

## Article

# Modified King's Family for Multiple Zeros of Scalar Nonlinear Functions

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**Abstract:** There is no doubt that there is plethora of optimal fourth-order iterative approaches available to estimate the simple zeros of nonlinear functions. We can extend these method/methods for multiple zeros but the main issue is to preserve the same convergence order. Therefore, numerous optimal and non-optimal modifications have been introduced in the literature to preserve the order of convergence. Such count of methods that can estimate the multiple zeros are limited in the scientific literature. With this point, a new optimal fourth-order scheme is presented for multiple zeros with known multiplicity. The proposed scheme is based on the weight function strategy involving functions in ratio. Moreover, the scheme is optimal as it satisfies the hypothesis of Kung–Traub conjecture. An exhaustive study of the convergence is shown to determine the fourth order of the methods under certain conditions. To demonstrate the validity and appropriateness for the proposed family, several numerical experiments have been performed. The numerical comparison highlights the effectiveness of scheme in terms of accuracy, stability, and CPU time.

**Keywords:** scalar nonlinear equations; multiple zeros; king's family; optimal method; Kung–Traub conjecture

## 1. Introduction

With the rapid growth of the numerical field, various physical and technical applications [1–3] are justifying the importance for solving the nonlinear equations. Such problems arise in various fields of natural and physical sciences, including the heat and fluid flow problems, initial and boundary value problems, as well as problems associated with global positioning systems (GPS). For retrieving the solution through an analytical approach is almost inconceivable for any nonlinear equation except for some of them. Thus, iterative approaches provide an attractive alternative for solving these kinds of problems. While discussing about the root finding of nonlinear equation of the form  $f(x) = 0$ , where  $f(x)$  is real function defined in a domain  $D \subseteq \mathbb{R}$ , we pictured the classical Newton's method and for multiple roots, the modified Newton method [4–6] (also known as Rall's method was introduced by E. Schröder in 1870) in mind. The modified Newton method is given by

$$x_{t+1} = x_t - m \frac{f(x_t)}{f'(x_t)}. \quad (1)$$

Equation (1) converges quadratically for multiple roots with given multiplicity  $m \geq 1$ . Graphically, the sketch of multiple root is visualized from Figure 1. While, there are several one-point

iterative approaches accessible in the literature, but they are not of practical relevance when presented from a real context, because of their theoretical shortcomings on convergence order and efficiency index. Moreover, most of the one-point approaches are computationally expensive and inefficient when evaluated on academic problems that arise from real life. Multipoint iterative approaches are therefore better choices to classify as appropriate solvers. One of the advantages of multipoint iterative methods without memory for scalar nonlinear equations is that we have a conjecture about their convergence order. For any multipoint method, requiring  $t$  functional evaluation can have at most  $2^{t-1}$  convergence order, according to the hypothesis of Kung–Traub conjecture [5]. For instance, modified Newton method evaluating function at two points, and it reaches to the order  $2^{2-1}$ , for given  $m \geq 1$ . Hence, the modified Newton method is optimal in the sense of Kung–Traub Conjecture. Thus, a board community of researchers suggested several optimal [7–19] and non-optimal [20,21] multipoint iterative methods for estimating the multiple zeros of a function on the basis of Kung–Traub conjecture. For instance, Li et al. [9] investigated on the fourth-order scheme for calculating the multiple roots of an equation as follows:

$$\begin{cases} y_t = x_t - \frac{2m}{m+2} \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = x_t - b_1 \frac{f(x_t)}{f'(x_t)} - \frac{f(x_t)}{b_2 f'(x_t) + b_3 f'(y_t)}, \end{cases} \quad (2)$$

where  $b_1 = m - \frac{m^2}{2}$ ,  $b_2 = -\frac{1}{m}$ ,  $b_3 = \frac{1}{m(\frac{m}{m+2})^m}$ , where  $m$  denotes the multiplicity of the desired zero of given function  $f$ .

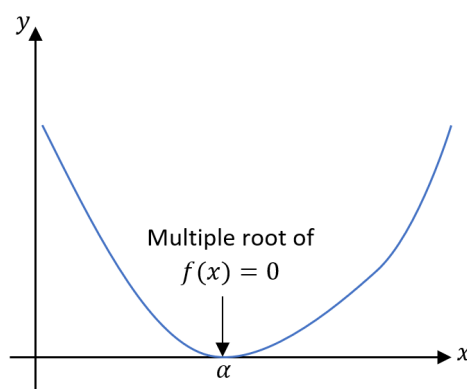


Figure 1. Multiple root  $\alpha$  of  $f(x) = 0$ .

Sharma and Sharma [11] suggested the multipoint iterative method of order four, defined below:

$$\begin{cases} y_t = x_t - \frac{2m}{m+2} \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = x_t - a_1 w_1(x_t) - a_2 w_2(x_t) - a_3 \frac{w_2^2(x_t)}{w_1(x_t)}, \end{cases} \quad (3)$$

where  $w_1(x_t) = \frac{f(x_t)}{f'(x_t)}$ ,  $w_2(x_t) = \frac{f(x_t)}{f'(y_t)}$ ,  $a_1 = \frac{m}{8}(m^3 - 4m + 8)$ ,  $a_2 = -\frac{1}{4}(\frac{m}{m+2})^{m+1}(m-1)(m+2)^3$ , and  $a_3 = \frac{1}{8}(m+2)^4(\frac{m}{m+2})^{2m+1}$ .

Moreover, Zhou et al. [12] developed the multipoint iterative method based on the weight function, and one of the particular form is:

$$\begin{cases} y_t = x_t - \frac{2m}{m+2} \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = x_t - \frac{m}{8} \left[ m^3 \left( \frac{m+2}{m} \right)^{2m} \left( \frac{f'(y_t)}{f'(x_t)} \right)^2 - 2m^2(m+3) \left( \frac{m+2}{m} \right)^m \frac{f'(y_t)}{f'(x_t)} \right. \\ \left. + (m^3 + 6m^2 + 8m + 8) \right] \frac{f(x_t)}{f'(x_t)}. \end{cases} \quad (4)$$

Recently, Behl and Hamdan [22] focused on the extension of Ostrowski's methods for finding the multiple zeros, and given as:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = x_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1-u}{1-2u} \right] Q(u), \end{cases} \quad (5)$$

where  $u = \left( \frac{f(y_t)}{f(x_t)} \right)^{\frac{1}{m}}$  and  $Q(u)$  is an weight function.

In literature, various researchers analyzed the variants of King's family for solving the scalar nonlinear equation with multiplicity  $m = 1$ . Recently, Behl and Hamdan [22] extended the Ostrowski's method for multiple zero of a function. Whereas Sharma and Sharma [11] focused on the Jarratt's method and modified it for computing the multiple roots. Till now for multiple zero function, the King's family were not introduced in literature.

It is a challenging problem in the field of numerical analysis, to construct an optimal scheme of King's family for approximating the multiple zero of a function. Thus, motivating from this idea, we have made an attempt to extend the King's family [23] to optimal multipoint iterative method for obtaining the desire multiple zero of an input function. For this, we used the weight function technique. Furthermore, we have shown that the new method illustrates the good coordination with the numerical section, as it offers the smaller residual errors while estimating the multiple zeros of a function.

The manuscript is organized as follows: Section 2 first introduces the construction of new fourth-order scheme in general framework and then its theoretical analysis is provided. Moreover, in Section 3, several special cases are included, depending on the different weight functions used in the developed family. Whereas, Section 4 is confined to the numerical experiments that highlight the scheme's effectiveness, accuracy, and stability on some intricate real-life problems. Section 5, presents the summary and conclusions.

## 2. Construction of the New Fourth-Order Multipoint Iterative Scheme

Consider the suggested iterative scheme as:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] u_t Q(u_t), \end{cases} \quad (6)$$

where,  $\beta \in \mathbb{R}$ , the weight function  $Q(u_t) : \mathbb{C} \rightarrow \mathbb{C}$  is an analytic/holomorphic map [24,25] in the neighborhood of origin (0) with  $u_t = \left( \frac{f(y_t)}{f(x_t)} \right)^{\frac{1}{m}}$ . The  $u_t$  is a multi-valued function. Considering the principal analytic branches of  $u_t$ , as it is a multi-valued function. Moreover, it reduces the labour-period, by considering  $u_t$  as a principal root known as  $u_t = \exp \left[ \frac{1}{m} \text{Log} \left( \frac{f(y_t)}{f(x_t)} \right) \right]$ , with  $\text{Log} \left( \frac{f(y_t)}{f(x_t)} \right) = \text{Log} \left| \frac{f(y_t)}{f(x_t)} \right| + i \text{Arg} \left( \frac{f(y_t)}{f(x_t)} \right)$  for  $-\pi < \text{Arg} \left( \frac{f(y_t)}{f(x_t)} \right) \leq \pi$ , where  $\text{Arg}(z)$  is denotes the principal argument of  $z$ .

Without evaluating the new functional value, Theorem 1 shows that the proposed scheme converges with at least fourth-order of convergence. Moreover, the weight function is restricted under certain conditions and discussed in following theorem.

**Theorem 1.** Consider the function  $f : \mathbb{D} \subseteq \mathbb{C} \rightarrow \mathbb{C}$  is an analytic inside the region  $\mathbb{D}$ , enclosing the desired zero (say  $x = \alpha_m$ ) of the function  $f$  with the multiplicity  $m \geq 1$ . Then, the proposed strategy in Equation (6) attains the fourth-order of convergence provided it employ the following criteria:

$$\mathcal{Q}(0) = 1, \quad \mathcal{Q}'(0) = 0, \quad |\mathcal{Q}''(0)| < \infty. \quad (7)$$

Moreover, the following error expression is satisfied by the new scheme in Equation (6):

$$e_{t+1} = \left( \frac{(1 + 4\beta + m - \mathcal{Q}''(0))c_1^3 - 2mc_1c_2}{2m^3} \right) e_t^4 + O(e_t^5), \quad (8)$$

where error at  $t^{\text{th}}$  step  $e_t = x_t - \alpha_m$  and  $c_k = \frac{m!}{(m+k)!} \frac{f^{(m+k)}(\alpha_m)}{f^{(m)}(\alpha_m)}$ ,  $k = 1, 2, 3, 4$ .

**Proof.** Let us assume that the scheme in Equation (6) has a multiple zero,  $x = \alpha_m$  with the known multiplicity  $m$ , greater than or equal to one. Expanding  $f(x_t)$  and  $f'(x_t)$  about a point  $x = \alpha_m$  via the Taylor series expansion as follow:

$$f(x_t) = \frac{f^{(m)}(\alpha_m)}{m!} e_t^m \left( 1 + \sum_{i=1}^4 c_i e_t^i + O(e_t^5) \right), \quad (9)$$

and

$$f'(x_t) = \frac{f^{(m)}(\alpha_m)}{(m-1)!} e_t^{m-1} \left( 1 + \sum_{i=1}^4 \frac{m+i}{m} c_i e_t^i + O(e_t^5) \right), \quad (10)$$

respectively.

Further, we have calculated the following expressions, with the use of Equations (9) and (10),

$$\frac{f(x_t)}{f'(x_t)} = \frac{1}{m} e_t - \frac{c_1}{m^2} e_t^2 + \frac{(1+m)c_1^2 - 2mc_2}{m^3} e_t^3 + O(e_t^4), \quad (11)$$

and

$$\begin{aligned} f(y_t) &= f\left(x_t - m \frac{f(x_t)}{f'(x_t)}\right), \\ &= \frac{f^{(m)}(\alpha_m)}{m!} \left(\frac{c_1}{m}\right)^m e_t^{2m} \left[ 1 + \frac{2mc_2 - (m+1)c_1^2}{c_1} e_t \right. \\ &\quad \left. + \frac{(m^3 + 3m^2 + 3m + 3)c_1^5 - 2m(2m^2 + 3m + 2)c_1^3c_2 + 4m^2(m-1)c_1c_2^2 + 6m^2c_1^2c_3}{2mc_1^3} e_t^2 + O(e_t^3) \right]. \end{aligned} \quad (12)$$

Using Equations (9) and (12), we obtain

$$\begin{aligned} u_t &= \left( \frac{f(y_t)}{f(x_t)} \right)^{\frac{1}{m}} = \frac{c_1}{m} e_t + \frac{2mc_2 - (m+2)c_1^2}{m^2} e_t^2 \\ &\quad + \frac{(2m^2 + 7m + 7)c_1^3 - 2m(3m+7)c_1c_2 + 6m^2c_3}{2m^3} e_t^3 + O(e_t^4). \end{aligned} \quad (13)$$

and

$$\left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] u_t = \frac{c_1}{m} e_t - \frac{c_1^2 - 2c_2}{m} e_t^2 + \frac{(2m^2 - m - 4\beta - 1)c_1^3 + 2m(1 - 3m)c_1c_2 + 6m^2c_3}{2m^3} e_t^3 + O(e_t^4). \quad (14)$$

We conclude from Equation (13) that  $u_t$  having linear order in  $e_t$ . Therefore, expanding  $\mathcal{Q}(u_t)$  in the neighborhood of origin (0) as follow:

$$\begin{aligned}\mathcal{Q}(u_t) &= \mathcal{Q}(0) + u_t \mathcal{Q}'(0) + \frac{1}{2!} u_t^2 \mathcal{Q}''(0) + \frac{1}{3!} u_t^3 \mathcal{Q}'''(0) + O(e_t^4) \\ &= \mathcal{Q}(0) + \frac{c_1 \mathcal{Q}'(0)}{m} e_t + \frac{c_1^2 (\mathcal{Q}''(0) - 2(m+2) \mathcal{Q}'(0)) + 4c_2 m \mathcal{Q}'(0)}{2m^2} e_t^2 \\ &\quad + \frac{c_1^3 (3(2m^2 + 7(m+1)) \mathcal{Q}'(0) - 6(m+2) \mathcal{Q}''(0) + \mathcal{Q}'''(0)) - 6c_1 c_2 m ((3m+7) \mathcal{Q}'(0) - 2\mathcal{Q}''(0)) + 18c_3 m^2 \mathcal{Q}'(0)}{6m^3} e_t^3 \\ &\quad + O(e_t^4),\end{aligned}\tag{15}$$

by adopting Taylor series expansion.

By substituting Equations (11), (14) and (15) in the scheme in Equation (6), we have

$$\begin{aligned}e_{t+1} &= y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] u_t \mathcal{Q}(u_t) \\ &= \frac{(1 - \mathcal{Q}(0))c_1}{m} e_t^2 + \frac{c_1^2 (m(\mathcal{Q}(0) - 1) + \mathcal{Q}(0) - \mathcal{Q}'(0) - 1) - 2c_2 m (\mathcal{Q}(0) - 1)}{m^2} e_t^3 \\ &\quad + \frac{1}{2m^3} \left[ c_1^3 (2m^2 (\mathcal{Q}(0) - 1) + m(3\mathcal{Q}(0) - 4\mathcal{Q}'(0) - 4) + (1 - 4\beta) \mathcal{Q}(0) - 6\mathcal{Q}'(0) + \mathcal{Q}''(0) - 2) \right. \\ &\quad \left. - 2c_1 c_2 m (3m(\mathcal{Q}(0) - 1) + 3\mathcal{Q}(0) - 4\mathcal{Q}'(0) - 4) + 6c_3 m^2 (\mathcal{Q}(0) - 1) \right] e_t^4 + O(e_t^5).\end{aligned}\tag{16}$$

For attaining the fourth-order of convergence by the presented method in Equation (6), the coefficients of terms  $e_t^2$ , and  $e_t^3$  should be zero at the same time. Thus, from Equation (16) the following system of equations is obtained that involve  $\mathcal{Q}(0)$ , and its first derivative  $\mathcal{Q}'(0)$ :

$$\begin{cases} 1 - \mathcal{Q}(0) = 0, \\ \mathcal{Q}'(0) = 0, \end{cases}\tag{17}$$

which further yields the following restriction on weight function  $\mathcal{Q}$ :

$$\mathcal{Q}(0) = 1, \quad \mathcal{Q}'(0) = 0.\tag{18}$$

By substituting Equation (18) in Equation (16), the following error equation and asymptotic constant is satisfied by scheme defined in Equation (6):

$$e_{t+1} = \frac{(1 + 4\beta + m - \mathcal{Q}''(0))c_1^3 - 2mc_1 c_2}{2m^3} e_t^4 + O(e_t^5),\tag{19}$$

where  $|\mathcal{Q}''(0)| < \infty$ .

Thus, the Equation (19) justifies that the presented family in Equation (6) reaches to the optimal convergence order by just evaluating the 3 functional ( $f(x)$ ,  $f(y)$ , and  $f'(x)$ ) at each iterate. This completes the proof.  $\square$

### 3. Variants of New Family in Equation (6)

It is straightforward to have from Theorem that one can get modified strategies of King's family by employing some particular values of  $\mathcal{Q}(u_t)$  and by introducing various forms of weight functions.

**Case 1.** Considering the following polynomial weight function:

$$\mathcal{Q}(u_t) = Au_t^3 + 1,\tag{20}$$

where  $\mathcal{Q}''(0) = 6A$ , where  $|A| < \infty$ .

In view of Equation (20) and scheme in Equation (6), the new fourth-order optimal family is obtained as follows:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] [Au_t^4 + u_t]. \end{cases} \quad (21)$$

**Some of the sub-special cases for Equation (21)**

(i) When  $\beta = 0$ , and  $A = 0$ , the proposed scheme in Equation (21) read as:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1}{1 - 2u_t} \right] u_t. \end{cases} \quad (22)$$

(ii) For  $\beta = -1$ , and  $A = \frac{1}{2}$ , the family in Equation (21) becomes:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - \frac{m}{2} \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 - u_t}{1 - 3u_t} \right] [u_t^4 + 2u_t]. \end{cases} \quad (23)$$

The above Equation (23) is another particular case of the scheme in Equation (6).

**Case 2.** Choosing the rational weight function as defined below:

$$\mathcal{Q}(u_t) = \frac{1 + A_1 u_t + A_2 u_t^2}{1 + A_1 u_t}, \text{ provided } A_1 u_t \neq -1, \quad (24)$$

and where  $\mathcal{Q}''(0) = 2A_2$ , here  $A_2$  is any finite real number. Adopting the weight function in Equation (24) in the proposed scheme in Equation (6), we have

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] \left[ \frac{u_t(1 + A_1 u_t + A_2 u_t^2)}{1 + A_1 u_t} \right], \end{cases} \quad (25)$$

which is another new type of multipoint family.

**Some of the sub-special cases for Equation (25)**

(i) For  $\beta = \frac{1}{2}$ ,  $A_1 = \frac{1}{10}$ , and  $A_2 = 2$ , the scheme in Equation (25) reads as

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{-u_t(u_t + 2)(20u_t^2 + u_t + 10)}{(u_t + 10)(3u_t - 2)} \right]. \end{cases} \quad (26)$$

In this way, we obtain another particular form of fourth-order optimal iterative technique.

(ii) For  $\beta = -2$ ,  $A_1 = 6.45$ , and  $A_2 = -10$ , the family in Equation (25) leads us

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{u_t(400u_t^3 - 458u_t^2 + 89u_t + 20)}{-516u_t^2 + 49u_t + 20} \right], \end{cases} \quad (27)$$

a new optimal 4th-order iterative scheme.

**Case 3.** Consider the another rational function that defines the weight function as:

$$Q(u_t) = \frac{1 + A_1 u_t}{1 + A_1 u_t + A_2 u_t^2}, \quad (28)$$

where  $Q''(0) = -2A_2$ , where  $A_2$  is the finite real value.

By substituting Equation (28) in Equation (6), the new optimal family of 4th-order can obtained as follows:

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{1 + \beta u_t}{1 + (\beta - 2)u_t} \right] \left[ \frac{u_t(1 + A_1 u_t)}{1 + A_1 u_t + A_2 u_t^2} \right]. \end{cases} \quad (29)$$

**Some of the sub-special cases for the scheme in Equation (29)**

(i) For  $\beta = -\frac{1}{4}$ ,  $A_1 = 0$ , and  $A_2 = \frac{1}{10}$ , the family in Equation (6) provides us the special case of Equation (29)

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \left[ \frac{10u_t(u_t - 4)}{(9u_t - 4)(u_t^2 + 10)} \right]. \end{cases} \quad (30)$$

(ii) For  $\beta = 10$ ,  $A_1 = 4$ , and  $A_2 = -21$ , the family in Equation (29) read as

$$\begin{cases} y_t = x_t - m \frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} = y_t - m \frac{f(x_t)}{f'(x_t)} \frac{u_t(4u_t + 1)(10u_t + 1)}{(8u_t + 1)(-21u_t^2 + 4u_t + 1)}, \end{cases} \quad (31)$$

is another new fourth-order optimal multipoint method.

#### 4. Numerical Testing and Discussions

This section is aimed to confirm the theoretical aspects by numerical examination. For this, an attempt is made to demonstrate the comparison of the new approach to practical and academic structures with the existing strategies. For justifying the proposed scheme in Equation (6), we have compared our new schemes defined in Equation (26), and Equation (30) denoted by MM1, and MM2, and compared with the existing methods defined in Equations (2)–(4), denoted by LM, SM, and ZM, respectively. Moreover, the results are compared with the expression in Equation (5), (for equation number RM (32) of article [22]).

The numerical outcomes are displayed in Tables 1–6, by comparing our techniques with existing methods in terms of approximate zeros ( $x_t$ ), absolute residual error of the considered function ( $|f(x_t)|$ ), absolute difference in two successive approximations  $|x_{t+1} - x_t|$ , computational order of convergence  $\rho \approx \frac{\log |f(x_{t+1})/f(x_t)|}{\log |f(x_t)/f(x_{t-1})|}$ ,  $t \geq 2$ , (see [26,27]), at the last iteration indices ( $t$ ), and computational time (Time)

in seconds. We have maintained 2000 significant digits of minimum precision to minimize the round off error.

As mentioned within the above description, we determine the value of all the functional residual and the constants till 2000 significant digits, however, we have displayed the value of obtained approximated zero of the function up-to twenty-five significant digits. Whereas, the absolute error in the successive approximations  $|x_{t+1} - x_t|$  and the residual error  $|f(x_t)|$  exhibited till two significant digits along with the exponent power. Moreover, the computational convergence order is shown up-to five digits. The results are obtained with the help of *Mathematica* software (version 11.1, Wolfram Research, Tokyo, Japan).

Note that for calculating the multiplicity  $m$  of a root, one may use the following ways:

(i) Traub in [5] proposed the following approximation formula that

$$m \approx \frac{\log |f(x)|}{\log |f(x)/f'(x)|},$$

when  $x$  is very close to the multiple root of  $f$ .

(ii) Lagouanelle in [28] introduced the following expression:

$$m \approx \frac{f'(x)^2}{f'(x)^2 - f(x)f''(x)},$$

where  $x$  is very close to the multiple root of  $f$ .

**Example 1.** (Van der Waals equation of state):

Consider the following Van der Waals equation of state:

$$\left(P + \frac{\bar{a}n^2}{V^2}\right)(V - n\bar{b}) = nRT,$$

where the parameter  $\bar{a}$  and  $\bar{b}$  (known as Van der Wall's constants) depends upon critical temperature, and critical pressure of the specified gas. Evaluate  $V$  (volume of the gas) with respect to known values of remaining variables by calculating the solution of the following equation:

$$PV^3 - (n\bar{b}P + nRT)V^2 + \bar{a}n^2V - \bar{a}\bar{b}n^2 = 0.$$

Thus, the above equations have at least one real root, as it is cubic polynomial. By using the specific values of the parameters, the following nonlinear function of  $V = x$  is obtained:

$$f(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675,$$

having the three zeros: 1.72, 1.75, and 1.75. Thus, the required root is  $\alpha = 1.75$  (as multiplicity is two).

The numerical performance presented in Table 1 shows the better outcomes of the presented methods MM1, and MM2 with respect to the precision in calculating the multiple root of  $f(x) = 0$ , whereas, MM2 is over-performing in terms of accuracy.



**Table 1.** Outcomes for comparison by testing schemes on Example 1.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	1.8	2.0(−4)	4.8(−2)		
	1	1.752482846514733146916889	2.0(−7)	2.5(−4)		
	2	10.7500004339982948258837904	5.7(−15)	4.3(−7)		
	3	1.750000000000000000000575	9.9(−45)	5.7(−22)	3.9415	0.374
ZM	0	1.8	2.0(−4)	4.7(−2)		
	1	1.752889533818870707034275	2.7(−7)	2.9(−3)		
	2	1.750000885938106677419627	2.4(−14)	8.9(−7)		
	3	1.7500000000000000000011407	3.9(−42)	1.1(−20)	3.9311	0.361
LI1	0	1.8	2.0(−4)	4.7(−2)		
	1	1.752523213564796124343964	2.1(−7)	2.5(−3)		
	2	1.750000409392763577418605	5.0(−15)	4.1(−7)		
	3	1.750000000000000000000390	4.6(−45)	3.9(−22)	3.9453	0.375
SM	0	1.8	2.0(−4)	4.7(−2)		
	1	1.752635974832068545325330	2.3(−7)	2.6(−3)		
	2	1.750000520632870976970862	8.1(−15)	5.2(−7)		
	3	1.7500000000000000000001105	3.7(−44)	1.1(−21)	3.9415	0.376
MM1	0	1.8	2.0(−4)	4.8(−2)		
	1	1.751727697259551849018861	9.4(−8)	1.7(−3)		
	2	1.750000022800442863424761	1.6(−17)	2.3(−8)		
	3	1.750000000000000000000000	1.2(−56)	6.3(−28)	3.9990	0.313
MM2	0	1.8	2.0(−4)	4.8(−2)		
	1	1.751675437187118274346379	8.9(−8)	1.7(−3)		
	2	1.750000034386502521339945	3.5(−17)	3.4(−8)		
	3	1.750000000000000000000000	1.5(−54)	7.1(−27)	3.9757	0.344

**Example 2.** (Problem of Planck's radiation law):

Now, consider the defined below problem of Planck's radiation law which measures the spectral density of electromagnetic radiations released by a black-body at a given temperature, at thermal equilibrium [29] as:

$$\Phi(\lambda) = \left( \frac{8\pi hc}{\lambda^5} \right) \left( \exp \left( \frac{ch}{\lambda k_B T} \right) - 1 \right)^{-1}, \quad (32)$$

where  $T$ ,  $\lambda$ ,  $k_B$ ,  $h$ , and  $c$  denotes the absolute temperature of the black-body, wavelength of radiation, Boltzmann constant, Planck's constant, and speed of light in the medium (vacuum), respectively. We are interested to determine the wavelength  $\lambda$  which results to the maximum energy density  $\Phi(\lambda)$ .

Further, the first derivative of  $\Phi$  is equated to zero, which corresponds to the maximum value of  $\Phi$  at:

$$\frac{\frac{ch}{\lambda k_B T} \exp \left( \frac{ch}{\lambda k_B T} \right)}{\exp \left( \frac{ch}{\lambda k_B T} \right) - 1} = 5. \quad (33)$$

If  $x = \frac{ch}{\lambda k_B T}$ , then Equation (33) is satisfied when

$$f(x) = 5 \exp(-x) + x - 5 = 0. \quad (34)$$

Thus, the solutions of Equation (34), results the maximum values of  $\lambda$ , and is means by the given below expression:

$$\lambda \approx \frac{ch}{\alpha k_B T}, \quad (35)$$

where  $\alpha$  is a solution of Equation (34). Our desired root is  $x = 4.9651142317442$  with multiplicity  $m = 1$ .

The numerical outcomes for Equation (34) are illustrated in Table 2. It can be declared by observing the results that the MM1 and MM2 methods have smaller residual errors in contrast to the existing methods when the accuracy of root is computed in multi-precision arithmetic. Moreover, the time consumed by new methods while computing the results is lesser with respect to other techniques, which justifying the attempt of developing the new scheme.

**Table 2.** Outcomes for comparison by testing schemes on Example 2.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.469
	1	4.965114231923295299252763	3.5(−11)	1.8(−10)		
	2	4.965114231744276303698759	2.5(−44)	1.3(−43)		
	3	4.965114231744276303698759	7.0(−177)	3.6(−176)		
ZM	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.547
	1	4.965114232182517130938556	8.5(−11)	4.4(−10)		
	2	4.965114231744276303698759	2.2(−42)	1.1(−41)		
	3	4.965114231744276303698759	9.5(−169)	4.9(−168)		
LI1	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.531
	1	4.9651142321500576086511740	7.8(−11)	4.1(−10)		
	2	4.965114231744276303698759	1.5(−42)	7.6(−24)		
	3	4.965114231744276303698759	1.9(−169)	9.7(−169)		
SM	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.422
	1	4.965114232153670143191299	7.9(−11)	4.1(−10)		
	2	4.965114231744276303698759	1.5(−42)	8.0(−42)		
	3	4.965114231744276303698759	2.3(−169)	1.2(−168)		
MM1	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.375
	1	4.965114231898958327178771	3.0(−11)	1.5(−10)		
	2	4.965114231744276303698759	1.2(−44)	6.2(−44)		
	3	4.965114231744276303698759	3.2(−178)	1.6(−177)		
MM2	0	5.0	6.7(−3)	3.5(−2)	4.0000	0.390
	1	4.965114231903813303678618	3.1(−11)	1.6(−10)		
	2	4.965114231744276303698759	1.4(−44)	7.3(−44)		
	3	4.965114231744276303698759	6.1(−178)	3.2(−177)		

**Example 3.** (Fractional conversion of a reactant in chemical reactor):

Considering the fractional conversion of a given species A (please, refer to [30] for further details of this problem) in terms of  $x$  as

$$f(x) = \frac{-x}{x-1} + 5 \log \left[ \frac{0.4 - 0.5x}{0.4(1-x)} \right] + 4.45977. \quad (36)$$

If  $x > 0$  or  $x < 1$ , then there is no significant meaning of the above defined fractional conversion. Which indicates that the  $x$  is bounded in its complementary region i.e.,  $0 \leq x \leq 1$ . Moreover, this problem is not defined for the  $x \in [0.8, 1]$ , and this region is nearly to the required root  $\alpha \approx 0.757396246253753879459641297929$ . Furthermore, this function has some other properties, which make the solution tougher to estimate. For instance, when  $x = 1.098$ , the test problem has an infeasible solution, whereas its derivative is very close to zero for  $0 \leq x \leq 0.5$ .

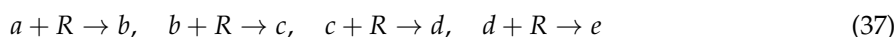
The outcomes for this test function are determined in Table 3. Clearly, it indicates that the proposed scheme works faster than others with greater accuracy. Although, LI show higher accuracy but results with higher computational time, whereas our methods results earlier.

**Table 3.** Outcomes for comparison by testing schemes on Example 3.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	0.75	0.53	7.4 (−3)	3.9999	0.423
	1	0.7574254324545007554731111	2.3 (−3)	2.9 (−5)		
	2	0.7573962462537595049877312	4.5 (−13)	5.6 (−15)		
	3	0.7573962462537538794596413	6.2 (−52)	7.8 (−54)		
ZM	0	0.75	0.53	7.5 (−3)	3.9997	0.374
	1	0.7574561453539206869998600	4.8 (−3)	6.0 (−5)		
	2	0.7573962462539628865065368	1.7 (−11)	2.1 (−13)		
	3	0.7573962462537538794596413	2.5 (−45)	3.1 (−47)		
LI	0	0.75	0.53	7.4 (−3)	4.0000	0.414
	1	0.7573971767293504371600571	7.4 (−5)	9.3 (−7)		
	2	0.7573962462537538794598735	1.9 (−20)	2.3 (−22)		
	3	0.7573962462537538794596413	7.2 (−83)	9.0 (−85)		
SM	0	0.75	0.53	7.4 (−3)	4.0000	0.391
	1	0.7574019238603881881123733	4.5 (−4)	5.7 (−6)		
	2	0.7573962462537538816233893	1.7 (−16)	2.2 (−18)		
	3	.7573962462537538794596413	3.6 (−66)	4.6 (−68)		
MM1	0	0.75	0.53	7.4 (−3)	4.0001	0.297
	1	0.7573785486502861269668355	1.4 (−3)	1.8 (−5)		
	2	0.7573962462537534666510670	3.3 (−14)	4.1 (−16)		
	3	0.7573962462537538794596413	9.7 (−57)	1.2 (−58)		
MM2	0	0.75	0.53	7.4 (−3)	4.0000	0.281
	1	0.7573915463347229318601339	3.7 (−4)	4.7 (−6)		
	2	0.7573962462537538785756882	7.1 (−17)	8.8 (−19)		
	3	0.7573962462537538794596413	8.8 (−68)	1.1 (−69)		

**Example 4.** Continuous stirred tank reactor:

Consider the following sequence of reaction taken place in reactor (for further study, refer [31])



In the above sequence, the components  $a$  and  $R$  are supplied with the rate  $t$  and  $q - t$ , respectively to the reactor. For draft of simple feedback from control system, Douglas [32] analyzed this problem in detail and introduced the following expression to study the transfer function of the reactor.

$$K_c \frac{2.98(x + 2.25)}{4.35 + x)(1.45 + x)(2.85 + x)^2} = -1, \quad (38)$$

where  $K_c$  stands for the gain of the proportional controller. The control system is stable for values of  $K_c$  that yields roots of the transfer function having negative real part. If we choose  $K_c = 0$ , we get the poles of the open-loop transfer function as roots of the nonlinear equation:

$$f(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875, \quad (39)$$

given as:  $x = 1.45, 2.85, 2.85, 4.35$ . Thus, one of root have multiplicity is equal to two. Hence, it is our desired root.

From Table 4, we can observe that the new methods MM1, MM2, and the existing RM method show equivalent accuracy of the desired result and resulting well with lesser residual error and difference in successive obtained approximations in comparison to the other considered methods. Along with this, numerical convergence order is justifying our theoretical analysis.

**Example 5.** The characteristic polynomial of non-singular matrix

$$B = \begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix},$$

is defined as a function:

$$f(x) = (x - 2)^4(x + 1). \quad (40)$$

Clearly, the characteristic function in Equation (40) has one of the zeros with multiplicity  $m = 4$ . Thus, the desired root is  $\alpha = 2$ , while testing the schemes.

The numerical outcomes of the nonlinear function in Equation (40) depicts that the newly proposed schemes MM1 and MM2 have splendid outcomes in the view of less residual error and estimation of the convergence order. Moreover, the results are achieved faster than other techniques.

**Table 4.** Outcomes for comparison by testing schemes on Example 4.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	−2.89	34 (−3)	4.0 (−2)	4.0000	0.235
	1	−2.850000002897936091688825	1.8 (−17)	2.9 (−9)		
	2	−2.850000000000000000000000	3.4 (−73)	4.0 (−37)		
	3	−2.850000000000000000000000	4.8 (−296)	1.5 (−148)		
ZM	0	−2.89	34 (−3)	4.0 (−2)	3.9999	0.266
	1	−2.850000005791740889942378	7.0 (−17)	5.8 (−9)		
	2	−2.850000000000000000000000	3.5 (−70)	1.3 (−35)		
	3	−2.850000000000000000000000	2.0 (−283)	3.1 (−142)		
LI	0	−2.89	34 (−3)	4.0 (−2)	4.0000	0.282
	1	−2.850000005791466192809057	7.0 (−17)	5.8 (−9)		
	2	−2.850000000000000000000000	3.4 (−70)	1.3 (−35)		
	3	−2.850000000000000000000000	4.0 (−283)	3.1 (−142)		
SM	0	−2.89	34 (−3)	4.0 (−2)	4.0000	0.250
	1	−2.850000005791534880147705	7.0 (−17)	5.8 (−9)		
	2	−2.850000000000000000000000	3.4 (−70)	1.3 (−35)		
	3	−2.850000000000000000000000	2.0 (−283)	3.1 (−142)		
MM1	0	−2.89	34 (−3)	4.0 (−2)	4.0000	0.234
	1	−2.850000002897111461553972	1.8 (−17)	2.9 (−9)		
	2	−2.850000000000000000000000	3.4 (−73)	4.0 (−37)		
	3	−2.850000000000000000000000	4.4 (−296)	1.5 (−148)		
MM2	0	−2.89	34 (−3)	4.0 (−2)	4.0000	0.219
	1	−2.850000002897276257647646	1.8 (−17)	2.9 (−9)		
	2	−2.850000000000000000000000	3.4 (−73)	4.0 (−37)		
	3	−2.850000000000000000000000	4.5 (−296)	1.5 (−148)		

**Table 5.** Outcomes for comparison by testing schemes on Example 5.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	2.5	9.4 (−2)	5.0 (−1)	3.9999	0.297
	1	2.001161286804071669996589	1.8 (−12)	1.2 (−3)		
	2	2.000000000000127370423081	2.6 (−52)	1.3 (−13)		
	3	2.000000000000000000000000	1.2 (−211)	1.9 (−53)		
ZM	0	2.5	9.4 (−2)	5.0 (−1)	3.9995	0.421
	1	2.002099196572935106235141	1.9 (−11)	2.1 (−3)		
	2	2.0000000000002635107113493	4.8 (−47)	2.6 (−12)		
	3	2.000000000000000000000000	1.9 (−189)	6.6 (−48)		
LI	0	2.5	9.4 (−2)	5.0 (−1)	3.9995	0.422
	1	2.002046745900269802911623	1.8 (−11)	2.0 (−3)		
	2	2.0000000000002313832097351	2.9 (−47)	2.3 (−12)		
	3	2.000000000000000000000000	2.1 (−190)	3.8 (−48)		
SM	0	2.5	2.2 (−1)	5.0 (−1)	3.9995	0.266
	1	2.002070658506889233693277	1.8 (−11)	2.1 (−3)		
	2	2.0000000000002455359338550	3.6 (−47)	2.5 (−12)		
	3	2.000000000000000000000000	5.7 (−190)	4.9 (−48)		
MM1	0	2.5	9.4 (−2)	5.0 (−1)	3.9999	0.188
	1	2.000631307074392969408765	1.6 (−13)	6.3 (−4)		
	2	2.000000000000003719387941	1.9 (−58)	3.7 (−15)		
	3	2.000000000000000000000000	4.0 (−238)	4.5 (−60)		
MM2	0	2.5	9.4 (−2)	5.0 (−1)	3.9999	0.201
	1	2.000698820155283179400838	2.4 (−13)	7.0 (−4)		
	2	2.000000000000007811109281	3.7 (−57)	7.8 (−15)		
	3	2.000000000000000000000000	2.2 (−232)	1.2 (−58)		

**Example 6.** Consider the another nonlinear testing sample from [6], as follows:

$$f(x) = -(x - \cos(x))^5.$$

The above test function has one of the multiple zero at  $\alpha = 0.739085133215161$  with  $m = 5$ .

The results of this test problem is shown in Table 6. We can observe from the numerical tests showed in this table that results by presented methods achieve are much effective in minimum time period than its competitors.

**Table 6.** Outcomes for comparison by testing schemes on Example 6.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.422
	1	0.7392038427966442566755797	3.1 (−19)	1.2 (−4)		
	2	0.7390851332151606510075209	9.4 (−85)	9.4 (−18)		
	3	0.7390851332151606416553121	8.0 (−347)	3.6 (−70)		
ZM	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.375
	1	0.7392757027128889858592349	3.3 (−18)	1.9 (−4)		
	2	0.7390851332151607350417478	9.3 (−80)	9.3 (−17)		
	3	0.7390851332151606416553121	6.0 (−326)	5.4 (−66)		
LI	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.390
	1	0.7392723206615554833661534	3.0 (−18)	1.9 (−4)		
	2	0.7390851332151607264738339	5.8 (−80)	8.5 (−17)		
	3	0.7390851332151606416553121	7.7 (−327)	3.6 (−66)		
SM	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.312
	1	0.7392740635455635715487264	3.2 (−18)	1.9 (−4)		
	2	0.7390851332151607307959473	7.4 (−80)	8.9 (−17)		
	3	0.7390851332151606416553121	2.2 (−326)	4.4 (−66)		
MM1	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.266
	1	0.7391483908290041120587025	1.3 (−20)	6.3 (−5)		
	2	0.7390851332151606418924910	9.9 (−93)	2.4 (−19)		
	3	0.7390851332151606416553121	3.0 (−381)	4.7 (−77)		
MM2	0	1.0	2.1 (−2)	2.6 (−1)	4.0000	0.250
	1	0.7391573362095670075029553	2.6 (−20)	7.2 (−5)		
	2	0.7390851332151606422333790	8.5 (−91)	5.8 (−19)		
	3	0.7390851332151606416553121	9.9 (−373)	2.4 (−75)		

**Example 7.** Kepler's law of planetary motion says that a plant revolve around sun in a elliptic order. Assume that the point  $(x, y)$  defines the position of plant at time  $t$ , which can be evaluated by following expressions:

$$x = a \cos(E - e),$$

$$y = a\sqrt{1 - e^2} \sin(E),$$

where  $E$  stands for eccentric anomaly and  $e$  stands for the eccentricity of the ellipse.

To determine the position  $(x, y)$ , we need to compute  $E$ , which can be calculated by the use of Kepler's equation of motion:

$$M = E - e \sin(E), \quad 0 < e < 1,$$

where  $M$  is the mean anomaly. The above equation relates the mean anomaly  $M$  to the eccentric anomaly  $E$  of an elliptic orbit with eccentricity  $e$ . Thus to compute  $E$ , one can solve the following nonlinear function

$$f(E) = M - E + e \sin(E). \quad (41)$$

For testing the various iterative methods, we consider  $M = 0.01$ ,  $e = 0.9995$ , and initial approximation as  $M + e$ . The approximate zero of the function defined in above equation is  $\alpha \approx 0.3899777749463621$ . The results are demonstrated in Table 7, which manifest that the proposed methods are winning in each aspects of comparisons.

**Table 7.** Outcomes for comparison by testing schemes on Example 7.

Methods	$t$	$x_t$	$ f(x_t) $	$ x_{t+1} - x_t $	$\rho$	CPU Time
RM	0	1.0095	1.5 (−1)	5.0 (−1)	3.4161	0.500
	1	0.5136967453894586609159624	1.3 (−2)	1.2 (−1)		
	2	0.3927585763241378068015195	2.1 (−4)	2.8 (−3)		
	3	0.3899777774038898109598081	1.9 (−10)	2.5 (−9)		
ZM	0	1.0095	1.5 (−1)	4.7 (−1)	3.1292	0.470
	1	0.5430573479124553561348711	1.7 (−2)	1.5 (−1)		
	2	0.3977607771502995641956331	6.0 (−4)	7.8 (−3)		
	3	0.3899780202671430167582084	1.9 (−8)	2.5 (−7)		
LI	0	1.0095	1.5 (−1)	5.3 (−1)	3.7385	0.500
	1	0.4837367205082459147025189	8.9 (−3)	9.3 (−2)		
	2	0.3904615029476423930256901	3.7 (−5)	4.8 (−4)		
	3	0.3899777749469492864763083	4.4 (−14)	5.9 (−13)		
SM	0	1.0095	1.5 (−1)	5.1 (−1)	3.6262	0.531
	1	0.5000749322403258106055786	1.1 (−2)	1.1 (−1)		
	2	0.3912225515544090451760855	9.4 (−5)	1.2 (−3)		
	3	0.3899777749887829300491702	3.2 (−12)	4.2 (−11)		
MM1	0	1.0095	1.5 (−1)	5.3 (−1)	3.7860	0.390
	1	0.4840808276844397285896050	8.9 (−3)	9.4 (−2)		
	2	0.3902438322588099864282668	2.0 (−5)	2.7 (−4)		
	3	0.3899777749463368088536943	1.9 (−15)	2.5 (−14)		
MM2	0	1.0095	1.5 (−1)	5.4 (−1)	3.8534	0.422
	1	0.4665777847560329405454156	7.0 (−3)	7.6 (−2)		
	2	0.3900984734994537560943818	9.1 (−6)	1.2 (−4)		
	3	0.3899777749463631190000837	7.1 (−17)	9.4 (−16)		

## 5. Summary and Conclusions

The main remarks of the study are:

1. A new optimal variant of King's family for finding the desired multiple zero of a given function is developed.
2. The presented multipoint iterative scheme uses function at two values and involves only the first derivative at one point of each iteration.
3. The structure of the scheme is dependent upon the weight function under certain conditions. With the aid of weight function, various fourth order schemes can be obtained.
4. A wide range of real-life applications is introduced in the numerical segment to validate the efficacy of the suggested modified family as compared to the alternative methods.
  - (a) From the experimental outcomes, we can point out that the new tools provide superior results for the considered test functions in terms of precision and accuracy.
  - (b) Moreover, the time used for evaluating the desired results by new methods indicates the effectiveness over the other techniques.

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