \right|, \quad (13)$$

where x' is a fixed element of X' with $\|x'\| \leq 1$, we have:

$$\left(\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu|(t) \right|^p \right)^{1/p} \leq \sup_{k \in \mathbb{N}} \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu|(t) \right|. \quad (14)$$

Taking the supremum in (14) as $x' \in X'$, $\|x'\| \leq 1$, we obtain:

$$\begin{aligned} \|f\|_{KSP[\mu]} &= \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right|^p \right]^{1/p} \right\} \\ &\leq \sup_{x' \in X', \|x'\| \leq 1} \left[\sup_{k \in \mathbb{N}} \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right| \right] = \|f\|_{KS^\infty[\mu]}, \end{aligned}$$

getting the assertion. $\square$

Now, we prove that:

Theorem 1. The map $f \mapsto \|f\|_{KSP[\mu]}$ defined in (11) is a norm.

Proof. Observe that, by definition, $\|f\|_{KSP[\mu]} \geq 0$ for every $f \in L^1[\mu]$. Let $f \in L^1[\mu]$ with $\|f\|_{KSP[\mu]} = 0$. We prove that $f = 0$ μ , almost everywhere. It is enough to take $1 \leq p < \infty$, since the case $p = \infty$ will follow from (12). For $k \in \mathbb{N}$, let a_k be as in (13). As the η_k 's are strictly positive, from:

$$\left(\sum_{k=1}^{\infty} \eta_k a_k^p \right)^{1/p} = 0$$

it follows that $a_k = 0$ for every $k \in \mathbb{N}$. Hence,

$$\left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu|(t) \right| = 0 \quad \text{for each } k \in \mathbb{N} \text{ and } x' \in X' \text{ with } \|x'\| \leq 1. \quad (15)$$

Proceeding by contradiction, suppose that $f \neq 0$ μ , almost everywhere. If $E^+ = f^{-1}([0, +\infty[)$, $E^- = f^{-1}(]-\infty, 0])$, then $E^+, E^- \in \Sigma$, since f is Σ -measurable, and we have $\mu(E^+) \neq 0$ or $\mu(E^-) \neq 0$.

Suppose that $\mu(E^+) \neq 0$. By the Hahn–Banach theorem, there is $x'_0 \in X'$ with $\|x'_0\| \leq 1$, $x'_0 \mu(E^+) \neq 0$, and hence, $|x'_0 \mu(E^+)| > 0$. Moreover, if $f^*(t) = \min\{f(t), 1\}$, $t \in T$, then $E^+ = \{t \in T : f^*(t) > 0\}$. For each $n \in \mathbb{N}$, set:

$$E_n^+ = \left\{ t \in T : \frac{1}{n+1} < f^*(t) \leq \frac{1}{n} \right\}.$$

Since $E^+ = \bigcup_{n=1}^{\infty} E_n^+$ and $x'_0 \mu$ is σ -additive, there is $\bar{n} \in \mathbb{N}$ with $|x'_0 \mu|(E_{\bar{n}}^+) > 0$. Put $\bar{B} = E_{\bar{n}}^+$, and choose $\bar{\varepsilon}$ such that:

$$0 < \bar{\varepsilon} < \min \left\{ \frac{1}{\bar{n}+1} |x'_0 \mu|(\bar{B}), 1 \right\}. \quad (16)$$

By the separability of μ , in correspondence with $\bar{\varepsilon}$ and $\bar{B}$, there is $B_{k_0} \in \mathbb{B}$ satisfying (3), that is:

$$\|\mu\|(\bar{B} \Delta B_{k_0}) = \sup_{x' \in X', \|x'\| \leq 1} [|x' \mu|(\bar{B} \Delta B_{k_0})] \leq \bar{\varepsilon}. \quad (17)$$

From (16) and (17), we deduce:

$$\|\mu\|(B_{k_0}) \leq \|\mu\|(\bar{B}) + \|\mu\|(\bar{B} \Delta B_{k_0}) < \|\mu\|(T) + 1 = M,$$

so that $B_{k_0} \in \mathbb{B}$, and:

$$\begin{aligned} \left| (L) \int_T \chi_{B_{k_0}}(t) f(t) d|x'_0 \mu|(t) \right| &\geq (L) \int_T \mathcal{E}_{k_0}(t) f(t) d|x'_0 \mu|(t) \\ &= (L) \int_{B_{k_0}} f(t) d|x'_0 \mu|(t) \geq (L) \int_{B_{k_0}} f^*(t) d|x'_0 \mu|(t) \\ &\geq (L) \int_{\bar{B}} f^*(t) d|x'_0 \mu|(t) - (L) \int_{\bar{B} \Delta B_{k_0}} f^*(t) d|x'_0 \mu|(t) \\ &\geq \frac{1}{\bar{n}+1} |x'_0 \mu|(\bar{B}) - |x'_0 \mu|(\bar{B} \Delta B_{k_0}) \geq \frac{1}{\bar{n}+1} |x'_0 \mu|(\bar{B}) - \bar{\varepsilon} > 0, \end{aligned} \quad (18)$$

which contradicts (15). Therefore, $\mu(E^+) = 0$.

Now, suppose that $\mu(E^-) \neq 0$. By proceeding analogously as in (18), replacing f with $-f$ and f^* with the function f_* defined by $f_*(t) = \min\{-f(t), 1\}$, $t \in T$, we find an $x'_1 \in X'$ with $\|x'_1\| \leq 1$, an $\bar{n} \in \mathbb{N}$, a $\bar{B} \in \Sigma$, an $\bar{\varepsilon} > 0$, and a $B_{k_1} \in \mathbb{B}$ with $\|\mu\|(B_{k_1}) < M$, and:

$$\left| (L) \int_T \chi_{B_{k_1}}(t) f(t) d|x'_1 \mu|(t) \right| \geq (L) \int_{B_{k_1}} f_*(t) d|x'_1 \mu|(t) \geq \frac{1}{\bar{n}+1} |x'_1 \mu|(\bar{B}) - \bar{\varepsilon} > 0,$$

getting again a contradiction with (15). Thus, $\mu(E^-) = 0$, and $f = 0$ almost everywhere.

The triangular property of the norm can be deduced from Proposition 4 for $1 \leq p < \infty$, and it is not difficult to see for $p = \infty$; the other properties are easy to check. $\square$

For $1 \leq p \leq \infty$, the Kuelbs–Steadman space $KS^p[\mu]$ (resp. $KS_w^p[\mu]$) is the completion of $L^1[\mu]$ (resp. $L_w^1[\mu]$) with respect to the norm defined in (11) (see also [2–4,35–37]). Observe that, to avoid ambiguity, we take the completion of $L^1[\mu]$ rather than that of $L^p[\mu]$, but since the embeddings in Theorem 2 are continuous and dense, the two methods are substantially equivalent.

By proceeding similarly as in [2] (Theorem 3.26), we prove the following relations between the spaces $L^q[\mu]$ and $KS^p[\mu]$.

Theorem 2. For every p, q with $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, it is $L^q[\mu] \subset KS^p[\mu]$ continuously and densely. Moreover, the space of all Σ -simple functions is dense in $KS^p[\mu]$.

Proof. We first consider the case $1 \leq p < \infty$. Let $f \in L^q[\mu]$, with $1 \leq q < \infty$, and M be as in (4). Note that $M^{\frac{q-1}{q}} \leq M$, since $M \geq 1$. As $|\mathcal{E}_k(t)| = \mathcal{E}_k(t) \leq 1$ and $|\mathcal{E}_k(t)|^q \leq \mathcal{E}_k(t)$ for any $k \in \mathbb{N}$ and $t \in T$, taking into account (9) and applying Jensen's inequality to the function $t \mapsto |t|^q$ (see also [23] (Exercise 4.9)), we deduce:

$$\begin{aligned} \|f\|_{KS^p[\mu]} &= \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right|^{\frac{pq}{q}} \right]^{1/p} \right\} \\ &\leq \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left((|x'\mu|(B_k))^{q-1} \cdot (L) \int_T \mathcal{E}_k(t) |f(t)|^q d|x'\mu| \right)^{p/q} \right]^{1/p} \right\} \quad (19) \\ &\leq M^{\frac{q-1}{q}} \sup_{x' \in X', \|x'\| \leq 1} \left[\sup_{k \in \mathbb{N}} \left((L) \int_T \mathcal{E}_k(t) |f(t)|^q d|x'\mu| \right)^{1/q} \right] \\ &\leq M \sup_{x' \in X', \|x'\| \leq 1} \left[\left((L) \int_T |f(t)|^q d|x'\mu| \right)^{1/q} \right] = M \|f\|_{L^q[\mu]}, \end{aligned}$$

where M is as in (4). Now, let $1 \leq p < \infty$ and $q = \infty$. We have:

$$\begin{aligned} \|f\|_{KS^p[\mu]} &= \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right|^p \right]^{1/p} \right\} \\ &\leq \sup_{x' \in X', \|x'\| \leq 1} [(|x'\mu|(B_k))^p \cdot \text{ess sup} |f|^p]^{1/p} \leq M \cdot \|f\|_{L^\infty[\mu]}. \quad (20) \end{aligned}$$

The proof of the case $p = \infty$ is analogous to that of the case $1 \leq p < \infty$. Therefore, $f \in KS^p[\mu]$, and the embeddings in (19) and (20) are continuous.

Moreover, observe that every Σ -simple function belongs to $L^q[\mu]$, and the space of all Σ -simple functions is dense in $L^1[\mu]$ with respect to $\|\cdot\|_{L^1[\mu]}$ (see also [21] (Corollary 2.1.10)). Moreover, since $KS^p[\mu]$ is the completion of $L^1[\mu]$ with respect to the norm $\|\cdot\|_{KS^p[\mu]}$, the space $L^1[\mu]$ is dense in $KS^p[\mu]$ with respect to the norm $\|\cdot\|_{KS^p[\mu]}$ (see also [42] (§4.4)).

Choose arbitrarily $\varepsilon > 0$ and $f \in KS^p[\mu]$. There is $g \in L^1[\mu]$ with $\|g - f\|_{KS^p[\mu]} \leq \frac{\varepsilon}{M+1}$. Moreover, in correspondence with ε and g , we find a Σ -simple function s , with $\|s - g\|_{L^1[\mu]} \leq \frac{\varepsilon}{M+1}$. By (19) and (20), $\|\cdot\|_{KS^p[\mu]} \leq M\|\cdot\|_{L^1[\mu]}$, and hence, we obtain:

$$\begin{aligned} \|s - f\|_{KS^p[\mu]} &\leq \|s - g\|_{KS^p[\mu]} + \|g - f\|_{KS^p[\mu]} \\ &\leq M\|s - g\|_{L^1[\mu]} + \|g - f\|_{KS^p[\mu]} \leq \frac{M\varepsilon}{M+1} + \frac{\varepsilon}{M+1} = \varepsilon, \end{aligned}$$

getting the last part of the assertion. Thus, the embeddings in (19) and (20) are dense. $\square$

Proposition 6. $KS^\infty[\mu] \subset KS^p[\mu]$ for every $p \geq 1$.

Proof. The assertion follows from (12), since $KS^p[\mu]$ (resp. $KS^\infty[\mu]$) is the completion of $L^1[\mu]$ with respect to $\|f\|_{KS^p[\mu]}$ (resp. $\|f\|_{KS^\infty[\mu]}$). $\square$

Remark 2. (a) Notice that, for $q \neq \infty$, by Theorem 2 and Proposition 6, this holds also when $L^q[\mu]$ and $KS^p[\mu]$ are replaced by $L_w^q[\mu]$ and $KS_w^p[\mu]$, respectively.

(b) If f is (HKL) -integrable, then for each $x' \in X'$ and $k \in \mathbb{N}$, $\mathcal{E}_k f$ is both Henstock–Kurzweil and Lebesgue integrable with respect to $|x'\mu|$, since f is Σ -measurable, and the two integrals coincide, thanks to the (HK) -integrability of the characteristic function χ_E for each $E \in \Sigma$ and the monotone convergence theorem (see also [25,33]). Thus, taking into account (14), for every p with $1 \leq p < \infty$, we have:

$$\begin{aligned} & \sup_{x' \in X', \|x'\| \leq 1} \left[\left(\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right|^p \right)^{1/p} \right] \leq \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{k \in \mathbb{N}} \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right| \right) \\ &= \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{k \in \mathbb{N}} \left| (HK) \int_T \mathcal{E}_k(t) f(t) d|x'\mu| \right| \right) = \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{k \in \mathbb{N}} \left| (HK) \int_{B_k} f(t) d|x'\mu| \right| \right) \\ &\leq \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{A \in \Sigma} \left| (HK) \int_A f(t) d|x'\mu| \right| \right) = \|f\|_{HKL}. \quad \square \end{aligned}$$

The next result deals with the separability of Kuelbs–Steadman spaces, which holds even for $p = \infty$, differently from L^p spaces.

Proposition 7. For $1 \leq p \leq \infty$, the space $KSP[\mu]$ is separable.

Proof. Observe that, by our assumptions, μ is separable, and this is equivalent to the separability of the spaces $L^p[\mu]$ for all $1 \leq p < \infty$ (see also [35] (Proposition 2.3), [22] (Propositions 1A and 3)).

Now, let $\mathcal{H} = \{h_n : n \in \mathbb{N}\}$ be a countable subset of L^1 , dense in $L^1[\mu]$ with respect to the norm $\|\cdot\|_{L^1[\mu]}$. By Theorem 2, $\mathcal{H} \subset KSP[\mu]$. We claim that $\mathcal{H}$ is dense in $KSP[\mu]$. Pick arbitrarily $\varepsilon > 0$ and $f \in KSP[\mu]$. There is $g \in L^1[\mu]$ with $\|g - f\|_{KSP[\mu]} \leq \frac{\varepsilon}{M+1}$. In correspondence with ε and g , there exists $n_0 \in \mathbb{N}$ such that $\|h_{n_0} - g\|_{L^1[\mu]} \leq \frac{\varepsilon}{M+1}$. By (19), $\|\cdot\|_{KSP[\mu]} \leq M\|\cdot\|_{L^1[\mu]}$, and hence:

$$\begin{aligned} \|h_{n_0} - f\|_{KSP[\mu]} &\leq \|h_{n_0} - g\|_{KSP[\mu]} + \|g - f\|_{KSP[\mu]} \leq M\|h_{n_0} - g\|_{L^1[\mu]} + \|g - f\|_{KSP[\mu]} \\ &\leq \frac{M\varepsilon}{M+1} + \frac{\varepsilon}{M+1} = \varepsilon, \end{aligned}$$

getting the claim. $\square$

Now, we prove the following.

Theorem 3. For $1 \leq p, q < \infty$, the embeddings in (19) are completely continuous, namely map weakly convergent sequences in $L^q[\mu]$ into norm convergent sequences in $KSP[\mu]$.

Proof. Pick arbitrarily $1 \leq q < \infty$, and let $(f_n)_n$ be a sequence of elements of $L^q[\mu]$, weakly convergent in $L^q[\mu]$. Then, we get:

$$V = \sup_{n \in \mathbb{N}} \|f_n - f\|_{L^q[\mu]} < +\infty \quad (21)$$

(see also [23] (Proposition 3.5 (iii))) and:

$$\lim_{n \rightarrow +\infty} (KL) \int_T \chi_A(t) (f_n(t) - f(t)) d\mu = 0 \quad \text{for every } A \in \Sigma \quad (22)$$

(see also [14,15]). Now, let us consider the family of operators $W_k : L^q[\mu] \rightarrow X$, $k \in \mathbb{N}$, defined by:

$$W_k(g) = (KL) \int_T \mathcal{E}_k(t) g(t) d\mu, \quad g \in L^q[\mu].$$

It is not difficult to check that W_k is well defined and is a linear operator for every $k \in \mathbb{N}$. Moreover, since $0 \leq \mathcal{E}_k(t) \leq 1$ for all $k \in \mathbb{N}$ and $t \in T$ and taking into account [21] (Theorem 2.1.5 (iii)), for every $g \in L^q[\mu]$, we get:

$$\sup_{x' \in X', \|x'\| \leq 1} \left| (L) \int_T \mathcal{E}_k(t) g(t) d|x'\mu| \right|^q \leq \sup_{x' \in X', \|x'\| \leq 1} \left((L) \int_T |g(t)|^q d|x'\mu| \right) = \|g\|_{L^q[\mu]}^q < +\infty, \quad (23)$$

and hence, $\sup_k \|W_k(g)\|_X < +\infty$. From (23), it follows also that W_k is a continuous operator for every $k \in \mathbb{N}$. From (21) and the uniform boundedness principle, we deduce:

$$+\infty > W = \sup_{k,n} \|W_k(f_n - f)\|_X = \sup_{x' \in X', \|x'\| \leq 1} \left(\sup_{k,n} \left| (L) \int_T \mathcal{E}_k(t) (f_n(t) - f(t)) d|x'\mu| \right| \right). \quad (24)$$

Now, choose arbitrarily $\varepsilon > 0$ and $1 \leq p < \infty$. Note that, by Theorem 2, $f, f_n \in KS^p[\mu]$ for all $n \in \mathbb{N}$. By arguing similarly as in [14] (Appendix 2.3), we find a positive integer K_0 such that $\sum_{k=K_0+1}^{\infty} \eta_k \leq \varepsilon$. Taking into account (9), from (24), we obtain:

$$\sum_{k=K_0+1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) (f_n(t) - f(t)) d|x'\mu| \right|^p \leq \varepsilon W^p \quad (25)$$

for each $n \in \mathbb{N}$ and $x' \in X'$ with $\|x'\| \leq 1$. Moreover, by (22) used with $A = B_k, k = 1, 2, \dots, K_0$, we find a positive integer n^* with:

$$\sum_{k=1}^{K_0} \eta_k \left| (L) \int_T \mathcal{E}_k(t) (f_n(t) - f(t)) d|x'\mu| \right|^p \leq \varepsilon \quad (26)$$

whenever $n \geq n^*$ and $x' \in X', \|x'\| \leq 1$. From (25) and (26), we obtain:

$$\|f_n - f\|_{KS^p[\mu]} = \sup_{x' \in X', \|x'\| \leq 1} \left\{ \left[\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) (f_n(t) - f(t)) d|x'\mu| \right|^p \right]^{1/p} \right\} \leq \varepsilon^{1/p} (1 + W^p)^{1/p}$$

for all $n \geq n^*$. Thus, the sequence $(f_n)_n$ norm converges in $KS^p[\mu]$. This ends the proof. $\square$

Now, we prove that $KS^p[\mu]$ spaces are Banach lattices and Köthe function spaces. First, we recall some properties of such spaces (see also [43,44]).

A partially ordered Banach space Y , which is also a vector lattice, is a Banach lattice if $\|x\| \leq \|y\|$ for every $x, y \in Y$ with $|x| \leq |y|$.

A weak order unit of Y is a positive element $e \in Y$ such that, if $x \in Y$ and $x \wedge e = 0$, then $x = 0$.

Let Y be a Banach lattice and $\emptyset \neq A \subset B \subset Y$. We say that A is solid in B if for each x, y with $x \in B, y \in A$ and $|x| \leq |y|$, it is $x \in A$.

Let λ be an extended real-valued measure on Σ . A Banach space Y consisting of (classes of equivalence of) λ -measurable functions is called a Köthe function space with respect to λ if, for every $g \in Y$ and for each measurable function f with $|f| \leq |g|$ λ , almost everywhere, it is $f \in Y$ and $\|f\| \leq \|g\|$, and $\chi_A \in Y$ for every $A \in \Sigma$ with $\lambda(A) < +\infty$.

Theorem 4. *If $p \geq 1$, then $KS^p[\mu]$ is a Banach lattice with a weak order unit and a Köthe function space with respect to a control measure λ of μ .*

Proof. By the Rybakov theorem (see also [17] (Theorem IX.2.2)), there is $x'_0 \in X'$ with $\|x'_0\| \leq 1$, such that $\lambda = x'_0 \mu$ is a control measure of μ . If $f, g \in KS^p[\mu]$, $|f| \leq |g| \lambda$, almost everywhere, $k \in \mathbb{N}$ and $x' \in X'$ with $\|x'\| \leq 1$, then:

$$\left((L) \int_T \mathcal{E}_k(t) |f(t)| d|x' \mu| \right)^p \leq \left((L) \int_T \mathcal{E}_k(t) |g(t)| d|x' \mu| \right)^p \quad (27)$$

(see also [16] (Proposition 5)), and hence, $\|f\|_{KS^p[\mu]} \leq \|g\|_{KS^p[\mu]}$. By (27), we can deduce that $KS^p[\mu]$ is a Banach lattice, because $KS^p[\mu]$ is the completion of $L^1[\mu]$ with respect to $\|\cdot\|_{KS^p[\mu]}$, $L^1[\mu]$ is a Banach lattice, and the lattice operations are continuous with respect to the norms (see also [44] (Proposition 1.1.6 (i))). Since $L^1[\mu]$ is solid with respect to the space of λ -measurable functions (see also [21]) and the closure of every solid subset of a Banach lattice is solid (see also [44] (Proposition 1.2.3 (i))), arguing similarly as in (27), we obtain that, if f is λ -measurable, $g \in KS^p[\mu]$, and $|f| \leq |g| \mu$, almost everywhere, then $g \in KS^p[\mu]$.

If $A \in \Sigma$, then $\lambda(A) < +\infty$ and $\chi_A \in L^1[\mu]$ (see also [16] (Proposition 5)), and hence, $\chi_A \in KS^p[\mu]$. Therefore, $KS^p[\mu]$ is a Köthe function space.

Finally, we prove that χ_T is a weak order unit of $KS^p[\mu]$. First, note that $\chi_T \in L^p[\mu]$, and hence, $\chi_T \in KS^p[\mu]$. Let $f \in KS^p[\mu]$ be such that $f^* = f \wedge \chi_T = 0 \mu$, almost everywhere. We get:

$$\{t \in T : f^*(t) = 0\} = \{t \in T : f(t) = 0\},$$

and hence, $f = 0 \mu$, almost everywhere. This ends the proof. $\square$

Note that, by the definition of the (KL)-integral, the norm defined in (11) corresponds, in a certain sense, to the topology associated with the norm convergence of the integrals (μ -topology; see also [14] (Theorem 2.2.2)). However, with this norm, it is not natural to define an inner product in the space KS^2 , since m is vector-valued.

On the other hand, when X' is separable and $\{x'_h : h \in \mathbb{N}\}$ is a countable dense subset of X' , with $\|x'_h\| \leq 1$ for every h , it is possible to deal with the topology related to the weak convergence of integrals (weak μ -topology; see also [14] (Proposition 2.1.1)), whose corresponding norm is given by:

$$\|f\|_{KS^p[w\mu]} = \begin{cases} \left[\sum_{h=1}^{\infty} \omega_h \left(\sum_{k=1}^{\infty} \eta_k \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'_h \mu| \right|^p \right) \right]^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{h \in \mathbb{N}} \left[\sup_{k \in \mathbb{N}} \left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'_h \mu| \right| \right] & \text{if } p = \infty, \end{cases} \quad (28)$$

where $\mathcal{E}_k, k \in \mathbb{N}$, is as in (11) and $(\eta_k)_k, (\omega_h)_h$ are two fixed sequences of strictly positive real numbers, such that $\sum_{k=1}^{\infty} \eta_k = \sum_{h=1}^{\infty} \omega_h = 1$. Note that, in general, a weak μ -topology does not coincide with a μ -topology, but there are some cases in which they are equal (see also [16] (Theorem 14)). Analogously, in Proposition 5, it is possible to prove the following:

Proposition 8. For each $f \in L^1[\mu]$ and $p \geq 1$, it is:

$$\|f\|_{KS^p[w\mu]} \leq \|f\|_{KS^\infty[w\mu]}. \quad (29)$$

Now, we give the next fundamental result.

Theorem 5. The map $f \mapsto \|f\|_{KS^p[w\mu]}$ defined in (28) is a norm.

Proof. First of all, note that $\|f\|_{KS^p[\mu]} \geq 0$ for any $f \in L^1[\mu]$. Let $f \in L^1[\mu]$ be such that $\|f\|_{KS^p[\mu]} = 0$. We prove that $f = 0$ μ , almost everywhere. It will be enough to prove the assertion for $1 \leq p < \infty$, since the case $p = \infty$ follows from (29). Arguing analogously as in (15), we get:

$$\left| (L) \int_T \mathcal{E}_k(t) f(t) d|x'_h \mu|(t) \right| = 0 \quad \text{for every } h, k \in \mathbb{N}.$$

By contradiction, suppose that $f \neq 0$ μ , almost everywhere. If $E^+ = f^{-1}(]0, +\infty[)$, $E^- = f^{-1}(]-\infty, 0[)$, then $E^+, E^- \in \Sigma$, since f is Σ -measurable, and we have $\mu(E^+) \neq 0$ or $\mu(E^-) \neq 0$. Suppose that $\mu(E^+) \neq 0$. By the Hahn–Banach theorem, there is $x'_0 \in X'$ with $\|x'_0\| \leq 1$, $x'_0 \mu(E^+) \neq 0$, and hence, $|x'_0 \mu(E^+)| > 0$. Since the set $\{x'_h : h \in \mathbb{N}\}$ is dense in X' with respect to the norm of X' , there is a positive integer h_0 with:

$$|x'_{h_0} \mu(E^+)| > 0. \quad (30)$$

Without loss of generality, we can assume $\|x'_{h_0}\| \leq 1$. Now, it is enough to proceed analogously as in Theorem 1, by replacing the linear continuous functional x'_0 in (18) with the element x'_{h_0} found in (30), by finding another element $x'_{h_1} \in X'$ with $|x'_{h_1} \mu(E^-)| > 0$, and by arguing again as in (18).

The triangular property of the norm is straightforward for $p = \infty$ and for $1 \leq p < \infty$ is a consequence of the inequality:

$$\begin{aligned} \left[\sum_{h=1}^{\infty} \omega_h \left(\sum_{k=1}^{\infty} \eta_k |b_{k,h} + c_{k,h}|^p \right) \right]^{1/p} &\leq \left[\sum_{h=1}^{\infty} \omega_h \left(\sum_{k=1}^{\infty} \eta_k (|b_{k,h}| + |c_{k,h}|)^p \right) \right]^{1/p} \\ &\leq \left[\sum_{h=1}^{\infty} \omega_h \left(\sum_{k=1}^{\infty} \eta_k |b_{k,h}|^p \right) \right]^{1/p} + \left[\sum_{h=1}^{\infty} \omega_h \left(\sum_{k=1}^{\infty} \eta_k |c_{k,h}|^p \right) \right]^{1/p} \end{aligned} \quad (31)$$

which holds whenever $(b_{k,h})_{k,h}, (c_{k,h})_{k,h}$ are two double sequences of real numbers and $(\eta_k)_k, (\omega_h)_h$ are two sequences of positive real numbers, such that $\sum_{h=1}^{\infty} \omega_h = \sum_{k=1}^{\infty} \eta_k = 1$. The inequality in (31), as that in (10), follows from Minkowski's inequality. The other properties are easy to check. $\square$

Now, in correspondence with the norm defined in (28), we define the following bilinear functional $\langle \cdot, \cdot \rangle : L^1[\mu] \times L^1[\mu] \rightarrow \mathbb{R}$ by:

$$\langle f, g \rangle_{KS^2[w\mu]} = \sum_{h=1}^{\infty} \omega_h \left[\sum_{k=1}^{\infty} \eta_k \left((L) \int_T \mathcal{E}_k(t) f(t) d|x'_h \mu|(t) \right) \left((L) \int_T \mathcal{E}_k(s) g(s) d|x'_h \mu|(s) \right) \right]. \quad (32)$$

Arguing similarly as in Theorem 5, it is possible to see that the functional $\langle \cdot, \cdot \rangle_{KS^2[w\mu]}$ in (32) is an inner product, and:

$$\|\cdot\|_{KS^2[w\mu]} = (\langle \cdot, \cdot \rangle_{KS^2[w\mu]})^{1/2}.$$

For $1 \leq p \leq \infty$, the Kuelbs–Steadman space $KS^p[w\mu]$ is the completion of $L^1[\mu]$ with respect to the norm defined in (28). Observe that, using Proposition 3, we can see that:

$$\|\cdot\|_{KS^p[w\mu]} \leq \|\cdot\|_{KS^p[\mu]} \text{ and } \|\cdot\|_{KS^p[w\mu]} \leq \|\cdot\|_{HKL} \text{ for } 1 \leq p \leq \infty.$$

As in Theorems 2 and 3, it is possible to prove the following:

Theorem 6. For each p, q with $1 \leq p, q \leq \infty$, it is $L^q[\mu] \subset KS^p[w\mu]$ with continuous and dense embedding, and the space of all Σ -simple functions is dense in $KS^p[w\mu]$. Moreover, if $1 \leq p, q < \infty$, the embedding is completely continuous. Furthermore, $KS^p[w\mu]$ is a separable Banach lattice with a weak order unit and a Köthe function space with respect to a control measure λ of μ .

Since $(KS^2[w\mu], \langle \cdot, \cdot \rangle_{KS^2[w\mu]})$ is a separable Hilbert space, by applying [2] (Theorems 5.15 and 8.7), it is possible to consider operators like, for instance, convolution and Fourier transform and to extend the theory there studied to the context of vector-valued measures (see also [45], [2] (Remark 5.16)).

4. Conclusions

We introduced Kuelbs–Steadman spaces related to the integration for scalar-valued functions with respect to a σ -additive measure μ , taking values in a Banach space X . We endowed them with the structure of the Banach space, both in connection with the norm convergence of integrals and in connection with the weak convergence of integrals ($KS^p[\mu]$ and $KS^p[w\mu]$, respectively). A fundamental role is played by the separability of μ . We proved that these spaces are separable Banach lattices and Köthe function spaces. Moreover, we saw that the embeddings of $L^q[\mu]$ into $KS^p[\mu]$ ($KS^p[w\mu]$) are continuous and dense, and also completely continuous when $1 \leq p, q < \infty$. When X' is separable, we endowed $KS^2[w\mu]$ with an inner product. In this case, $KS^2[w\mu]$ is a separable Hilbert space, and hence, it is possible to deal with operators like convolution and Fourier transform and to extend to Banach space-valued measures the theory investigated in [1–3].

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References

- Gill, T. L.; Myers, T. Constructive Analysis on Banach Spaces. *Real Anal. Exch.* **2019**, *44*, 1–36. [\[CrossRef\]](#)
- Gill, T.L.; Zachary, W.W. *Functional Analysis and the Feynman Operator Calculus*; Springer: New York, NY, USA, 2016.
- Kalita, H.; Hazarika, B.; Myers, T. Construction of the KS^p spaces on $\mathbb{R}^\infty$ and separable Banach spaces. *arXiv* **2020**, arXiv:2002.11512v2.
- Kuelbs, J. Gaussian measures on a Banach space. *J. Funct. Anal.* **1970**, *5*, 354–367. [\[CrossRef\]](#)
- Boccuto, A.; Candeloro, D.; Sambucini, A.R. Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures. *Rend. Lincei Mat. Appl.* **2015**, *26*, 363–383. [\[CrossRef\]](#)
- Candeloro, D.; Croitoru, A.; Gavrilut, A.; Sambucini, A.R. An extension of the Birkhoff integrability for multifunctions. *Mediterr. J. Math.* **2016**, *13*, 2551–2575. [\[CrossRef\]](#)
- Candeloro, D.; Di Piazza, L.; Musiał, K.; Sambucini, A.R. Gauge integrals and selections of weakly compact valued multifunctions. *J. Math. Anal. Appl.* **2016**, *441*, 293–308. [\[CrossRef\]](#)
- Candeloro, D.; Di Piazza, L.; Musiał, K.; Sambucini, A.R. Some new results on integration for multifunctions. *Ric. Mat.* **2018**, *67*, 361–372. [\[CrossRef\]](#)
- Candeloro, D.; Sambucini, A.R. Comparison between some norm and order gauge integrals in Banach lattices. *PanAm. Math. J.* **2015**, arXiv:1503.04968.
- Caponetti, D.; Marraffa, V.; Naralnikov, K. On the integration of Riemann-measurable vector-valued functions. *Monatsh. Math.* **2017**, *182*, 513–536. [\[CrossRef\]](#)
- Di Piazza, L.; Marraffa, V.; Musiał, K. Variational Henstock integrability of Banach space valued functions. *Math. Bohem.* **2016**, *141*, 287–296. [\[CrossRef\]](#)
- Di Piazza, L.; Musiał, K. Set valued Kurzweil–Henstock–Pettis integral. *Set-Valued Anal.* **2005**, *13*, 167–169. [\[CrossRef\]](#)
- Di Piazza, L.; Musiał, K. Decompositions of Weakly Compact Valued Integrable Multifunctions. *Mathematics* **2020**, *8*, 863. [\[CrossRef\]](#)

14. Ferrando Palomares, I. Duality in Spaces of p -Integrable Functions with Respect to a Vector Measure. Ph.D. Thesis, Universidad Politécnica de Valencia, Valencia, Spain, 2009.
15. Okada, S. The dual space of $\mathcal{L}^1(\mu)$ for a vector measure μ . *J. Math. Anal. Appl.* **1993**, *177*, 583–599. [[CrossRef](#)]
16. Sánchez-Pérez, E.A. Compactness arguments for spaces of p -integrable functions. *Ill. J. Math.* **2001**, *45*, 907–923.
17. Diestel, J.; Uhl, J.J. *Vector Measures*; American Mathematical Society: Providence, RI, USA, 1977.
18. Curbera, G.P.; Ricker, W.J. Vector Measures, Integration and Applications. In *Trends in Mathematics, Positivity*; Birkhäuser: Basel, Switzerland, 2007; pp. 127–160.
19. Kluvánek, I.; Knowles, G. *Vector Measures and Control Systems*; Mathematics Studies; North Holland Publ. Co.: Amsterdam, The Netherlands, 1975.
20. Lewis, D.R. Integration with respect to vector measures. *Pac. J. Math.* **1970**, *33*, 157–165. [[CrossRef](#)]
21. Panchapagesan, T.V. *The Bartle-Dunford-Schwartz Integral—Integration with Respect to a Sigma-Additive Vector Measure*; Birkhäuser: Basel, Switzerland; Boston, MA, USA; Berlin, Germany, 2008.
22. Ricker, W. Separability of the L^1 -space of a vector measure. *Glasgow Math. J.* **1992**, *34*, 1–9. [[CrossRef](#)]
23. Brézis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; Springer: New York, NY, USA; Dordrecht, The Netherlands; Heidelberg, Germany; London, UK, 2011.
24. Džamonja, M.; Kunen, K. Properties of the class of measure separable compact spaces. *Fund. Math.* **1995**, *147*, 261–277.
25. Boccuto, A.; Riečan, B.; Vrabelová, M. *Kurzweil–Henstock Integral in Riesz Spaces*; Bentham Sci. Publ.: Sharjah, UAE, 2009.
26. Candeloro, D.; Sambucini, A.R. Order-type Henstock and Mc Shane integrals in Banach lattice setting. In Proceedings of the SISY 2014—IEEE 12th International Symposium on Intelligent Systems and Informatics, Subotica, Serbia, 11–13 September 2014; pp. 55–59.
27. Cao, S. The Henstock integral for Banach-valued functions. *SEA Bull. Math.* **1992**, *16*, 35–40.
28. Federson, M. Some Peculiarities of the HK-integrals of Banach space-valued functions. *Real Anal. Exch.* **2004**, *29*, 439–460. [[CrossRef](#)]
29. Fremlin, D.H.; Mendoza, J. On the integration of vector-valued functions. *Ill. J. Math.* **1994**, *38*, 127–147. [[CrossRef](#)]
30. Gordon, R. *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*; American Mathematical Society: Providence, RI, USA, 1994.
31. Lee, P.Y.; Výborný, R. *The Integral: An Easy Approach after Kurzweil and Henstock*; Cambridge University Press: Cambridge, UK, 2000.
32. Pfeffer, W.F. The multidimensional fundamental theorem of Calculus. *J. Austral. Math. Soc. (Ser. A)* **1987**, *43*, 143–170. [[CrossRef](#)]
33. Riečan, B. On the Kurzweil integral in compact topological spaces. *Radovi Mat.* **1986**, *2*, 151–163.
34. Curbera, G.P. Banach space properties of L^1 of a vector measure. *Proc. Am. Math. Soc.* **1995**, *123*, 3797–3806.
35. Fernández, A.; Mayoral, F.; Naranjo, F.; Sáez, C.; Sánchez-Pérez, E.A. Spaces of p -integrable functions with respect to a vector measure. *Positivity* **2006**, *10*, 1–6. [[CrossRef](#)]
36. Stefansson, G.F. L^1 of a vector measure. *Le Matematiche* **1993**, *48*, 219–234.
37. Boccuto, A.; Candeloro, D.; Sambucini, A.R. L^p -spaces in vector lattices and applications. *Math. Slovaca* **2017**, *67*, 1409–1426. [[CrossRef](#)]
38. Alexiewicz, A. Linear functionals on Denjoy integrable functions. *Colloq. Math.* **1948**, *1*, 289–293. [[CrossRef](#)]
39. Swartz, C. *Introduction to Gauge Integral*; World Scientific Publ.: Singapore, 2004.
40. McLeod, R.M. *The Generalized Riemann Integral*; Carus Math. Monograph 20; Mathematical Association of America: New York, NY, USA, 1980.
41. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1934.
42. Kantorovich, L.A.; Akilov, G.P. *Functional Analysis*, 2nd ed.; Pergamon Press: Oxford, UK, 1982.
43. Lindenstrauss, J.; Tzafriri, L. *Classical Banach Spaces II, Function Spaces*; A Series of Modern Surveys in Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1979.

- 44. Meyer-Nieberg, P. *Banach Lattices*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1991.
- 45. Delgado, O.; Miana, P.J. Algebra structure for L^p of a vector measure. *J. Math. Anal. Appl.* **2009**, *358*, 355–363. [\[CrossRef\]](#)



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