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Optimization Based Methods for Solving the Equilibrium Problems with Applications in Variational Inequality Problems and Solution of Nash Equilibrium Models

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Abstract: In this paper, we propose two modified two-step proximal methods that are formed through the proximal-like mapping and inertial effect for solving two classes of equilibrium problems. A weak convergence theorem for the first method and the strong convergence result of the second method are well established based on the mild condition on a bifunction. Such methods have the advantage of not involving any line search procedure or any knowledge of the Lipschitz-type constants of the bifunction. One practical reason is that the stepsize involving in these methods is updated based on some previous iterations or uses a stepsize sequence that is non-summable. We consider the well-known Nash–Cournot equilibrium models to support our well-established convergence results and see the advantage of the proposed methods over other well-known methods.

Keywords: subgradient extragradient methods; weak convergence; strong convergence; Hilbert spaces; convex quadratic optimization; variational inequality problems

1. Introduction

Let K to be a nonempty convex, closed subset of a Hilbert space $\mathbb{E}$ and $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ be a bifunction with $f(u, u) = 0$ for each $u \in K$. The equilibrium problem for f upon K is defined as follows:

$$\text{Find } p^* \in K \text{ such that } f(p^*, v) \geq 0, \forall v \in K. \quad (1)$$

The equilibrium problem, pioneered by Blum and Oettli [1] as a unifying feature, is found to be involved more and more actively in a number of applications, such as poroelasticity for petroleum engineering [2], porous materials [3,4], financial analysis in economics [5,6], the reconstruction of images in imaging processing [7–9], telecommunication networks or public roads [10,11], and

noncooperative games with the corresponding equilibrium concept by Nash [12]. The problem (EP) was also known as Ky Fan's inequality [13] due to his contributions to this area of research. In fact, this problem did not receive sufficient consideration before this specific format. Nikaido characterized Nash equilibria [14] as the solutions of the problem (EP), and Gwinner designed it just as a tool to solve optimization and variational inequalities [15]; however, they did not deal with the problem itself in a separate setup.

The equilibrium problem involves many mathematical problems as a particular case, i.e., the variational inequality problems (VIP), fixed point problems, complementarity problems, optimization problems, saddle point problems, the Nash equilibrium of non-cooperative games, and the vector and scalar minimization problems (see [1,16]). Moreover, iterative methods are efficient tools to evaluate an approximate solution to an equilibrium problem. Several methods are well known for solving the problem (EP), for instance the proximal point method [17,18], projection methods [19], extragradient methods [20–22], the subgradient method [23–25], methods using the Bregman distance [26], and others [27–30].

The extragradient method was originally established by Korpelevich [31] for solving the variational inequality problem, which is a specific case of equilibrium problems. Korpelevich proved the weak convergence of the generated sequence under the hypotheses of Lipschitz continuity and pseudo-monotonicity on the operator. It involves determining two projections onto a closed convex set in each iteration of the method. If the closed convex set is simple enough, so that projections onto it are explicitly computed, then the extragradient method is exceptionally suitable. The modification of this method is the subgradient extragradient method [32], where the second projection is not taken over onto the closed convex set, but on a half space and provides a simple calculation.

In 2008, Tran et al. [33] proposed an extragradient method extension of the Korpelevich extragradient method [31] for dealing with the pseudo-monotone equilibrium problem in finite-dimensional spaces. It is crucial to determine two minimization problems on a closed convex set in each iteration of the method, and there is a reasonable fixed stepsize in minimization problems. Precisely, the iterative sequence $\{u_n\}$ is determined as follows:

$$\begin{cases} u_0 \in C, \\ v_n = \arg \min_{y \in K} \{ \lambda f(u_n, y) + \frac{1}{2} \|u_n - y\|^2 \}, \\ u_{n+1} = \arg \min_{y \in K} \{ \lambda f(v_n, y) + \frac{1}{2} \|u_n - y\|^2 \}, \end{cases} \quad (2)$$

where $0 < \lambda < \min\{\frac{1}{2k_1}, \frac{1}{2k_2}\}$, and k_1, k_2 are Lipschitz constants. In 2016, Lyashko et al. [34] proposed an extension of the method (2) motivated by the result in [35]. Precisely, this iterative sequence $\{u_n\}$ is defined as follows:

$$\begin{cases} u_0, v_0 \in C, \\ u_{n+1} = \arg \min_{y \in K} \{ \lambda f(v_n, y) + \frac{1}{2} \|u_n - y\|^2 \}, \\ v_{n+1} = \arg \min_{y \in K} \{ \lambda f(v_n, y) + \frac{1}{2} \|u_{n+1} - y\|^2 \}, \end{cases} \quad (3)$$

where $0 < \lambda < \frac{1}{2k_2 + 4k_1}$ and k_1, k_2 are Lipschitz constants.

On the other hand, inertial-type methods are also important, based on the technique of the heavy-ball methods of the second-order time dynamic system. Polyak [36] began by taking inertial extrapolation as an acceleration method to solve smooth convex optimization problems. These methods are two-step iterative schemes, while the next iteration is evaluated by the use of the previous two iterations, and it could be viewed as an accelerating step of the iterative sequence. Several inertial-like algorithms for special cases of the problem (EP) can be found, for instance in [37–40].

This manuscript aims to introduce two modifications for the result proposed by Lyashko et al. [34] motivated by the results in [32,41,42]. These modifications are carried out by applying the inertial and subgradient strategy to speed up the iteration process and reduce the numerical computation. The first

result incorporates the Lyashko two-step extragradient method with the inertial term, and a variable stepsize is followed that is updated on every single iteration by using the previous iterations. We prove a weak convergence theorem for our first proposed method through the standard assumptions on the cost bifunction. Furthermore, we come up with an alternative inertial-type method, which is another variant of the first method. The second method does not require any knowledge about the strongly pseudomonotone and Lipschitz-type constants of a bifunction, and the strong convergence of the method is achieved.

This paper is arranged as follows: Section 2 includes a number of definitions and basic results. Sections 3 and 4 contain both of our methods involving the pseudomonotone and strongly pseudomonotone bifunction, as well as the convergence theorem. Section 5 covers the applications of our proposed results to the variational inequality problems. Section 6 illustrates numerical experiments in comparison with other existing algorithms using the Nash–Cournot equilibrium models to display the efficiency of our proposed algorithms.

2. Preliminaries

We make use of K as a closed, convex subset of the Hilbert space $\mathbb{E}$. We denote $u_n \rightarrow p^*$ and $u_n \rightharpoonup p^*$ as the sequence $\{u_n\}$ strongly converges to p^* and $\{u_n\}$ weakly converges to p^* , respectively. In addition, $EP(f, K)$ indicates the set of solutions of the equilibrium problem within K and $p^* \in EP(f, K)$.

Definition 1. Let a convex function $g : K \rightarrow \mathbb{R}$, and a subdifferential of g at $u \in K$ is:

$$\partial g(u) = \{w \in \mathbb{E} : g(v) - g(u) \geq \langle w, v - u \rangle, \text{ for all } v \in K\}.$$

Definition 2. The normal cone of K at $u \in K$ is:

$$N_K(u) = \{w \in \mathbb{E} : \langle w, v - u \rangle \leq 0, \text{ for all } v \in K\}.$$

Definition 3. The metric projection $P_K(u)$ of u onto a closed, convex subset K of $\mathbb{E}$ is defined as:

$$P_K(u) = \arg \min \{\|v - u\| : v \in K\}.$$

Now, we consider the concepts of the monotonicity of a bifunction (see [1,43]).

Definition 4. Let a bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ on K for $\gamma > 0$ be said to be:

(1) strongly monotone if:

$$f(u, v) + f(v, u) \leq -\gamma \|u - v\|^2, \forall u, v \in K;$$

(2) monotone if:

$$f(u, v) + f(v, u) \leq 0, \forall u, v \in K;$$

(3) strongly pseudomonotone if:

$$f(u, v) \geq 0 \implies f(v, u) \leq -\gamma \|u - v\|^2, \forall u, v \in K;$$

(4) pseudomonotone if:

$$f(u, v) \geq 0 \implies f(v, u) \leq 0, \forall u, v \in K;$$

(5) satisfying the Lipschitz-type condition on K if there exist two numbers $k_1, k_2 > 0$, such that:

$$f(u, z) \leq f(u, v) + f(v, z) + k_1 \|u - v\|^2 + k_2 \|v - z\|^2, \forall u, v, z \in K.$$

Lemma 1 ([44]). Let K be a nonempty, closed, and convex subset of a Hilbert space $\mathbb{E}$ and $g : K \rightarrow \mathbb{R}$ be a convex, subdifferentiable, and lower semi-continuous function. An element $u \in K$ is a minimizer of a function g if and only if $0 \in \partial g(u) + N_K(u)$ where $\partial g(u)$ and $N_K(u)$ denote the subdifferential of g at u and the normal cone of K at u , respectively.

Lemma 2 ([45]). Let $a, b \in \mathbb{E}$ and $\mu \in \mathbb{R}$, then the following is true:

$$\|\mu a + (1 - \mu)b\|^2 = \mu\|a\|^2 + (1 - \mu)\|b\|^2 - \mu(1 - \mu)\|a - b\|^2.$$

Lemma 3 ([46]). Let a_n, b_n , and c_n be a sequence in $[0, +\infty)$ such that:

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + c_n, \quad \forall n \geq 1, \quad \text{with} \quad \sum_{n=1}^{+\infty} c_n < +\infty,$$

and also, there exists $b > 0$ such that $0 \leq b_n \leq b < 1, \forall n \in \mathbb{N}$. Then, we have:

- (i) $\sum_{n=1}^{+\infty} [a_n - a_{n-1}]_+ < \infty$, with $[s]_+ := \max\{s, 0\}$;
- (ii) $\lim_{n \rightarrow +\infty} a_n = a^* \in [0, \infty)$.

Lemma 4 ([47]). Let $\{\xi_n\}$ belong in $\mathbb{E}$ and $K \subset \mathbb{E}$ satisfying:

- (i) For $\xi \in K$, $\lim_{n \rightarrow \infty} \|\xi_n - \xi\|$ exists;
- (ii) Every weak cluster point of the sequence $\{\xi_n\}$ belongs to K .

Then, $\{\xi_n\}$ converges weakly to an element of K .

Lemma 5 ([48]). Let $\{p_n\}, \{q_n\} \subset [0, \infty)$ be sequences and $\sum_{n=1}^{+\infty} p_n = \infty$ with $\sum_{n=1}^{+\infty} p_n q_n < \infty$, then $\liminf_{n \rightarrow \infty} q_n = 0$.

3. An Algorithm for the Pseudomonotone Equilibrium Problem and Its Convergence Analysis

The first method is developed by adding an inertial term to Algorithm 1 ([34], p. 4). It is important to note that the algorithm in the paper [34] cannot be used practically without knowledge of the Lipschitz-type constants of the bifunction. To overcome this situation, we suggest a new step-size rule that does not depend on Lipschitz-type constants. Rather, it depends on certain previous iterations and updates on each iteration. In a situation where Lipschitz constants are unknown or difficult to calculate, this approach is very useful. The following is our first algorithm in more detail:

Algorithm 1 Two-step proximal iterative method for pseudomonotone EP.

Initialization: Choose $u_{-1}, v_{-1} \in K$, $\lambda_0 > 0$ and $\mu \in [0, g(\alpha))$. Set:

$$u_0 = \arg \min_{y \in K} \{ \lambda_0 f(v_{-1}, y) + \frac{1}{2} \|u_{-1} - y\|^2 \} \text{ and } v_0 = \arg \min_{y \in K} \{ \lambda_0 f(v_{-1}, y) + \frac{1}{2} \|u_0 - y\|^2 \}.$$

Iterative steps: Now, $v_{n-1}, v_n, u_{n-1}, u_n \in K$ and λ_n are known for $n \geq 0$.

Step 1: Construct a half space:

$$\Pi_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n \omega_n - v_n, z - v_n \rangle \leq 0\}$$

where $\omega_n \in \partial f(v_{n-1}, v_n)$, and then, compute:

$$u_{n+1} = \text{Prox}_{\lambda_n f(v_n, \cdot)} t_n = \arg \min_{y \in \Pi_n} \{ \lambda_n f(v_n, y) + \frac{1}{2} \|t_n - y\|^2 \},$$

where $t_n = u_n + \alpha_n(u_n - u_{n-1})$ and $\alpha_n \subset [0, \frac{1}{6})$ is a nondecreasing sequence.

Step 2: Set $d = f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1})$ with:

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n, \frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2d} \right\} & \text{if } d > 0; \\ \lambda_n & \text{otherwise.} \end{cases}$$

Compute:

$$v_{n+1} = \text{Prox}_{\lambda_{n+1} f(v_n, \cdot)} u_{n+1} = \arg \min_{y \in K} \{ \lambda_{n+1} f(v_n, y) + \frac{1}{2} \|u_{n+1} - y\|^2 \}.$$

Step 3: If $u_{n+1} = v_n = t_n$, STOP; otherwise, set $n := n + 1$, and return back to **Step 1**.

Assumption 1. Let a bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ satisfy the following conditions:

- (A1) f is pseudomonotone on K with $f(u, u) = 0$, for all $u \in K$;
- (A2) f satisfies the Lipschitz-type condition on $\mathbb{E}$ with k_1 and k_2 ;
- (A3) $\lim_{n \rightarrow \infty} \sup f(u_n, y) \leq f(p, y)$ for each $y \in K$ and $\{u_n\} \subset K$ with $u_n \rightarrow p$;
- (A4) $f(u, \cdot)$ is subdifferentiable and convex on $\mathbb{E}$ for each $u \in \mathbb{E}$.

Lemma 6. We have the following inequality from Algorithm 1.

$$\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle t_n - u_{n+1}, y - u_{n+1} \rangle, \forall y \in \Pi_n.$$

Proof. By the value of u_{n+1} with Lemma 1, we have:

$$0 \in \partial_2 \left\{ \lambda_n f(v_n, y) + \frac{1}{2} \|t_n - y\|^2 \right\} (u_{n+1}) + N_{\Pi_n}(u_{n+1}).$$

Thus, for $\omega \in \partial_2 f(v_n, u_{n+1})$, there exists $\bar{\omega} \in N_{\Pi_n}(u_{n+1})$ such that:

$$\lambda_n \omega + u_{n+1} - t_n + \bar{\omega} = 0,$$

which gives:

$$\langle t_n - u_{n+1}, y - u_{n+1} \rangle = \lambda_n \langle \omega, y - u_{n+1} \rangle + \langle \bar{\omega}, y - u_{n+1} \rangle, \forall y \in \Pi_n.$$

By $\bar{\omega} \in N_{\Pi_n}(u_{n+1})$, we have $\langle \bar{\omega}, y - u_{n+1} \rangle \leq 0, \forall y \in \Pi_n$. This implies that:

$$\langle t_n - u_{n+1}, y - u_{n+1} \rangle \leq \lambda_n \langle \omega, y - u_{n+1} \rangle, \forall y \in \Pi_n. \quad (4)$$

From $\omega \in \partial f(v_n, u_{n+1})$ and by the subdifferential definition, we have:

$$f(v_n, y) - f(v_n, u_{n+1}) \geq \langle \omega, y - u_{n+1} \rangle, \forall y \in \mathbb{E}. \quad (5)$$

Combining (4) and (5), we obtain:

$$\lambda_n f(v_n, y) - \lambda_n f(v_n, u_{n+1}) \geq \langle t_n - u_{n+1}, y - u_{n+1} \rangle, \forall y \in \Pi_n.$$

□

Lemma 7. We also have the following expression from Algorithm 1.

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle u_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \forall y \in K.$$

Proof. The proof is identical to the proof of Lemma 6. □

Remark 1. By taking $u_{n+1} = v_n = t_n$ and $u_{n+1} = v_n = v_{n+1}$ in Algorithm 1, we can get $v_n \in EP(f, K)$ from Lemma 6 and Lemma 7, respectively.

Lemma 8. We have the following relationship from Algorithm 1.

$$\lambda_n \{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle u_n - v_n, u_{n+1} - v_n \rangle.$$

Proof. Since $u_{n+1} \in \Pi_n$, this gives that $\langle u_n - \lambda_n \omega_n - v_n, u_{n+1} - v_n \rangle \leq 0$. Thus, we get:

$$\lambda_n \langle \omega_n, u_{n+1} - v_n \rangle \geq \langle u_n - v_n, u_{n+1} - v_n \rangle. \quad (6)$$

By $\omega_n \in \partial f(v_{n-1}, v_n)$ and the definition of subdifferential, we get:

$$f(v_{n-1}, y) - f(v_{n-1}, v_n) \geq \langle \omega_n, y - v_n \rangle, \forall y \in \mathbb{E}.$$

Substituting $y = u_{n+1}$ in the above inequality, we get:

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \geq \langle \omega_n, u_{n+1} - v_n \rangle, \forall y \in \mathbb{E}. \quad (7)$$

Combining (6) and (7), we have:

$$\lambda_n \{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle u_n - v_n, u_{n+1} - v_n \rangle.$$

□

Lemma 9. Let $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ satisfy Assumption 1. Then, for each $p^* \in EP(f, K) \neq \emptyset$, we have:

$$\begin{aligned} \|u_{n+1} - p^*\|^2 &\leq \|t_n - p^*\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right) \|u_n - v_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|u_{n+1} - v_n\|^2 \\ &\quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_n - u_{n+1}\|^2. \end{aligned}$$

Proof. It follows from Lemma 6 with $y = p^*$ that we have:

$$\lambda_n f(v_n, p^*) - \lambda_n f(v_n, u_{n+1}) \geq \langle t_n - u_{n+1}, p^* - u_{n+1} \rangle. \quad (8)$$

Thus, $f(p^*, v_n) \geq 0$, and due to Condition (A1), we have $f(v_n, p^*) \leq 0$. This implies that:

$$\langle t_n - u_{n+1}, u_{n+1} - p^* \rangle \geq \lambda_n f(v_n, u_{n+1}). \quad (9)$$

By the definition, λ_{n+1} implies that:

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - f(v_n, u_{n+1}) \leq \frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\lambda_{n+1}}$$

both sides have been multiplied with $\lambda_n > 0$, such that:

$$\lambda_n f(v_n, u_{n+1}) \geq \lambda_n f(v_{n-1}, u_{n+1}) - \lambda_n f(v_{n-1}, v_n) - \frac{\lambda_n \mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\lambda_{n+1}} \quad (10)$$

By combining (9) and (10), we have:

$$\langle t_n - u_{n+1}, u_{n+1} - p^* \rangle \geq \lambda_n \{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|v_{n-1} - v_n\|^2 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|u_{n+1} - v_n\|^2. \quad (11)$$

Since $u_{n+1} \in \Pi_n$, then from Lemma 8, we have:

$$\lambda_n \{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle u_n - v_n, u_{n+1} - v_n \rangle. \quad (12)$$

From (11) and (12), we obtain:

$$\langle t_n - u_{n+1}, u_{n+1} - p^* \rangle \geq \langle u_n - v_n, u_{n+1} - v_n \rangle - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|v_{n-1} - v_n\|^2 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|u_{n+1} - v_n\|^2. \quad (13)$$

We have the following facts:

$$2\langle t_n - u_{n+1}, u_{n+1} - p^* \rangle = \|t_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 - \|u_{n+1} - p^*\|^2,$$

$$2\langle v_n - u_n, v_n - u_{n+1} \rangle = \|u_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|u_n - u_{n+1}\|^2,$$

and:

$$\|v_{n-1} - v_n\|^2 \leq (\|v_{n-1} - u_n\| + \|u_n - v_n\|)^2 \leq 2\|v_{n-1} - u_n\|^2 + 2\|u_n - v_n\|^2.$$

From above facts with Expression (13) complete the proof. $\square$

Theorem 1. The sequences $\{t_n\}$, $\{v_n\}$, and $\{u_n\}$ are generated by Algorithm 1 and weakly converge to p^* , where:

$$0 < \mu < \frac{1-6\alpha}{3} \quad \text{and} \quad 0 \leq \alpha_n \leq \alpha < \frac{1}{6}.$$

Proof. Adding to both sides the value $\frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2$ in Lemma 9, we have:

$$\begin{aligned} & \|u_{n+1} - p^*\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2 \\ & \leq \|t_n - p^*\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right) \|u_n - v_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|u_{n+1} - v_n\|^2 \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2 \end{aligned} \quad (14)$$

$$\begin{aligned} & \leq \|t_n - p^*\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right) \|u_n - v_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \|u_{n+1} - v_n\|^2 \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (15)$$

$$\begin{aligned} & \leq \|t_n - p^*\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \left[\|u_{n+1} - v_n\|^2 + \|u_n - v_n\|^2\right] \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (16)$$

$$\begin{aligned} & \leq \|t_n - p^*\|^2 - \frac{1}{2} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \left[\|u_{n+1} - v_n\| + \|u_n - v_n\|\right]^2 \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (17)$$

$$\begin{aligned} & \leq \|t_n - p^*\|^2 - \frac{1}{2} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \|u_{n+1} - u_n\|^2 \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2. \end{aligned} \quad (18)$$

By the definition of t_n in Algorithm 1, we get:

$$\begin{aligned} \|t_n - p^*\|^2 &= \|u_n + \alpha_n(u_n - u_{n-1}) - p^*\|^2 \\ &= \|(1 + \alpha_n)(u_n - p^*) - \alpha_n(u_{n-1} - p^*)\|^2 \\ &= (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \alpha_n(1 + \alpha_n)\|u_n - u_{n-1}\|^2. \end{aligned} \quad (19)$$

The value of u_{n+1} with Lemma 2 gives:

$$\begin{aligned} \|u_{n+1} - t_n\|^2 &= \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \alpha_n^2 \|u_n - u_{n-1}\|^2 - 2\alpha_n \langle u_{n+1} - u_n, u_n - u_{n-1} \rangle \end{aligned} \quad (20)$$

$$\begin{aligned} & \geq \|u_{n+1} - u_n\|^2 + \alpha_n^2 \|u_n - u_{n-1}\|^2 - 2\alpha_n \|u_{n+1} - u_n\| \|u_n - u_{n-1}\| \\ & \geq \|u_{n+1} - u_n\|^2 + \alpha_n^2 \|u_n - u_{n-1}\|^2 - \alpha_n \|u_{n+1} - u_n\|^2 - \alpha_n \|u_n - u_{n-1}\|^2 \\ & \geq (1 - \alpha_n) \|u_{n+1} - u_n\|^2 + (\alpha_n^2 - \alpha_n) \|u_n - u_{n-1}\|^2. \end{aligned} \quad (21)$$

Thus, Expression (18) with (19) and (21) turns into:

$$\begin{aligned} & \|u_{n+1} - p^*\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2 \\ & \leq (1 + \alpha_n) \|u_n - p^*\|^2 - \alpha_n \|u_{n-1} - p^*\|^2 + \alpha_n (1 + \alpha_n) \|u_n - u_{n-1}\|^2 \\ & \quad - \frac{1}{2} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \|u_{n+1} - u_n\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 \\ & \quad + \|u_{n+1} - u_n\|^2 - (1 - \alpha_n) \|u_{n+1} - u_n\|^2 - (\alpha_n^2 - \alpha_n) \|u_n - u_{n-1}\|^2 \end{aligned} \quad (22)$$

$$\begin{aligned} & \leq (1 + \alpha_n) \|u_n - p^*\|^2 - \alpha_n \|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 \\ & \quad - \left(\frac{1}{2} - \frac{\mu\lambda_n}{2\lambda_{n+1}} - \frac{\mu\lambda_n}{\lambda_{n+2}} - \alpha_n\right) \|u_{n+1} - u_n\|^2 + 2\alpha_n \|u_n - u_{n-1}\|^2 \end{aligned} \quad (23)$$

$$\begin{aligned} & \leq (1 + \alpha_n) \|u_n - p^*\|^2 - \alpha_n \|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 \\ & \quad - \kappa_n \|u_{n+1} - u_n\|^2 + \xi_n \|u_n - u_{n-1}\|^2, \end{aligned} \quad (24)$$

where $\kappa_n := \left(\frac{1}{2} - \frac{\mu\lambda_n}{2\lambda_{n+1}} - \frac{\mu\lambda_n}{\lambda_{n+2}} - \alpha_n\right)$ and $\xi_n := 2\alpha_n$. By Substituting:

$$\Gamma_n = \|u_n - p^*\|^2 - \alpha_n \|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 + \xi_n \|u_n - u_{n-1}\|^2.$$

Next, we compute:

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n &= \|u_{n+1} - p^*\|^2 - \alpha_{n+1} \|u_n - p^*\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2 + \xi_{n+1} \|u_{n+1} - u_n\|^2 \\ & \quad - \|u_n - p^*\|^2 + \alpha_n \|u_{n-1} - p^*\|^2 - \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2 - \xi_n \|u_n - u_{n-1}\|^2 \\ & \leq \|u_{n+1} - p^*\|^2 - (1 + \alpha_n) \|u_n - p^*\|^2 + \alpha_n \|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}} \|u_{n+1} - v_n\|^2 \\ & \quad + \xi_{n+1} \|u_{n+1} - u_n\|^2 - \xi_n \|u_n - u_{n-1}\|^2 - \frac{2\mu\lambda_n}{\lambda_{n+1}} \|u_n - v_{n-1}\|^2. \end{aligned} \quad (25)$$

From Expression (24) with (25), we obtain:

$$\Gamma_{n+1} - \Gamma_n \leq -\kappa_n \|u_{n+1} - u_n\|^2 + \xi_{n+1} \|u_{n+1} - u_n\|^2 = -(\kappa_n - \xi_{n+1}) \|u_{n+1} - u_n\|^2, \quad (26)$$

and we continue to evaluate:

$$\kappa_n - \xi_{n+1} = \frac{1}{2} - \frac{\mu\lambda_n}{2\lambda_{n+1}} - \frac{\mu\lambda_n}{\lambda_{n+2}} - \alpha_n - 2\alpha_{n+1} \geq \left(\frac{1}{2} - 3\alpha\right) - \mu \left(\frac{\lambda_n}{2\lambda_{n+1}} + \frac{\lambda_n}{\lambda_{n+2}}\right). \quad (27)$$

Due to $\lambda_n \rightarrow \lambda$, the above implies that:

$$\left(\frac{1}{2} - 3\alpha\right) - \mu \left(\frac{\lambda_n}{2\lambda_{n+1}} + \frac{\lambda_n}{\lambda_{n+2}}\right) \longrightarrow \frac{1}{2} - 3\alpha - \frac{3}{2}\mu > 0 \quad \text{as } n \rightarrow \infty.$$

Due to our supposition and from the above, there exists an $\delta \geq 0$ and $n_0 \in \mathbb{N}$ such that:

$$\kappa_n - \xi_{n+1} \geq \delta \geq 0, \quad \forall n \geq n_0. \quad (28)$$

Thus, Expression (26) with (28) implies that:

$$\Gamma_{n+1} - \Gamma_n \leq -\delta \|u_{n+1} - u_n\|^2 \leq 0. \quad (29)$$

The sequence $\{\Gamma_n\}$ is non-increasing for $n \geq n_0$. By the value of Γ_{n+1} , we have:

$$\begin{aligned}\Gamma_{n+1} &= \|u_{n+1} - p^*\|^2 - \alpha_{n+1}\|u_n - p^*\|^2 + \frac{2\mu\lambda_{n+1}}{\lambda_{n+2}}\|u_{n+1} - v_n\|^2 + \xi_{n+1}\|u_{n+1} - u_n\|^2 \\ &\geq -\alpha_{n+1}\|u_n - p^*\|^2.\end{aligned}\quad (30)$$

From the value Γ_n , we have:

$$\begin{aligned}\Gamma_n &= \|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}}\|u_n - v_{n-1}\|^2 + \xi_n\|u_n - u_{n-1}\|^2 \\ &\geq \|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2.\end{aligned}\quad (31)$$

The expression (31) for $n \geq n_0$ implies that:

$$\begin{aligned}\|u_n - p^*\|^2 &\leq \Gamma_n + \alpha_n\|u_{n-1} - p^*\|^2 \\ &\leq \Gamma_{n_0} + \alpha\|u_{n-1} - p^*\|^2 \\ &\leq \dots \leq \Gamma_{n_0}(\alpha^{n-n_0} + \dots + 1) + \alpha^{n-n_0}\|u_{n_0} - p^*\|^2 \\ &\leq \frac{\Gamma_{n_0}}{1-\alpha} + \alpha^{n-n_0}\|u_{n_0} - p^*\|^2.\end{aligned}\quad (32)$$

It follows from Expressions (30) and (32) that:

$$-\Gamma_{n+1} \leq \alpha_{n+1}\|u_n - p^*\|^2 \leq \alpha\|u_n - p^*\|^2 \leq \alpha\frac{\Gamma_{n_0}}{1-\alpha} + \alpha^{n-n_0+1}\|u_{n_0} - p^*\|^2. \quad (33)$$

It follows from Expressions (29) and (33) that:

$$\begin{aligned}\delta \sum_{n=n_0}^k \|u_{n+1} - u_n\|^2 &\leq \Gamma_{n_0} - \Gamma_{n+1} \\ &\leq \Gamma_{n_0} + \alpha\frac{\Gamma_{n_0}}{1-\alpha} + \alpha^{n-n_0+1}\|u_{n_0} - p^*\|^2 \\ &\leq \frac{\Gamma_{n_0}}{1-\alpha} + \|u_{n_0} - p^*\|^2,\end{aligned}\quad (34)$$

and letting $k \rightarrow \infty$ in Expression (34), we obtain:

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|^2 < +\infty, \text{ implies that } \|u_{n+1} - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (35)$$

From Expressions (20) and (35), we have:

$$\lim_{n \rightarrow \infty} \|u_{n+1} - t_n\| \rightarrow 0, \quad (36)$$

and also to continue:

$$0 \leq \|u_n - t_n\| \leq \|u_n - u_{n+1}\| + \|u_{n+1} - t_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (37)$$

Let $\Delta_n = \|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \frac{2\mu\lambda_n}{\lambda_{n+1}}\|u_n - v_{n-1}\|^2$. Expression (33) converts into:

$$-\Delta_{n+1} \leq \alpha\frac{\Gamma_{n_0}}{1-\alpha} + \alpha^{n-n_0+1}\|u_{n_0} - p^*\|^2 + \xi_{n+1}\|u_n - u_{n-1}\|^2. \quad (38)$$

By Expressions (16) and (19) with (38) for $n \geq n_0$, we have:

$$\begin{aligned} & \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \left[\|u_n - v_n\|^2 + \|u_{n+1} - v_n\|^2\right] \\ & \leq \Delta_n - \Delta_{n+1} + \alpha_n(1 + \alpha_n)\|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \\ & \leq \Delta_n - \Delta_{n+1} + \alpha(1 + \alpha)\|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2. \end{aligned} \quad (39)$$

Due to the assumption on μ for some $\epsilon > 0$,

$$\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \frac{2\mu\lambda_n}{\lambda_{n+2}}\right) \geq \epsilon > 0, \text{ as } n \geq n_0.$$

Let us fix $k \geq n_0$ and Expressions (38) and (39) for $n = n_0, n_0 + 1, \dots, k$. Summing them, we have:

$$\begin{aligned} & \epsilon \sum_{n=n_0}^k \|u_n - v_n\|^2 + \epsilon \sum_{n=n_0}^k \|u_{n+1} - v_n\|^2 \\ & \leq \Delta_{n_0} - \Delta_{k+1} + \alpha(1 + \alpha) \sum_{n=n_0}^k \|u_n - u_{n-1}\|^2 + \sum_{n=n_0}^k \|u_{n+1} - u_n\|^2 \\ & \leq \Delta_{n_0} + \alpha \frac{\Delta_{n_0}}{1 - \alpha} + \alpha^{n-n_0+1} \|u_{n_0} - p^*\|^2 + \xi_{k+1} \|u_k - u_{k-1}\|^2 \\ & \quad + \alpha(1 + \alpha) \sum_{n=n_0}^k \|u_n - u_{n-1}\|^2 + \sum_{n=n_0}^k \|u_{n+1} - u_n\|^2 \end{aligned} \quad (40)$$

and letting $k \rightarrow \infty$ in the above expression leads to:

$$\sum_n \|u_n - v_n\|^2 < \infty, \sum_n \|u_{n+1} - v_n\|^2 < \infty. \quad (41)$$

The above expression also implies that:

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = \lim_{n \rightarrow \infty} \|u_{n+1} - v_n\| = 0. \quad (42)$$

By using the triangle inequality with Expression (42), we have:

$$0 \leq \|v_n - v_{n-1}\| \leq \|u_n - v_n\| + \|u_n - v_{n-1}\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (43)$$

Next, from Expression (14) with (19):

$$\begin{aligned} \|u_{n+1} - p^*\|^2 & \leq (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \alpha(1 + \alpha)\|u_n - u_{n-1}\|^2 \\ & \quad + \frac{2\mu\lambda_n}{\lambda_{n+1}}\|u_n - v_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (44)$$

and the above expression with (35), (41), and Lemma 3 implies that the limit of $\|u_n - p^*\|$, $\|t_n - p^*\|$ and $\|v_n - p^*\|$ exists for each $p^* \in EP(f, K)$, meaning that the sequences $\{u_n\}$, $\{t_n\}$, and $\{v_n\}$ are bounded. Next, we have to prove that each weak sequential limit point of the sequence $\{u_n\}$ belongs to $EP(f, K)$. Let z be any sequential weak cluster point of the sequence $\{u_n\}$, i.e., a weak convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to z , which also implies that $\{v_{n_k}\}$ also converge weakly to z .

Now, our aim to show that $z \in EP(f, K)$. Using Lemma 6, the definition of λ_{n+1} (10), and Lemma 8, we obtain:

$$\begin{aligned} \lambda_{n_k} f(v_{n_k}, y) &\geq \lambda_{n_k} f(v_{n_k}, u_{n_k+1}) + \langle t_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle \\ &\geq \lambda_{n_k} f(v_{n_k-1}, u_{n_k+1}) - \lambda_{n_k} f(v_{n_k-1}, v_{n_k}) - \frac{\mu \lambda_{n_k}}{2\lambda_{n_k+1}} \|v_{n_k} - v_{n_k-1}\|^2 \\ &\quad - \frac{\mu \lambda_{n_k}}{2\lambda_{n_k+1}} \|v_{n_k} - u_{n_k+1}\|^2 + \langle t_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle \\ &\geq \langle u_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k} \rangle - \frac{\mu \lambda_{n_k}}{2\lambda_{n_k+1}} \|v_{n_k} - v_{n_k-1}\|^2 \\ &\quad - \frac{\mu \lambda_{n_k}}{2\lambda_{n_k+1}} \|v_{n_k} - u_{n_k+1}\|^2 + \langle t_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle, \end{aligned} \quad (45)$$

whereas y is an any element in Π_n . As a result, with (36), (37), (42), and (43) and due to the boundness of $\{u_n\}$, the above inequality goes to zero. By given $\lambda > 0$, the assumption (A3), and $v_{n_k} \rightharpoonup z$, we get:

$$0 \leq \limsup_{k \rightarrow \infty} f(v_{n_k}, y) \leq f(z, y), \quad \forall y \in \Pi_n.$$

Since $z \in K \subset \Pi_n$, then $f(z, y) \geq 0, \forall y \in K$. It gives that z belongs to $EP(f, K)$. By Lemma 4, we show that $\{t_n\}$, $\{u_n\}$, and $\{v_n\}$ converge weakly to p^* as $n \rightarrow \infty$. $\square$

4. An Algorithm for the Strongly Pseudomonotone Equilibrium Problem and Its Convergence Analysis

The second algorithm is also another variant of Algorithm 1 [34] that can solve the strongly pseudomonotone equilibrium problem. However, the advantage of this algorithm is that it provides strong convergence by using a diminishing stepsize sequence. Let $\{\lambda_n\} \subset (0, +\infty)$ be a sequence satisfying the following:

$$(S_1) : \lim_{n \rightarrow \infty} \lambda_n = 0 \text{ and } (S_2) : \sum_{n=1}^{\infty} \lambda_n = +\infty. \quad (46)$$

Assumption 2. Let a bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ satisfy the following conditions:

- (B1) f is strongly pseudomonotone on K with $f(u, u) = 0$, for all $u \in K$;
- (B2) f satisfies the Lipschitz-type condition on $\mathbb{E}$ with k_1 and k_2 ;
- (B3) $f(u, \cdot)$ is sub-differentiable and convex on $\mathbb{E}$ for each fixed $u \in \mathbb{E}$.

The Algorithm 2 is described below.

Algorithm 2 Two-step proximal iterative method for strongly pseudomonotone EP.

Initialization: Choose $u_{-1}, v_{-1} \in \mathbb{E}, \alpha_n \in [0, \frac{1}{6})$ with λ_n satisfying (46). Set:

$$u_0 = \text{Prox}_{\lambda_0 f(v_{-1}, \cdot)} u_{-1} = \arg \min_{y \in K} \{ \lambda_0 f(v_{-1}, y) + \frac{1}{2} \|u_{-1} - y\|^2 \},$$

and:

$$v_0 = \text{Prox}_{\lambda_0 f(v_{-1}, \cdot)} u_0 = \arg \min_{y \in K} \{ \lambda_0 f(v_{-1}, y) + \frac{1}{2} \|u_0 - y\|^2 \}.$$

Iterative steps: Given u_{n-1}, v_{n-1}, u_n , and v_n for $n \geq 0$, construct a half space:

$$\Pi_n = \{ z \in \mathbb{E} : \langle u_n - \lambda_n \omega_{n-1} - v_n, z - v_n \rangle \leq 0 \},$$

where $\omega_{n-1} \in \partial f(v_{n-1}, v_n)$ and $t_n = u_n + \alpha_n(u_n - u_{n-1})$.

Step 1: Compute:

$$u_{n+1} = \text{Prox}_{\lambda_n f(v_n, \cdot)} t_n = \arg \min_{y \in \Pi_n} \{ \lambda_n f(v_n, y) + \frac{1}{2} \|t_n - y\|^2 \}.$$

Step 2: Compute:

$$v_{n+1} = \text{Prox}_{\lambda_{n+1} f(v_n, \cdot)} u_{n+1} = \arg \min_{y \in K} \{ \lambda_{n+1} f(v_n, y) + \frac{1}{2} \|u_{n+1} - y\|^2 \}.$$

Step 3: If $u_{n+1} = v_n = t_n$, STOP; otherwise, set $n := n + 1$, and go back to Step 1.

Lemma 10. Assume $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ satisfying Assumption 2. For each $p^* \in EP(f, K) \neq \emptyset$, we have:

$$\begin{aligned} \|u_{n+1} - p^*\|^2 &\leq \|t_n - p^*\|^2 - (1 - 4k_1\lambda_n)\|u_n - v_n\|^2 - (1 - 2k_2\lambda_n)\|u_{n+1} - v_n\|^2 \\ &\quad + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2. \end{aligned}$$

Proof. Follow the same step as in the proof of Lemma 9, and term $2\gamma\lambda_n\|v_n - p^*\|^2$ will appear due to the strongly pseudomonotonicity of a bifunction. $\square$

Theorem 2. Assume that a bifunction $f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$ with Assumption 2. Let $\{u_n\}$ be sequences in $\mathbb{E}$ generated by Algorithm 2, where the sequence α_n is non-decreasing and $0 \leq \alpha_n \leq \alpha < \frac{1}{6}$. Then, the sequence $\{u_n\}$, $\{v_n\}$, and $\{t_n\}$ strongly converges to p^* in $EP(f, K)$.

Proof. By Lemma 10 and adding $4k_1\lambda_n\|u_{n+1} - v_n\|^2$ on both sides:

$$\begin{aligned} &\|u_{n+1} - p^*\|^2 + 4k_1\lambda_n\|u_{n+1} - v_n\|^2 \\ &\leq \|t_n - p^*\|^2 - (1 - 4k_1\lambda_n)\|u_n - v_n\|^2 - (1 - 2k_2\lambda_n)\|u_{n+1} - v_n\|^2 + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 \\ &\quad - 2\gamma\lambda_n\|v_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 + 4k_1\lambda_n\|u_{n+1} - v_n\|^2 \end{aligned} \quad (47)$$

$$\begin{aligned} &= \|t_n - p^*\|^2 - (1 - 4k_1\lambda_n)\|u_n - v_n\|^2 - (1 - 2k_2\lambda_n - 4k_1\lambda_n)\|u_{n+1} - v_n\|^2 \\ &\quad + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (48)$$

$$\begin{aligned} &= \|t_n - p^*\|^2 - \frac{1}{2}(1 - 2k_2\lambda_n - 4k_1\lambda_n)[2\|u_{n+1} - v_n\|^2 + 2\|u_n - v_n\|^2] \\ &\quad + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (49)$$

$$\begin{aligned} &\leq \|t_n - p^*\|^2 - \frac{1}{2}(1 - 2k_2\lambda_n - 4k_1\lambda_n)\|u_{n+1} - u_n\|^2 \\ &\quad + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 - \|u_{n+1} - t_n\|^2 + \|u_{n+1} - u_n\|^2. \end{aligned} \quad (50)$$

From the value of t_n in Algorithm 2, we have:

$$\|t_n - p^*\|^2 = (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \alpha_n(1 + \alpha_n)\|u_n - u_{n-1}\|^2. \quad (51)$$

By the definition of t_n , we have:

$$\|u_{n+1} - t_n\|^2 = \|u_{n+1} - u_n\|^2 + \alpha_n^2\|u_n - u_{n-1}\|^2 - 2\alpha_n\langle u_{n+1} - u_n, u_n - u_{n-1} \rangle \quad (52)$$

$$\geq (1 - \alpha_n)\|u_{n+1} - u_n\|^2 + (\alpha_n^2 - \alpha_n)\|u_n - u_{n-1}\|^2, \quad (53)$$

Thus, Expression (50) with (51) and (53) leads to:

$$\begin{aligned} & \|u_{n+1} - p^*\|^2 + 4k_1\lambda_n\|u_{n+1} - v_n\|^2 \\ & \leq (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + \alpha_n(1 + \alpha_n)\|u_n - u_{n-1}\|^2 \\ & \quad - \frac{1}{2}(1 - 2k_2\lambda_n - 4k_1\lambda_n)\|u_{n+1} - u_n\|^2 + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 \\ & \quad - (1 - \alpha_n)\|u_{n+1} - u_n\|^2 - (\alpha_n^2 - \alpha_n)\|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \end{aligned} \quad (54)$$

$$\begin{aligned} & \leq (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 \\ & \quad - \left(\frac{1}{2} - k_2\lambda_n - 2k_1\lambda_n - \alpha_n\right)\|u_{n+1} - u_n\|^2 + 2\alpha_n\|u_n - u_{n-1}\|^2 \end{aligned} \quad (55)$$

$$\begin{aligned} & \leq (1 + \alpha_n)\|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + 4k_1\lambda_n\|u_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2 \\ & \quad - Q_n\|u_{n+1} - u_n\|^2 + R_n\|u_n - u_{n-1}\|^2, \end{aligned} \quad (56)$$

where $Q_n = \left(\frac{1}{2} - k_2\lambda_n - 2k_1\lambda_n - \alpha_n\right)$ with $R_n = 2\alpha_n$. Substitute:

$$\Phi_n = \|u_n - p^*\|^2 - \alpha_n\|u_{n-1} - p^*\|^2 + 4k_1\lambda_n\|u_n - v_{n-1}\|^2.$$

From the above substitution, Expression (56) turns into:

$$\Phi_{n+1} \leq \Phi_n - 2\gamma\lambda_n\|v_n - p^*\|^2 - Q_n\|u_{n+1} - u_n\|^2 + R_n\|u_n - u_{n-1}\|^2 \quad (57)$$

Next, we substitute:

$$\Psi_n = \Phi_n + R_n\|u_n - u_{n-1}\|^2.$$

By the above expression and (57):

$$\Psi_{n+1} - \Psi_n = -(Q_n - R_{n+1})\|u_{n+1} - u_n\|^2 - 2\gamma\lambda_n\|v_n - p^*\|^2. \quad (58)$$

It follows that:

$$\begin{aligned} Q_n - R_{n+1} &= \frac{1}{2} - k_2\lambda_n - 2k_1\lambda_n - \alpha_n - 2\alpha_{n+1} \\ &\geq \frac{1}{2} - k_2\lambda_n - 2k_1\lambda_n - 3\alpha \\ &= \frac{1}{2} - \lambda_n(k_2 + 2k_1) - 3\alpha \end{aligned} \quad (59)$$

Since $\lambda_n \rightarrow 0$, then there exist an $N_0 \in \mathbb{N}$, such that:

$$0 < \lambda_n < \frac{\frac{1}{2} - 3\alpha}{k_2 + 2k_1}, \quad \forall n \geq N_0.$$

Due to the above condition, we have:

$$Q_n - R_{n+1} \geq 0, \quad \text{for all } n \geq N_0. \quad (60)$$

Thus, Expressions (58) and (60) imply that:

$$\Psi_{n+1} - \Psi_n = -\delta \|u_{n+1} - u_n\|^2 \leq 0, \quad n \geq N_0, \quad \text{for some } \delta \geq 0. \quad (61)$$

The above implies that the sequence $\{\Psi_n\}$ is nonincreasing for $n \geq N_0$. From the value of Ψ_n , we have:

$$\begin{aligned} \|u_n - p^*\|^2 &\leq \Psi_n + \alpha \|u_{n-1} - p^*\|^2 \\ &\leq \Psi_{N_0} + \alpha \|u_{n-1} - p^*\|^2 \\ &\leq \dots \leq \Psi_{N_0} (\alpha^{n-N_0} + \dots + 1) + \alpha^{n-N_0} \|u_{N_0} - p^*\|^2 \\ &\leq \frac{\Psi_{N_0}}{1-\alpha} + \alpha^{n-N_0} \|u_{N_0} - p^*\|^2. \end{aligned} \quad (62)$$

Similarly, for the value of Ψ_{n+1} with the above expression, we have:

$$\begin{aligned} -\Psi_{n+1} &\leq \alpha_{n+1} \|u_n - p^*\|^2 \\ &\leq \alpha \|u_n - p^*\|^2 \\ &\leq \alpha \frac{\Psi_{N_0}}{1-\alpha} + \alpha^{n-N_0+1} \|u_{N_0} - p^*\|^2 \\ &\leq \alpha \frac{\Psi_{N_0}}{1-\alpha} + \|u_{N_0} - p^*\|^2. \end{aligned} \quad (63)$$

From Expressions (61) and (63), such that

$$\begin{aligned} \delta \sum_{n=N_0}^k \|u_{n+1} - u_n\|^2 &\leq \Psi_{N_0} - \Psi_{k+1} \\ &\leq \Psi_{N_0} + \alpha \frac{\Psi_{N_0}}{1-\alpha} + \|u_{N_0} - p^*\|^2 \\ &\leq \frac{\Psi_{N_0}}{1-\alpha} + \|u_{N_0} - p^*\|^2, \end{aligned} \quad (64)$$

letting $k \rightarrow \infty$ in the above expression, we have:

$$\sum_n \|u_{n+1} - u_n\|^2 < +\infty, \quad \text{implies that } \lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \quad (65)$$

From Expressions (52) and (65), we obtain:

$$\|u_{n+1} - t_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (66)$$

Next, Expression (63) implies that:

$$-\Phi_{n+1} \leq \alpha \frac{\Psi_{N_0}}{1-\alpha} + \|u_{N_0} - p^*\|^2 + R_{n+1} \|u_{n+1} - u_n\|^2. \quad (67)$$

Further, for Equations (49) and (51) for $n \geq N_0$, we have:

$$\begin{aligned} &(1 - 2k_2\lambda_n - 4k_1\lambda_n) \left[\|u_{n+1} - v_n\|^2 + \|u_n - v_n\|^2 \right] \\ &\leq \Phi_n - \Phi_{n+1} + \alpha(1+\alpha) \|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2. \end{aligned} \quad (68)$$

Now, we fix a natural number $k \geq N_0$ and consider the above inequality for all numbers $N_0, N_0 + 1, \dots, k$. Summing them using (67), we obtain:

$$\begin{aligned} & (1 - 2k_2\lambda_n - 4k_1\lambda_n) \sum_{n=N_0}^k [\|u_{n+1} - v_n\|^2 + \|u_n - v_n\|^2] \\ & \leq \Phi_{N_0} - \Phi_{k+1} + \alpha(1 + \alpha) \sum_{n=N_0}^k \|u_n - u_{n-1}\|^2 + \sum_{n=N_0}^k \|u_{n+1} - u_n\|^2 \\ & \leq \Phi_{N_0} + \alpha \frac{\Psi_{N_0}}{1 - \alpha} + \|u_{N_0} - p^*\|^2 + R_{k+1} \|u_{k+1} - u_k\|^2 \\ & \quad + \alpha(1 + \alpha) \sum_{n=N_0}^k \|u_n - u_{n-1}\|^2 + \sum_{n=N_0}^k \|u_{n+1} - u_n\|^2 \end{aligned} \quad (69)$$

and letting $k \rightarrow \infty$ in the above expression, we have:

$$\sum_n \|u_{n+1} - v_n\|^2 < +\infty \quad \text{and} \quad \sum_n \|u_n - v_n\|^2 < +\infty. \quad (70)$$

The above expression implies that:

$$\lim_{n \rightarrow \infty} \|u_{n+1} - v_n\| = \lim_{n \rightarrow \infty} \|u_n - v_n\| = 0. \quad (71)$$

We can easily derive the following by using Expressions (65), (66), and (71) such that:

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = \lim_{n \rightarrow \infty} \|u_n - t_n\| = \lim_{n \rightarrow \infty} \|v_{n-1} - v_n\| = 0. \quad (72)$$

Furthermore, Expression (54) with (65), (70), and Lemma 3 implies that:

$$\lim_{n \rightarrow \infty} \|u_n - p^*\| = l, \text{ for some } l \geq 0. \quad (73)$$

By Expression (72), we obtain:

$$\lim_{n \rightarrow \infty} \|t_n - p^*\| = \lim_{n \rightarrow \infty} \|v_n - p^*\| = l. \quad (74)$$

Now, we show that the sequence $\{u_n\}$ strongly converges to p^* . From the condition on λ_n for all $n \geq N_0$, the following still holds:

$$0 < \lambda_n < \frac{1}{2k_2 + 4k_1}, \quad \forall n \geq N_0.$$

It follows from the above condition and Lemma 10 that:

$$2\gamma\lambda_n \|v_n - p^*\|^2 \leq \|t_n - p^*\|^2 - \|u_{n+1} - p^*\|^2 + 4k_1\lambda_n \|u_n - v_{n-1}\|^2 + \|u_{n+1} - u_n\|^2, \quad \forall n \geq N_0. \quad (75)$$

From Expressions (51) and (75), we obtain:

$$\begin{aligned} 2\gamma\lambda_n \|v_n - p^*\|^2 & \leq -\|u_{n+1} - p^*\|^2 + (1 + \alpha_n) \|u_n - p^*\|^2 - \alpha_n \|u_{n-1} - p^*\|^2 \\ & \quad + \alpha_n(1 + \alpha_n) \|u_n - u_{n-1}\|^2 + 4k_1\lambda_n \|u_n - v_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \\ & \leq (\|u_n - p^*\|^2 - \|u_{n+1} - p^*\|^2) + 2\alpha \|u_n - u_{n-1}\|^2 + \|u_{n+1} - u_n\|^2 \\ & \quad + (\alpha_n \|u_n - p^*\|^2 - \alpha_{n-1} \|u_{n-1} - p^*\|^2) + 4k_1\lambda_n \|u_n - v_{n-1}\|^2. \end{aligned} \quad (76)$$

From the above expression with (65) and (70), this implies that:

$$\begin{aligned}
 & \sum_{n=N_0}^k 2\gamma\lambda_n \|v_n - p^*\|^2 \\
 & \leq (\|u_{N_0} - p^*\|^2 - \|u_{k+1} - p^*\|^2) + 2\alpha \sum_{n=N_0}^k \|u_n - u_{n-1}\|^2 + \sum_{n=N_0}^k \|u_{n+1} - u_n\|^2 \\
 & \quad + (\alpha_k \|u_k - p^*\|^2 - \alpha_{N_0-1} \|u_{N_0-1} - p^*\|^2) + \frac{4k_1}{2k_2 + 4k_1} \sum_{n=N_0}^k \|u_n - v_{n-1}\|^2 \\
 & \leq \|u_{N_0} - p^*\|^2 + \alpha \|u_k - p^*\|^2 + 2\alpha \sum_{n=N_0}^k \|u_n - u_{n-1}\|^2 + \frac{4k_1}{2k_2 + 4k_1} \sum_{n=N_0}^k \|u_n - v_{n-1}\|^2 \\
 & \quad + \sum_{n=N_0}^k \|u_{n+1} - u_n\|^2 \\
 & \leq M,
 \end{aligned} \tag{77}$$

for some $M \geq 0$. Now, letting $k \rightarrow \infty$, in the above expression, we obtain:

$$\sum_{n=1}^{\infty} 2\gamma\lambda_n \|v_n - p^*\|^2 < +\infty. \tag{78}$$

From Lemma 5 and Expression (78), this implies that:

$$\liminf \|v_n - p^*\| = 0. \tag{79}$$

Expressions (73) and (79) imply that $\lim_{n \rightarrow \infty} \|u_n - p^*\| = 0$. This completes the proof. $\square$

5. Application to Variational Inequality Problems

Now, we study the applications of our proposed methods to solve the pseudomonotone and strongly monotone variational inequality problems. An operator $G : \mathbb{E} \rightarrow \mathbb{E}$ is considered to be:

- (1) strongly pseudomonotone on K if $\langle G(u), y - u \rangle \geq 0 \Rightarrow \langle G(y), u - y \rangle \leq -\gamma \|u - y\|^2, \forall u, y \in K$;
- (2) pseudomonotone on K if $\langle G(u), y - u \rangle \geq 0 \Rightarrow \langle G(y), u - y \rangle \leq 0, \forall u, y \in K$;
- (3) satisfying being L -Lipschitz continuous on K if $\|G(u) - G(y)\| \leq L \|u - y\|, \forall u, y \in K$.

The variational inequality problem is defined as:

$$\text{Find } p^* \in K \text{ such that } \langle G(p^*), y - p^* \rangle \geq 0, \forall y \in K.$$

Note: Let bifunction $f(u, v) := \langle G(u), v - u \rangle$ for all $u, v \in K$. Then, the equilibrium problem converts into the above variational inequality problem with $L = 2k_1 = 2k_2$. It follows from u_{n+1} in Algorithm 1 and the above definition of bifunction f that:

$$\begin{aligned}
 u_{n+1} &= \arg \min_{y \in \Pi_n} \left\{ \lambda_n f(v_n, y) + \frac{1}{2} \|t_n - y\|^2 \right\} \\
 &= \arg \min_{y \in \Pi_n} \left\{ \lambda_n \langle G(v_n), y - v_n \rangle + \frac{1}{2} \|t_n - y\|^2 + \frac{\lambda_n^2}{2} \|G(v_n)\|^2 - \frac{\lambda_n^2}{2} \|G(v_n)\|^2 \right\} \\
 &= \arg \min_{y \in \Pi_n} \left\{ \lambda_n \langle G(v_n), y - t_n \rangle + \lambda_n \langle G(v_n), t_n - v_n \rangle + \frac{1}{2} \|t_n - y\|^2 + \frac{\lambda_n^2}{2} \|G(v_n)\|^2 - \frac{\lambda_n^2}{2} \|G(v_n)\|^2 \right\} \\
 &= \arg \min_{y \in \Pi_n} \left\{ \frac{1}{2} \|t_n - \lambda_n G(v_n) - y\|^2 \right\} \\
 &= P_{\Pi_n}(t_n - \lambda_n G(v_n)).
 \end{aligned} \tag{80}$$

The value of u_{n+1} reduces to the following projection:

$$v_{n+1} = P_K(u_{n+1} - \lambda_{n+1}G(v_n)).$$

Assumption 3. Let $G : K \rightarrow \mathbb{E}$ satisfy the following conditions:

- (G1) G is strongly pseudomonotone on K , and $VI(G, K)$ is a nonempty solution set;
- (G2) G is pseudomonotone on K , and $VI(G, K)$ is a nonempty solution set;
- (G3) G is L -Lipschitz continuous upon K for positive constant $L > 0$.
- (G4) $\limsup_{n \rightarrow \infty} \langle G(u_n), y - u_n \rangle \leq \langle G(p), y - p \rangle$ for every $y \in K$ and $\{u_n\} \subset K$ satisfying $u_n \rightharpoonup p$.

Corollary 1. Let $G : K \rightarrow \mathbb{E}$ satisfy (G2, G3, G4) as in Assumption 3. Assume $\{t_n\}$, $\{u_n\}$, and $\{v_n\}$ is the sequence generated in the following way:

- (i) Choose $u_{-1}, v_{-1} \in K$, $\lambda_0 > 0$, and $\mu \in [0, g(\alpha))$. Set:

$$u_0 = P_K(u_{-1} - \lambda_0 G(v_{-1})) \text{ and } v_0 = P_K(u_0 - \lambda_0 G(v_{-1})).$$

- (ii) Assume $v_{n-1}, v_n, u_{n-1}, u_n \in K$, and λ_n are known for $n \geq 0$. Construct a half space:

$$\Pi_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n G(v_{n-1}) - v_n, z - v_n \rangle \leq 0\}.$$

Compute:

$$\begin{cases} u_{n+1} = P_{\Pi_n}(t_n - \lambda_n G(v_n)), \text{ where } t_n = u_n + \alpha_n(u_n - u_{-1}), \\ v_{n+1} = P_K(u_{n+1} - \lambda_{n+1} G(v_n)). \end{cases}$$

The stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n, \frac{\mu(\|v_{n-1} - v_n\|^2 + \|u_{n+1} - v_n\|^2)}{2\langle G(v_{n-1}) - G(v_n), u_{n+1} - v_n \rangle} \right\} & \text{if } \langle G(v_{n-1}) - G(v_n), u_{n+1} - v_n \rangle > 0; \\ \lambda_n & \text{otherwise.} \end{cases}$$

Thus, the sequence $\{t_n\}$, $\{u_n\}$, and $\{v_n\}$ converges weakly to p^* of $VI(G, K)$.

Corollary 2. Let $G : K \rightarrow \mathbb{E}$ satisfy (G1, G3) as in Assumption 3. Assume $\{t_n\}$, $\{u_n\}$ and $\{v_n\}$ are the sequences generated as follows:

- (i) Choose $u_{-1}, v_{-1} \in K$, $\lambda_0 > 0$, and $\alpha_n \in [0, \frac{1}{6})$. Set:

$$u_0 = P_K(u_{-1} - \lambda_0 G(v_{-1})) \text{ and } v_0 = P_K(u_0 - \lambda_0 G(v_{-1})).$$

- (ii) Assume that $v_{n-1}, v_n, u_{n-1}, u_n \in K$ and λ_n are known for $n \geq 0$. Construct a half space:

$$\Pi_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n G(v_{n-1}) - v_n, z - v_n \rangle \leq 0\}.$$

Compute:

$$\begin{cases} u_{n+1} = P_{\Pi_n}(t_n - \lambda_n G(v_n)), \text{ where } t_n = u_n + \alpha_n(u_n - u_{-1}), \\ v_{n+1} = P_K(u_{n+1} - \lambda_{n+1} G(v_n)). \end{cases}$$

The stepsize sequence λ_n satisfies the following hypothesis:

$$(S_1) : \lim_{n \rightarrow \infty} \lambda_n = 0 \text{ and } (S_2) : \sum_{n=1}^{\infty} \lambda_n = +\infty. \quad (81)$$

Thus, the sequence $\{t_n\}$, $\{u_n\}$, and $\{v_n\}$ converges strongly to p^* of $VI(G, K)$.

6. Numerical Experiments

Now, we discuss two economy models to examine the efficiency of our proposed methods. For the numerical experiments, we wrote our algorithms using MATLAB programs (MATLAB R2018b) evaluated on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30 GHz 2.40 GHz, RAM 8.00 GB.

6.1. Nash–Cournot Equilibrium Model of Electricity Markets

We considered the equilibrium model of electricity markets [20]. We assumed that there were three companies that were generating electricity $i (i = 1, 2, 3)$. Companies 1, 2, and 3 had generating units named as $I_1 = \{1\}$, $I_2 = \{2, 3\}$, and $I_3 = \{4, 5, 6\}$, respectively. Suppose that u_j denotes the generating power of the unit for $j = \{1, 2, 3, 4, 5, 6\}$. We also assume that the price p of the electricity is defined as $p = 378.4 - 2 \sum_{j=1}^6 u_j$. The charge of the producing j unit is written as:

$$c_j(u_j) := \max\{\overset{\circ}{c}_j(u_j), \overset{\bullet}{c}_j(u_j)\},$$

with $\overset{\circ}{c}_j(u_j) := \frac{\overset{\circ}{\alpha}_j}{2} u_j^2 + \overset{\circ}{\beta}_j u_j + \overset{\circ}{\gamma}_j$ and $\overset{\bullet}{c}_j(u_j) := \overset{\bullet}{\alpha}_j u_j + \frac{\overset{\bullet}{\beta}_j}{\overset{\bullet}{\beta}_j + 1} \overset{\bullet}{\gamma}_j \overset{\bullet}{\beta}_j^{-1} (u_j)^{\frac{(\overset{\bullet}{\beta}_j + 1)}{\overset{\bullet}{\beta}_j}}$. The values of $\overset{\circ}{\alpha}_j$, $\overset{\circ}{\beta}_j$, $\overset{\circ}{\gamma}_j$, $\overset{\bullet}{\alpha}_j$, $\overset{\bullet}{\beta}_j$, and $\overset{\bullet}{\gamma}_j$ are set in Table 1. Consider that the profit of the firm i is:

$$f_i(u) := p \sum_{j \in I_i} u_j - \sum_{j \in I_i} c_j(u_j) = \left(378.4 - 2 \sum_{l=1}^6 u_l \right) \sum_{j \in I_i} u_j - \sum_{j \in I_i} c_j(u_j),$$

where $u = (u_1, \dots, u_6)^T$ and feasible set $K := \{u \in \mathbb{R}^6 : u_j^{\min} \leq u_j \leq u_j^{\max}\}$ with $u_j^{\min}$ and $u_j^{\max}$ given in Table 2. First, we describe the f equilibrium function as:

$$f(u, v) := \sum_{i=1}^3 (\phi_i(u, u) - \phi_i(u, v)),$$

where:

$$\phi_i(u, v) := \left[378.4 - 2 \left(\sum_{j \notin I_i} u_j + \sum_{j \in I_i} v_j \right) \right] \sum_{j \in I_i} v_j - \sum_{j \in I_i} c_j(v_j).$$

The above discussed model is viewed as the following equilibrium problem:

Find $p^* \in K$ such that $f(p^*, y) \geq 0, \forall y \in K$.

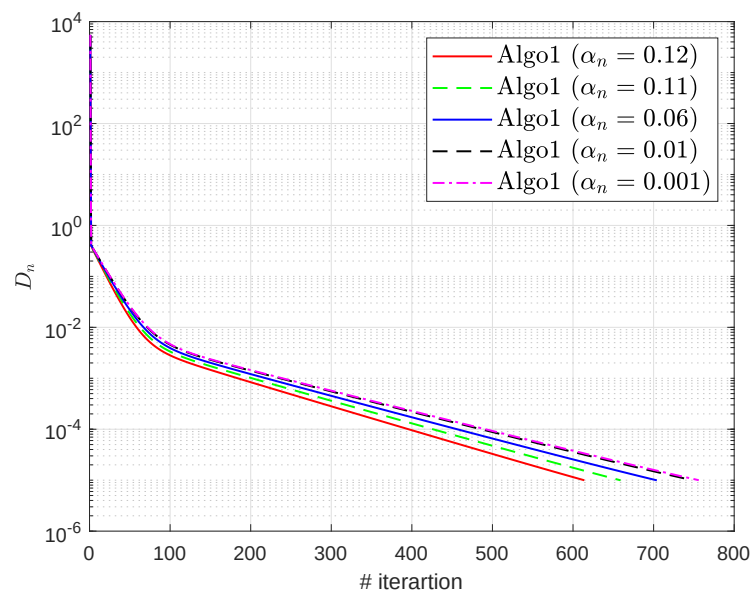
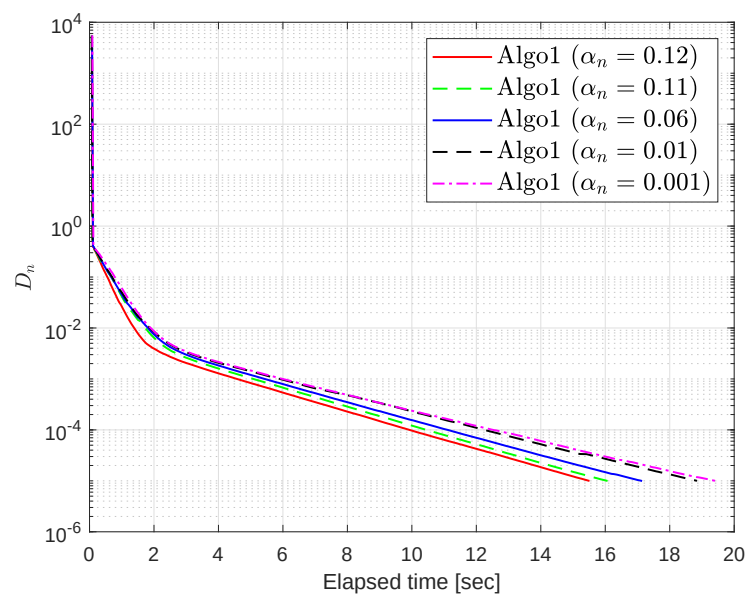
Table 1. The different values of the constants.

Unit j	$\overset{\circ}{\alpha}_j$	$\overset{\circ}{\beta}_j$	$\overset{\circ}{\gamma}_j$	$\overset{\bullet}{\alpha}_j$	$\overset{\bullet}{\beta}_j$	$\overset{\bullet}{\gamma}_j$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 2. The constraint set values.

j	1	2	3	4	5	6
u_j^{min}	0	0	0	0	0	0
u_j^{max}	80	80	50	55	30	40

For Algorithm 1 (Algo1), we use $u_{-1} = (10, 10, 20, 17, 8, 14)^T$ and $v_{-1} = (48, 48, 30, 28, 18, 24)^T$. Figures 1–4 and Tables 3 and 4 describe the numerical results for error term ($D_n = \|u_{n+1} - v_n\|^2 + \|t_n - v_n\|^2$) regarding different values of α_n and λ_0 .

**Figure 1.** Section 6.1: Algorithm 1's (Algo1) behavior with respect to different values of α_n .**Figure 2.** Section 6.1: Algorithm 1's behavior with respect to different values of α_n .

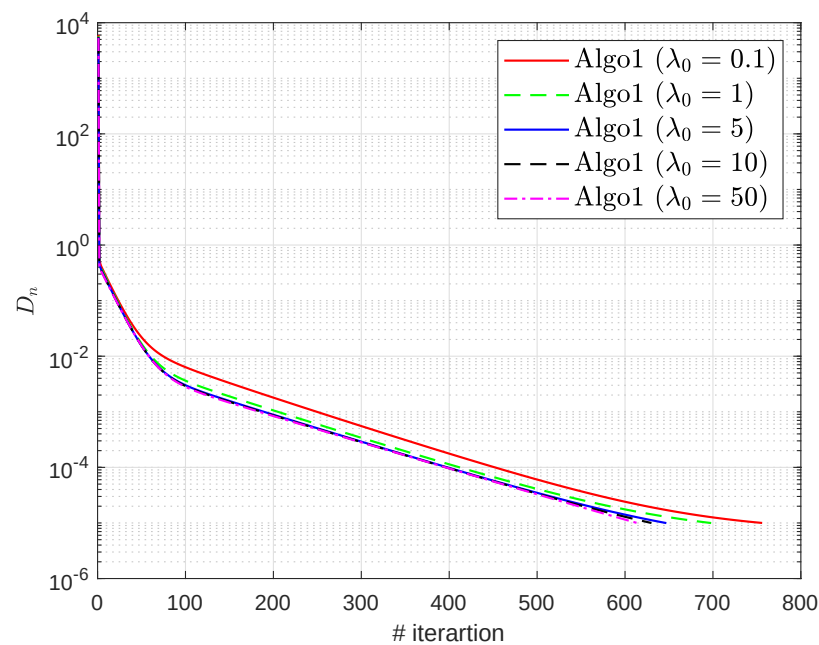


Figure 3. Section 6.1: Algorithm 1's behavior with respect to different values of λ_0 .

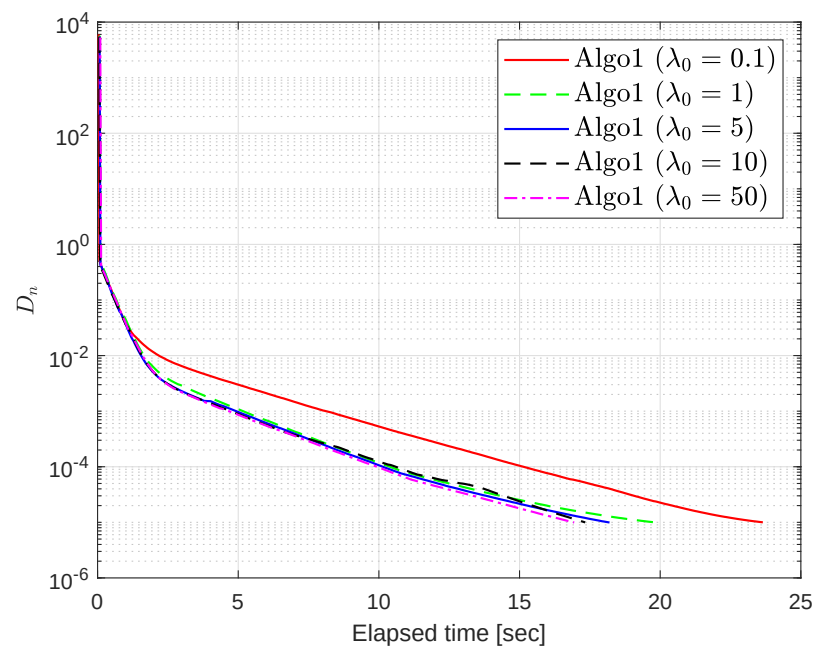


Figure 4. Section 6.1: Algorithm 1's behavior with respect to different values of λ_0 .

Table 3. Numerical results for Figures 1 and 2.

Algo.name	α_n	μ	λ_0	u	n	Time	Tolerance
Algo1	0.16	0.012	50	(46.0474, 28.9092, 18.4850, 20.4036, 12.5084, 14.1096) ^T	614	15.511285	$\epsilon = 10^{-5}$
Algo1	0.11	0.012	50	(46.0568, 28.4153, 18.9677, 20.2051, 12.6083, 14.2082) ^T	659	17.624634	$\epsilon = 10^{-5}$
Algo1	0.06	0.012	50	(46.0659, 27.9750, 19.3977, 20.0281, 12.6974, 14.2959) ^T	704	17.143282	$\epsilon = 10^{-5}$
Algo1	0.01	0.012	50	(46.0717, 27.5800, 19.7846, 19.8688, 12.7777, 14.3753) ^T	748	18.846679	$\epsilon = 10^{-5}$
Algo1	0.001	0.012	50	(46.0730, 27.5132, 19.8449, 19.8418, 12.7913, 14.3886) ^T	756	19.411861	$\epsilon = 10^{-5}$

Table 4. Numerical results for Figures 3 and 4.

Algo.name	α_n	μ	λ_0	u	n	Time	Tolerance
Algo1	0.16	0.012	0.1	$(46.4147, 20.5037, 26.6003, 21.2012, 8.8592, 16.9030)^T$	756	23.649384	$\epsilon = 10^{-5}$
Algo1	0.16	0.012	1	$(46.3512, 21.0528, 26.0886, 21.1505, 9.0529, 16.7769)^T$	698	19.827807	$\epsilon = 10^{-5}$
Algo1	0.16	0.012	5	$(46.2222, 23.2606, 23.9720, 21.0243, 9.8077, 16.1749)^T$	647	18.198626	$\epsilon = 10^{-5}$
Algo1	0.16	0.012	10	$(46.1499, 24.9832, 22.3073, 20.9005, 10.4870, 15.6306)^T$	630	17.335370	$\epsilon = 10^{-5}$
Algo1	0.16	0.012	50	$(46.0474, 28.9092, 18.4850, 20.4036, 12.5084, 14.1096)^T$	614	16.943758	$\epsilon = 10^{-5}$

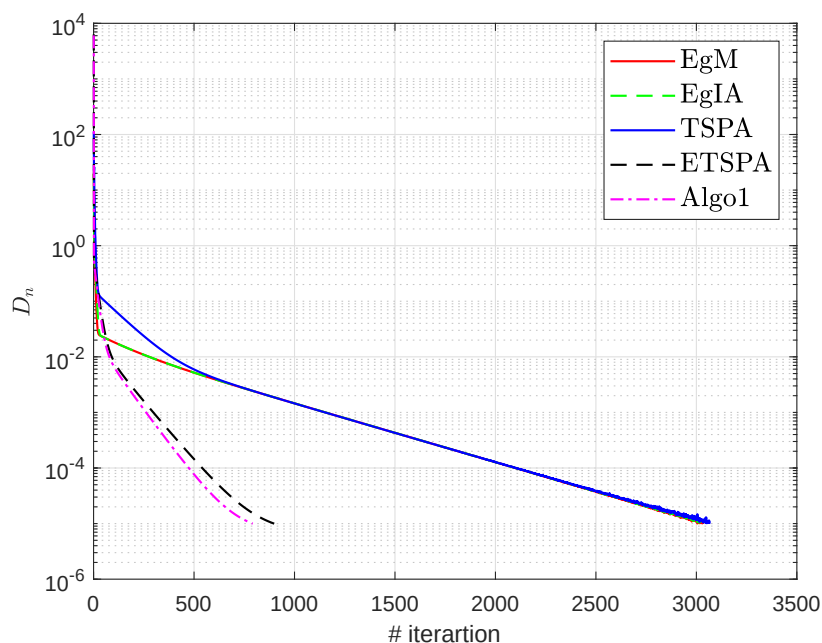
Algorithm 1 Comparison with Other Existing Methods

- (i) For the extragradient method (EgM) [33], $u_0 = (48, 48, 30, 28, 18, 24)^T$ and $D_n = \|u_n - v_n\|^2$.
- (ii) For Algorithm 1 (EgIA) [41], $u_0 = (48, 48, 30, 28, 18, 24)^T$ and $D_n = \|u_n - v_n\|^2$.
- (iii) For the two-step proximal algorithm (TSPA) [34], $D_n = \|u_{n+1} - v_n\|^2 + \|u_n - v_n\|^2$ and $u_0 = (10, 10, 20, 17, 8, 14)^T$, $v_0 = (48, 48, 30, 28, 18, 24)^T$.
- (iv) For Algorithm 2 (ETSPA) [41], $u_0 = (10, 10, 20, 17, 8, 14)^T$, $v_0 = (48, 48, 30, 28, 18, 24)^T$, $v_{-1} = (10, 20, 30, 10, 0, 1)^T$ and $D_n = \|u_{n+1} - v_n\|^2 + \|u_n - v_n\|^2$.
- (v) For Algorithm 1 (Algo1), $u_{-1} = (10, 10, 20, 17, 8, 14)^T$, $v_{-1} = (48, 48, 30, 28, 18, 24)^T$ and the error term $D_n = \|u_{n+1} - v_n\|^2 + \|t_n - v_n\|^2$.

Figures 5 and 6 and Table 5 illustrate the numerical findings for the stopping criterion.

Table 5. Numerical results for Figures 5 and 6. EgM, extragradient method; TSPA, two-step proximal algorithm.

Algo.name	α_n	μ	λ_0	u	n	Time	Tolerance
EgM	—	—	0.1	$(46.6523, 32.1467, 15.0011, 25.1431, 10.8357, 10.8357)^T$	3033	151.699125	$\epsilon = 10^{-5}$
EgIA	—	0.012	0.1	$(46.6523, 32.1467, 15.0011, 25.1430, 10.8358, 10.8358)^T$	3025	151.118686	$\epsilon = 10^{-5}$
TSPA	—	—	0.1	$(46.6523, 32.1467, 15.0011, 25.1433, 10.8356, 10.8356)^T$	3065	171.432413	$\epsilon = 10^{-5}$
ETSPA	—	0.012	0.1	$(46.3995, 20.4626, 26.6506, 19.9235, 10.7905, 16.2497)^T$	899	27.432681	$\epsilon = 10^{-5}$
Algo1	0.12	0.012	0.1	$(46.4110, 20.4926, 26.6137, 20.8384, 9.4075, 16.7174)^T$	792	19.730552	$\epsilon = 10^{-5}$

**Figure 5.** Section 6.1: Algorithm 1 comparison with existing methods.

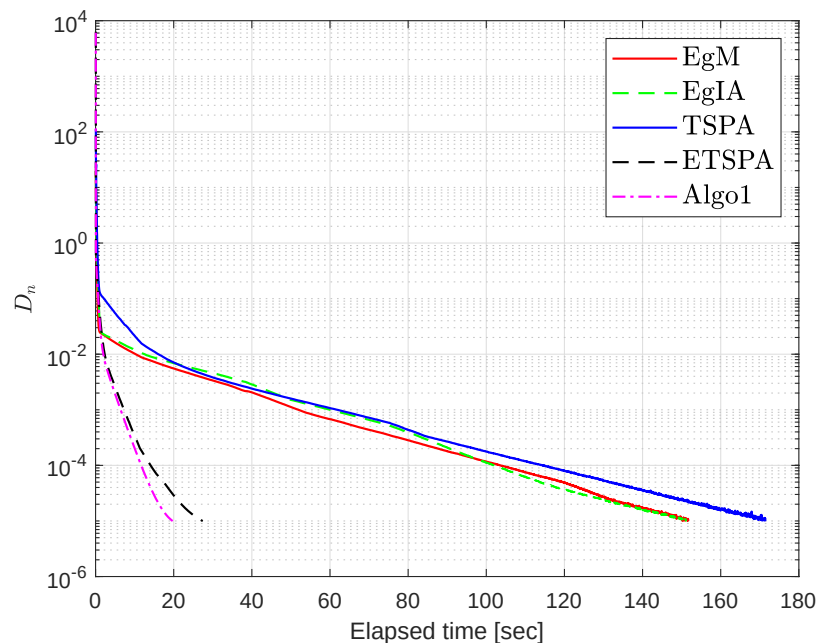


Figure 6. Section 6.1: Algorithm 1 comparison with existing methods.

6.2. Nash–Cournot Oligopolistic Equilibrium Model

We consider that there are n companies that are generating the same commodity. Suppose that u_i in vector u represents the amount of commodity production corresponding to each company i . The price function depends on the value of $S = \sum_{i=1}^m u_i$ such that $P_i(S) = \phi_i - \psi_i S$ where $\phi_i > 0$ and $\psi_i > 0$. This price function is affine and decreasing. The profit function $F_i(u) = P_i(S)u_i - t_i(u_i)$ with $t_i(u_i)$ is the tax and production charges for u_i . Suppose that $K := K_1 \times K_2 \times \cdots \times K_n$ with $K_i = [u_i^{\min}, u_i^{\max}]$ is the set of possible actions or strategies corresponding to each company i . In particular, each company seeks to achieve its maximum profit by considering the subsequent level of production on the premise that the production of the other companies is an input parameter. The technique also deals with this form of model based on the well-known Nash equilibrium principle. A point $p^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is the equilibrium point such that:

$$F_i(p^*) \geq F_i(p^*[u_i]) \quad \forall u_i \in K_i, \text{ for all } i = 1, 2, \dots, n.$$

where vector $p^*[u_i]$ means that u_i^* is replaced with u_i . Next, assume that $f(u, v) := \varphi(u, v) - \varphi(u, u)$ with $\varphi(u, v) := -\sum_{i=1}^n F_i(u[v_i])$, and the problem of finding a Nash equilibrium point of the model can be considered as follows:

$$\text{Find } p^* \in K \text{ such that } f(p^*, v) \geq 0, \forall v \in K.$$

It follows from [33] that the bifunction f becomes as follows:

$$f(u, v) = \langle Mu + Nv + r, v - u \rangle,$$

where:

$$N = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } M = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 3 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

with $r = (1, 2, -1, 2, -1)^T$ and $K = \{u \in \mathbb{R}^5 : -2 \leq u_i \leq 5\}$.

Algorithm 2 Comparison with Other Existing Methods:

- For the extragradient method (EgMSP) [49], $u_0 = (1, 3, 1, 1, 2)^T$ and $D_n = \|u_{n+1} - u_n\|$.
- For another method (EgIASP) [42], $u_0 = (1, 3, 1, 1, 2)^T$, $v_0 = (1, 0, 1, 0, 2)^T$ and $D_n = \|u_{n+1} - u_n\|$.
- For Algorithm 2 (Algo2), $u_{-1} = (1, 3, 1, 1, 2)^T$, $v_{-1} = (1, 0, 1, 0, 2)^T$ and $D_n = \|u_{n+1} - u_n\|$.

Figures 7–10 and Table 6 illustrate the numerical results for the stopping criterion.

Table 6. Numerical results for Figures 7–10.

Algo.name	α_n	λ_n	u	n	Time	Tolerance
EgMSP	–	$\frac{1}{\log(n+3)^5}$	$(-0.7721, 0.8680, 0.6003, -0.7293, 0.2312)^T$	207	4.835206	$\epsilon = 10^{-4}$
EgIASP	–	$\frac{1}{\log(n+3)^5}$	$(-0.8155, 0.8829, 0.6237, -0.7667, 0.2182)^T$	172	4.652686	$\epsilon = 10^{-4}$
Algo2	0.12	$\frac{1}{\log(n+3)^5}$	$(-0.7590, 0.8216, 0.6333, -0.7873, 0.1945)^T$	122	3.384679	$\epsilon = 10^{-4}$
EgMSP	–	$\frac{1}{n+1}$	$(-0.7256, 0.8033, 0.7197, -0.8664, 0.2000)^T$	102	2.427858	$\epsilon = 10^{-4}$
EgIASP	–	$\frac{1}{n+1}$	$(-0.7256, 0.8033, 0.7198, -0.8664, 0.2000)^T$	88	2.239737	$\epsilon = 10^{-4}$
Algo2	0.12	$\frac{1}{n+1}$	$(-0.7255, 0.8032, 0.7198, -0.8665, 0.2000)^T$	54	1.841815	$\epsilon = 10^{-4}$

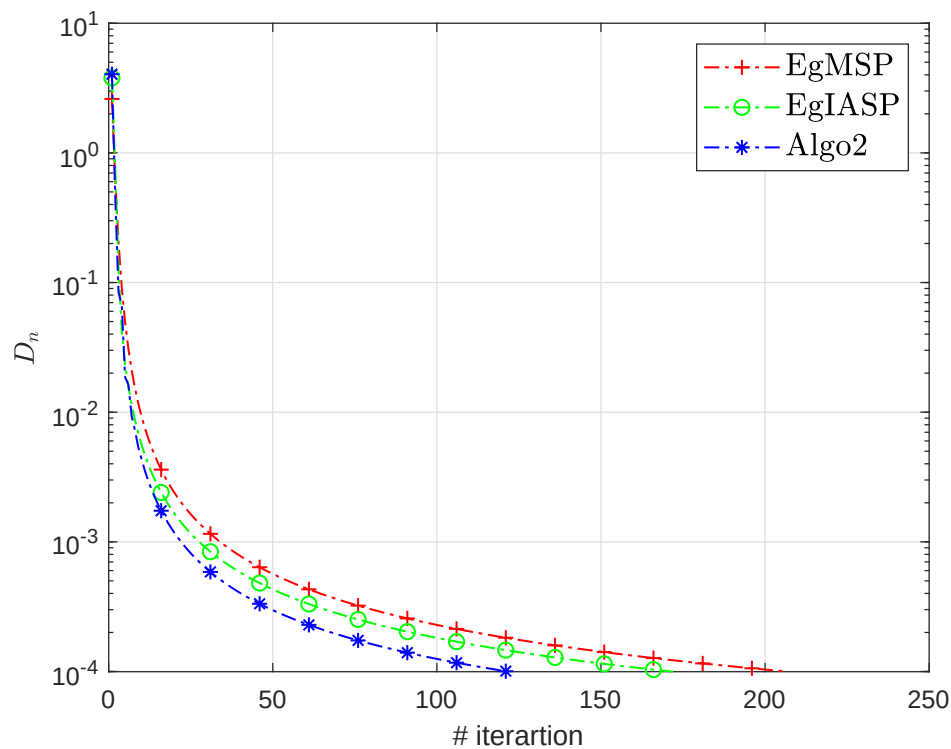


Figure 7. Section 6.2: Algorithm 2 comparison with existing methods.

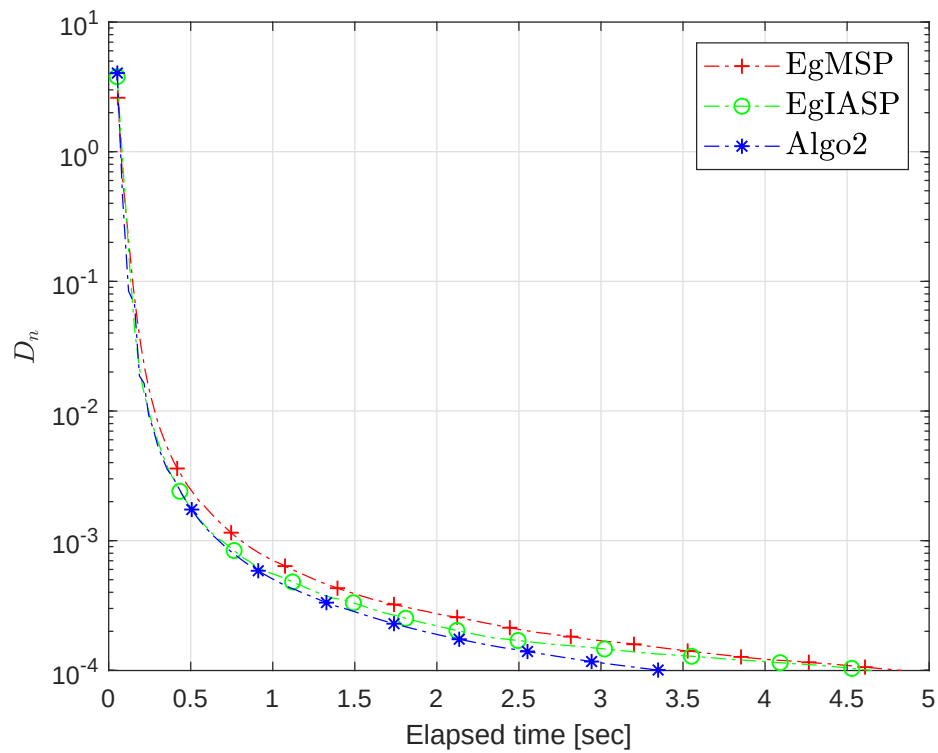


Figure 8. Section 6.2: Algorithm 2 comparison with existing methods.

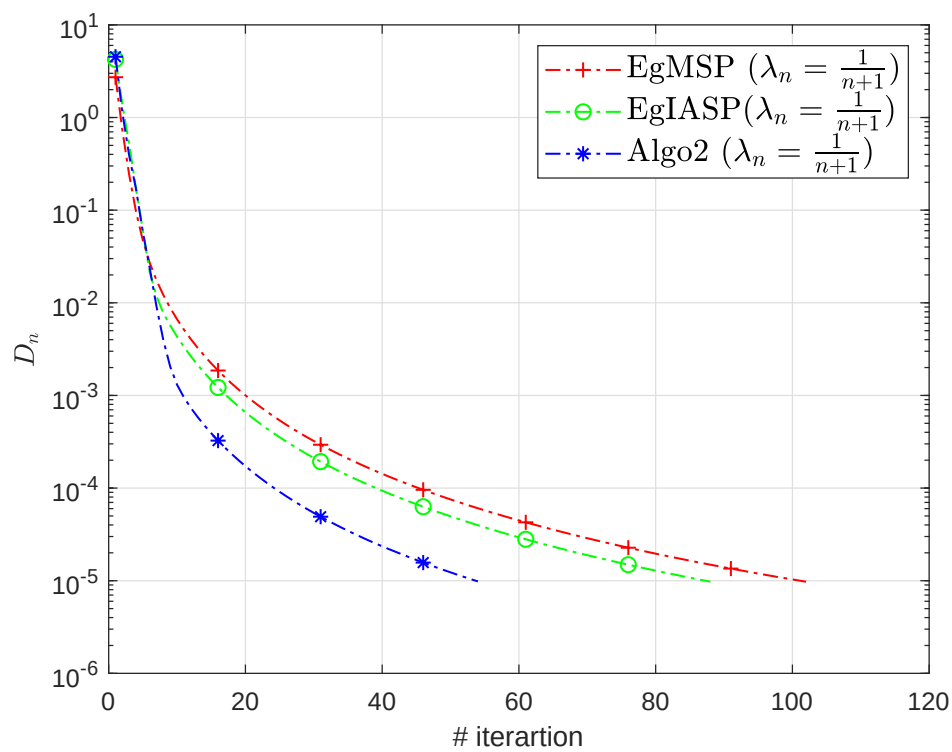


Figure 9. Section 6.2: Algorithm 2 comparison with existing methods.

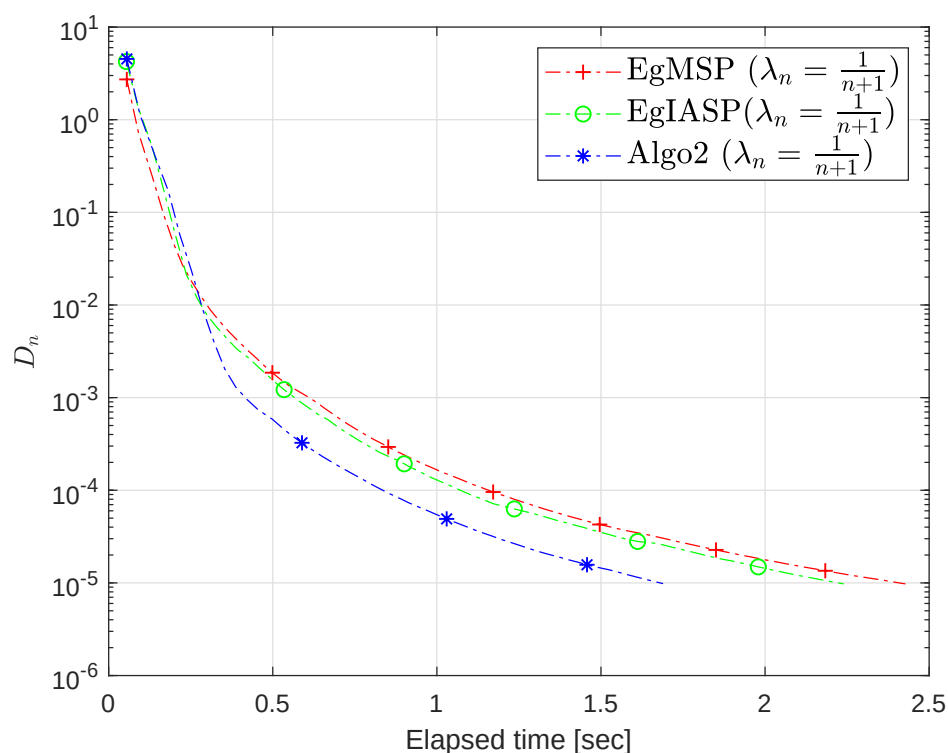


Figure 10. Section 6.2: Algorithm 2 comparison with existing methods.

7. Conclusions

In this study, two proximal-like algorithms were proposed to solve equilibrium problems involving a pseudomonotone and strongly pseudomonotone bifunction with the Lipschitz-type condition on a bifunction. We used a new step size rule that did not depend on the Lipschitz-type constant information. It was identified that the methods with an inertial term worked better than without an inertial term.

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