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Well-Posedness and Time Regularity for a System of Modified Korteweg-de Vries-Type Equations in Analytic Gevrey Spaces

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Abstract: Studies of modified Korteweg-de Vries-type equations are of considerable mathematical interest due to the importance of their applications in various branches of mechanics and physics. In this article, using trilinear estimate in Bourgain spaces, we show the local well-posedness of the initial value problem associated with a coupled system consisting of modified Korteweg-de Vries equations for given data. Furthermore, we prove that the unique solution belongs to Gevrey space $G^{\sigma} \times G^{\sigma}$ in x and $G^{3\sigma} \times G^{3\sigma}$ in t. This article is a continuation of recent studies reflected.

Keywords: modified Korteweg-de Vries equations; well-posedness; analytic Gevrey spaces; Bourgain spaces; trilinear estimates; time regularity

1. Introduction and Main Results

A single initial value problem for the Korteweg-de Vries (KdV) equation is written as

$$\begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x u^2 = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

For $x \in \mathbb{R}$, $t \in [0,T]$ and $u_0(x) \in H^s(\mathbb{R})$, it has been shown that (1) is locally well-posed for $s > \frac{-3}{4}$ in Sobolev spaces [1]. In [2], the authors extended results analyzed in Bourgain spaces [3] for $s \geq 0$ to the case $s > \frac{-3}{4}$ for solution of (1) on time interval $[0,\delta]$, $\delta > 0$. For $s \geq \frac{-1}{2}$, the local well-posedness of corresponding periodic $\mathbb{R}-$valued initial value problem (1) was shown with $x \in \mathbb{T}$. If we change the term u^2 in (1), the problem becomes a modified Korteweg-de Vries equation (mKdV).

For related problems in analytic Gevrey spaces, we review the results obtained in [4], where the Cauchy problem of the Ostrovsky equation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1} u + u\partial_x u = 0,$$

is considered by Boukarou et al. with data in analytic Gevrey spaces on the line and the circle. Based on bilinear estimates in Bourgain spaces, the local well-posedness is proved and Gevrey regularity of

the unique solution is provided. In [5], the question of well-posedness in Bourgain spaces for a class of Cauchy problem for fifth-order Kadomtsev-Petviashvili I equation has been treated in the system

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \\ u(x,y,0) = \varphi(x,y), \end{cases} \tag{2}$$

where $u = u(x,y,t)$, $(x,y) \in \mathbb{R}^2 \quad or \quad \mathbb{T}^2$, $(\alpha,t) \in \mathbb{R}^2$. The authors also obtained the regularity in t and x,y, where the solution is analytic in x,y and belongs to G^5 in t.

The dynamics of solutions in Korteweg-de Vries equation (KdV) and modified Korteweg-de Vries equation (mKdV) are well studied due to the complete integrability of these equations. A description, in the framework of modified Korteweg-de Vries equations, is given in many works [6–8] and for KdV the main results were published back in the 1970s, although many results have been obtained very recently [9]. We extend the previous results and propose a coupled system of modified Korteweg-de Vries equations on the line. Here we have a number of detailed articles and reviews, among which we note the work by [10], where the local and global well-posedness in $H^s(\mathbb{T})$ of the modified Korteweg-de Vries equation, for $s \geq 1/2$, have been studied extensively and also the global well-posedness in $L^2(\mathbb{T})$ was established in [2]. To begin with, we consider the problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(uv^2) = 0 \\ \partial_t v + \beta\partial_x^3 v + \partial_x(u^2 v) = 0, \quad x \in \mathbb{R}\,, t \in \mathbb{R},\ 0 < \beta < 1 \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x). \end{cases} \tag{3}$$

We are now in position to motivate our work. As a model example, we recall the following system

$$\begin{cases} \partial_t u + \partial_x^3 u + v\partial_x v = 0 \\ \partial_t v + \alpha\partial_x^3 v + \partial_x(uv) = 0, \end{cases} \tag{4}$$

where $(x,t) \in [0, 2\pi\lambda] \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}, \lambda \geq 1, 0 < \alpha \leq 1$. System (4) with $(u(x,0), v(x,0)) \in H^s([0,2\pi\lambda]) \times H^s([0,2\pi\lambda])$ or $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ is introduced in [11] to study the nonlinear resonant interactions of long wavelength equatorial Rossby waves and barotropic Rossby waves with a significant mid-latitude projection, in the presence of suitable horizontally and vertically sheared zonal mean flows.

For $\beta = 1$, the system (3) is reduced to a particular case of a large class of equations considered by [12]. In this case, the problems of well-posedness as well as the existence and the stability of the solitary waves for this type of system are widely studied by using the pioneer work in [13]. Oh [14] considered the problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(v^2) = 0, \quad u(x,0) = u_0(x) \\ \partial_t v + \beta\partial_x^3 v + \partial_x(uv) = 0, \quad v(x,0) = v_0(x), \quad 0 < \beta < 1, \end{cases} \tag{5}$$

where the local well posedness for data with regularity $s \geq 0$ is showed by using the Fourier transform restriction norm method.

For $0 < \beta < 1$, the author in [9] proved that the initial value problem (3) is locally well posed for given data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}), s > -\frac{1}{2}$.

The modified KdV systems are important in the study of dispersive equations and are considered widely in the literature, as it is known, in mathematical modeling of wave processes in many problems of plasma physics, solid state physics, hydrodynamics, quantum field theory, biophysics, chemical kinetics, fiber optics, etc.

The novelty of our work lies primarily in the use of trilinear estimate in Bourgain spaces, to show the local well-posedness of initial value problem associated with coupled system consisting modified

Korteweg-de Vries Equations (3) for given data. Furthermore, we prove that the unique solution belongs to Gevrey space $G^\sigma \times G^\sigma$ in x and $G^{3\sigma} \times G^{3\sigma}$ in t.

The first main result about the well-posedness of initial value problem related to coupled system of modified Korteweg-de Vries Equations (3) in analytic Gevrey spaces reads as follows.

Theorem 1. *Let $s > -\frac{1}{2}, 0 < \beta < 1, \sigma \geq 1, \delta \in \mathbb{R}_*^+$ and $(u_0, v_0) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s}$. Then for some real number $b > \frac{1}{2}$ and a constant $T = T(\|(u_0, v_0)\|_{G^{\sigma,\delta,s} \times G^{\sigma,\delta,s}})$, the Cauchy problem (3) admits a unique local in time solution*

$$(u, v) \in C\left([0, T], G^{\sigma,\delta,s}\right) \times C\left([0, T], G^{\sigma,\delta,s}\right). \tag{6}$$

Moreover, the map $(u_0, v_0) \to (u, v)$ is Lipschitz continuous from $G^{\sigma,\delta,s} \times G^{\sigma,\delta,s}$ to $C\left([0, T], G^{\sigma,\delta,s}\right) \times C\left([0, T], G^{\sigma,\delta,s}\right)$.

The analytic spaces $G^{\theta,s}$ have been introduced by Foias and Temam [15] by the norm

$$\|f\|^2_{G^{\theta,s}} = \int e^{2\theta(1+|\xi|)}(1 + |\xi|)^{2s}|\widehat{f}(\xi)|^2 d\xi < \infty,$$

and used by Grujic and Kalisch [16] to prove the well-posedness of non-periodic case for generalized Korteweg-de Vries equation.

Our second aim is to show the Gevrey's temporal regularity of unique solution, by using ideas inspired from [17–23].

Theorem 2. *Let $s > -\frac{1}{2}$, $0 < \beta < 1, \sigma \geq 1$ and $\delta \in \mathbb{R}_*^+$. If $(u_0, v_0) \in G^{\sigma,\delta,s} \times G^{\sigma,\delta,s}$, then the unique solution (6) belongs to the Gevrey class $G^{3\sigma} \times G^{3\sigma}$ in t. Furthermore, it is not belong to $G^d \times G^d, 1 \leq d < 3\sigma$ in time variable.*

The paper is organized as follows. Theorems 1 and 2 introduced in Section 1 are central. In Section 2, we define the function spaces, linear estimates and trilinear estimates. In Section 3, we prove Theorem 1, using the trilinear estimate and the linear estimate together with contraction mapping principle. The regularity in time variable is proved in the fourth section.

2. Preliminary Estimates and Function Spaces

2.1. Function Spaces

We define the needed spaces, where the analytic Gevrey spaces and analytic Gevrey Bourgain spaces will be central. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let

$$G^{\sigma,\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{R})} = \int e^{2\delta|\xi|^{1/\sigma}} \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty \right\}, \tag{7}$$

where $\langle\cdot\rangle = (1 + |\cdot|)$. We then define, for $b, s \in \mathbb{R}$, $\sigma \geq 1$ and $\delta \in \mathbb{R}_*^+$, the analytic Gevrey Bourgain spaces related to (3). $X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)$ is a completion of the Schwartz class $S(\mathbb{R}^2)$, subjected to the norm

$$\|w\|_{X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}} \langle\xi\rangle^{2s} \left\langle \tau - \beta\xi^3 \right\rangle^{2b} | \widehat{w}(\xi, \tau) |^2 d\xi d\tau \right)^{\frac{1}{2}}. \tag{8}$$

Sometimes, we use the notation $X^1_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b}$.

2.2. Linear Estimates

To present the proof of theorems, we start with trilinear estimate (8) defined in $X_{\sigma,\delta,s,b}(\mathbb{R}^2)$. Let us beginning by the embedded result.

Lemma 1. *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$, $\sigma \geq 1$ and $\delta > 0$. Then, for all $T > 0$ we have*

$$X_{\sigma,\delta,s,b} \hookrightarrow C\left([0,T], G^{\sigma,\delta,s}\right).$$

Proof. Define the operator A by

$$\widehat{Aw}^x(\xi,t) = e^{\delta|\xi|^{1/\sigma}} \widehat{w}^x(\xi,t). \tag{9}$$

It satisfies

$$\|w\|_{X_{\sigma,\delta,s,b}} = \|Aw\|_{X_{s,b}} \quad \text{and} \quad \|w\|_{G^{\sigma,\delta,s}} = \|Aw\|_{H^s}, \tag{10}$$

where $X_{s,b}$ is introduced in [9]. We observe that Aw belongs to $C(\mathbb{R}, H^s)$ and for some $C > 0$, we have

$$\| Aw \|_{C(\mathbb{R},H^s)} \leq C \| Aw \|_{X_{s,b}}. \tag{11}$$

Thus, it follows that $w \in \left([0,T], G^{\sigma,\delta,s}\right)$ and

$$\| w \|_{C\left([0,T],G^{\sigma,\delta,s}\right)} \leq C \| w \|_{X_{\sigma,\delta,s,b}}. \tag{12}$$

□

Owing to the Fourier transform with respect to the spatial variable x of the Cauchy problem (3), we obtain a differential equation and then solving it in t. We localize in t by using a cut-off function, satisfying $\psi \in C_0^\infty$, with $\psi = 1$ in $[-1,1]$, where supp$\psi \subset [-2,2]$, $\psi_T(t) = \psi(\frac{t}{T})$. We consider, for the operators Λ, Γ, the following integral system which is equivalent to Cauchy problem (3)

$$\begin{cases} \Lambda[u,v](t) = \psi(t)S(t)u_0 - \psi_T(t)\displaystyle\int_0^t S(t-\nu)F_1(\nu)d\nu \\ \Gamma[u,v](t) = \psi(t)S_\beta(t)v_0 - \psi_T(t)\displaystyle\int_0^t S_\beta(t-\nu)F_2(\nu)d\nu, \end{cases} \tag{13}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_\beta(t) = e^{-t\beta\partial_x^3}$, the unitary groups related to the linear problems are defined via Fourier transform as $[\widehat{S(t)f}](\xi) = e^{it\xi^3}\widehat{f}(\xi)$ and $[\widehat{S_\beta(t)f}](\xi) = e^{it\beta\xi^3}\widehat{f}(\xi)$. The nonlinear terms are defined by $F_1 = \partial_x(uv^2)$, $F_2 = \partial_x(u^2v)$.

Lemma 2. *Let $s, b \in \mathbb{R}$, $\delta \in \mathbb{R}_*^+$ and $\sigma \geq 1$. For some constant $C > 0$, we have*

$$\|\psi(t)S(t)u_0\|_{X_{\sigma,\delta,s,b}} \leq C \|u_0\|_{G^{\sigma,\delta,s}}, \tag{14}$$

and

$$\|\psi(t)S_\beta(t)v_0\|_{X_{\sigma,\delta,s,b}} \leq C \|v_0\|_{G^{\sigma,\delta,s}}, \tag{15}$$

for all $v_0, u_0 \in G^{\sigma,\delta,s}$.

Proof. By definition, we have

$$\begin{aligned}\psi(t)S(t)u_0 &= C\psi(t)\int_{\mathbb{R}} e^{i(x\xi+t\xi^3)}\widehat{u_0}(\xi)d\xi \\ &= C\int_{\mathbb{R}^2} e^{i(x\xi+t\tau)}\widehat{\psi}(\tau-\xi^3)\widehat{u_0}(\xi)d\xi d\tau.\end{aligned}$$

It follows that

$$\begin{aligned}&\| \psi(t)S(t)u_0 \|^2_{X_{\sigma,\delta,s,b}} \\ &= C\int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}}(1+|\xi|)^{2s}\left(1+|\tau-\xi^3|\right)^{2b}|\widehat{\psi}(\tau-\xi^3)|^2|\widehat{u_0}(\xi)|^2 d\xi d\tau \\ &= C\int_{\mathbb{R}} e^{2\delta|\xi|^{1/\sigma}}(1+|\xi|)^{2s}|\widehat{u_0}(\xi)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}(\tau-\xi^3)|^2\left(1+|\tau-\xi^3|\right)^{2b}d\tau\right)d\xi.\end{aligned}$$

We use the fact that $b > 1/2$ to get

$$\begin{aligned}&\int_{\mathbb{R}}|\widehat{\psi}(\tau-\xi^3)|^2\left(1+|\tau-\xi^3|\right)^{2b}d\tau \\ &\leq C'\int_{\mathbb{R}}|\widehat{\psi}(\tau-\xi^3)|^2 d\tau + C'\int_{\mathbb{R}}|\psi(\tau-\xi^3)|^2\left(1+|\tau-\xi^3|\right)^{2b}d\tau \leq C.\end{aligned} \tag{16}$$

□

Lemma 3. *Let $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leq 0 \leq b < b'+1$, $0 \leq T \leq 1$, $\delta > 0$ and $\sigma \geq 1$, then for some constant $C > 0$, we have*

$$\left\|\psi_T(t)\int_0^t S(t-\nu)F_1(x,\nu)\mathrm{d}\nu\right\|_{X_{\sigma,\delta,s,b}} \leq CT^{1-b+b'}\|F_1\|_{X_{\sigma,\delta,s,b'}}, \tag{17}$$

and

$$\left\|\psi_T(t)\int_0^t S_\beta(t-\nu)F_2(x,\nu)\mathrm{d}\nu\right\|_{X^\beta_{\sigma,\delta,s,b}} \leq CT^{1-b+b'}\|F_2\|_{X^\beta_{\sigma,\delta,s,b'}}. \tag{18}$$

Proof. Define

$$W = \psi_T(t)\int_0^t S(t-\nu)F_1(x,\nu)\mathrm{d}\nu.$$

Let us consider the operator A given by (9), then we have

$$\begin{aligned}\widehat{AW}^x(\xi,t) &= \psi_T(t)\int_0^t\left(e^{-i(t-\nu)\xi^3}\right)e^{\delta|\xi|^{1/\sigma}}\widehat{F_1}^x(\xi,\nu)\mathrm{d}\nu \\ &= \psi_T(t)\int_0^t[\widehat{S(t-\nu)(AF_1)}]^x(\xi,\nu)\mathrm{d}\nu.\end{aligned} \tag{19}$$

Thus

$$\| W \|_{X_{\sigma,\delta,s,b}} = \| AW \|_{X_{s,b}} = \left\|\psi_T(t)\int_0^t S(t-\nu)AF_1(x,\nu)d\nu\right\|_{X_{s,b}}.$$

Using Lemma 2.1 in [9], we have

$$\left\|\psi_T(t)\int_0^t S(t-\nu)AF_1(x,\nu)d\nu\right\|_{X_{s,b}} \leq CT^{1-b+b'}\|AF_1\|_{X_{s,b'}} = CT^{1-b+b'}\|F_1\|_{X_{\sigma,\delta,s,b'}}.$$

The inequality (18) is similar. □

2.3. Trilinear Estimates

The following Lemma states the desired trilinear estimate.

Lemma 4. *Let $s > -\frac{1}{2}$, $\sigma \geq 1$, $\delta > 0$, $b > \frac{1}{2}$ and b' be as in Lemma 3. Then*

$$\| \partial_x(uv^2) \|_{X_{\sigma,\delta,s,b'}} \leq C \| u \|_{X_{\sigma,\delta,s,b}} \| v \|^2_{X^{\beta}_{\sigma,\delta,s,b}}, \tag{20}$$

and

$$\| \partial_x(u^2v) \|_{X^{\beta}_{\sigma,\delta,s,b'}} \leq C \| u \|^2_{X_{\sigma,\delta,s,b}} \| v \|_{X^{\beta}_{\sigma,\delta,s,b}}. \tag{21}$$

Proof. We observe, by considering the operator A in (9), that

$$\begin{aligned} e^{\delta|\xi|^{1/\sigma}} \widehat{uvv} &= (2\pi)^{-2} e^{\delta|\xi|^{1/\sigma}} \widehat{u} * \widehat{v} * \widehat{v} \\ &\leq (2\pi)^{-2} \int_{\mathbb{R}^4} e^{\delta|\xi-\xi_1|^{1/\sigma}} \widehat{u}(\xi-\xi_1, \tau-\tau_1) e^{\delta|\xi_1-\xi_2|^{1/\sigma}} \widehat{v}(\xi_1-\xi_2, \tau_1-\tau_2) \\ &\quad e^{\delta|\xi_2|^{1/\sigma}} \widehat{v}(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\ &= A\widehat{uAv}Av, \end{aligned} \tag{22}$$

since $\delta \mid \xi \mid^{1/\sigma} \leq \delta \mid \xi - \xi_1 \mid^{1/\sigma} + \delta \mid \xi_1 - \xi_2 \mid^{1/\sigma} + \delta \mid \xi_2 \mid^{1/\sigma}, \quad \forall \sigma \geq 1$. Then

$$\begin{aligned} \| \partial_x(uv^2) \|_{X_{\sigma,\delta,s,b'}} &= \| e^{\delta|\xi|^{1/\sigma}} \langle \xi \rangle^s \left\langle \tau - \xi^3 \right\rangle^b \widehat{\partial_x(uvv)}(\xi, \tau) \|_{L^2_{\xi,\tau}} \\ &\leq \| \partial_x(AuAvAv) \|_{X_{s,b'}}. \end{aligned}$$

Now, by using Proposition 2.3 of [9], there exists $C > 0$ such that

$$\begin{aligned} \| \partial_x(AuAvAv) \|_{Xs,b'} &\leq C \| Au \|_{X_{s,b}} \| Av \|^2_{X^{\beta}_{s,b}} \\ &= C \| u \|_{X_{\sigma,\delta,s,b}} \| v \|^2_{X^{\beta}_{\sigma,\delta,s,b}}. \end{aligned}$$

□

3. Proof of Theorem 1

3.1. Existence of Solution

We are now ready to estimate all terms in (13) by using the trilinear estimates in the above Lemmas. We define the spaces

$$B_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b} \times X^{\beta}_{\sigma,\delta,s,b} \quad \text{and} \quad N^{\sigma,\delta,s} = G^{\sigma,\delta,s} \times G^{\sigma,\delta,s},$$

with norms

$$\|(u,v)\|_{B_{\sigma,\delta,s,b}} = max\{\|u\|_{X_{\sigma,\delta,s,b}}, \|v\|_{X^{\beta}_{\sigma,\delta,s,b}}\},$$

and similar for $N^{\delta,s}$.

Lemma 5. *Let $s > -\frac{1}{2}$, $\sigma \geq 1$ and $\delta \in \mathbb{R}_*^+$, $b > \frac{1}{2}$. Then, for all $(u_0, v_0) \in N^{\sigma,\delta,s}$ and $T \in (0,1)$, with some constant $C > 0$, we have*

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq C \left(\| (u_0, v_0) \|_{N^{\sigma,\delta,s}} + T^{\epsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}} \right), \tag{23}$$

and

$$\begin{aligned} &\|(\Lambda[u,v] - \Lambda[u^*,v^*], \Gamma[u,v] - \Gamma[u^*,v^*])\|_{B_{\sigma,\delta,s,b}} \\ &\leq CT^{\epsilon} \|(u-u^*, v-v^*)\|_{B_{\sigma,\delta,s,b}} \Big(\|(u,v)\|^2_{B_{\sigma,\delta,s,b}} \\ &+ \|(u,v)\|_{B_{\sigma,\delta,s,b}} \|(u^*,v^*)\|_{B_{\sigma,\delta,s,b}} + \|(v^*,v^*)\|^2_{B_{\sigma,\delta,s,b}} \Big), \forall (u,v), (u^*,v^*) \in B_{\sigma,\delta,s,b}. \end{aligned} \tag{24}$$

Proof. To estimate (23), it follows

$$\begin{aligned} \| \Lambda[u,v] \|_{X_{\sigma,\delta,s,b}} &\leq C \| u_0 \|_{G^{\sigma,\delta,s}} + CT^{\epsilon} \| u \|_{X_{\sigma,\delta,s,b}} \| v \|^2_{X^{\beta}_{\sigma,\delta,s,b}} \\ &\leq C \| (u_0, v_0) \|_{N^{\sigma,\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \| \Gamma[u,v] \|_{X^{\beta}_{\sigma,\delta,s,b}} &\leq C \| v_0 \|_{G^{\sigma,\delta,s}} + CT^{\epsilon} \| u \|^2_{X_{\sigma,\delta,s,b}} \| v \|_{X^{\beta}_{\sigma,\delta,s,b}} \\ &\leq C \| (u_0, v_0) \|_{N^{\sigma,\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}}. \end{aligned} \tag{26}$$

Therefore, from (25) and (26), we obtain

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq C \left(\| (u_0, v_0) \|_{N^{\sigma,\delta,s}} + T^{\epsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}} \right). \tag{27}$$

For the estimate (24), we observe that

$$\Lambda[u,v] - \Lambda[u^*,v^*] = \psi_T(t) \int_0^t S(t-\nu) \partial_x \left(uv^2 - u^* v^{*2} \right)(x,\nu) d\nu,$$

and

$$\Gamma[u,v] - \Gamma[u^*,v^*] = \psi_T(t) \int_0^t S_{\beta}(t-\nu) \partial_x \left(u^2 v - u^{*2} v^* \right)(x,\nu) d\nu,$$

where, by Proposition 2.3 of [9], we have

$$\omega = \partial_x (u^2 v - u^{*2} v^*) = \partial_x \left[v(u+u^*)(u-u^*) + u^{*2}(v-v^*) \right],$$

and

$$\omega' = \partial_x (uv^2 - u^* v^{*2}) = \partial_x \left[u(v+v^*)(v-v^*) + v^{*2}(u-u^*) \right].$$

□

We will show that $\Lambda \times \Gamma$ is a contraction on the ball $\mathbb{B}(0,R)$ to $\mathbb{B}(0,R)$.

Lemma 6. *Let $s \geq -\frac{1}{2}$, $\sigma \geq 1$, $\delta \in \mathbb{R}_*^+$ and $b > \frac{1}{2}$. Then, for all $(u_0, v_0) \in N^{\sigma,\delta,s}$, such that the map $\Lambda \times \Gamma : \mathbb{B}(0,R) \to \mathbb{B}(0,R)$ is a contraction, where $\mathbb{B}(0,R)$ is given by*

$$\mathbb{B}(0,R) = \{ (u,v) \in B_{\sigma,\delta,s,b}; \|(u,v)\|_{B_{\sigma,\delta,s,b}} \leq R \},$$

where $R = 2C\|(u_0,v_0)\|_{N^{\sigma,\delta,s}}$.

Proof. From Lemma 5, for all $(u,v) \in \mathbb{B}(0,R)$, we have

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq C \| (u_0,v_0) \|_{N^{\sigma,\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}} \leq \frac{R}{2} + CT^{\epsilon}R^3.$$

We choose sufficiently small T such that $T^{\epsilon} \leq \frac{1}{4CR^2}$. Hence,

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq R, \quad \forall (u,v) \in \mathbb{B}(0,R).$$

Thus, $\Lambda \times \Gamma : \Big(\mathbb{B}(0,R), \mathbb{B}(0,R)\Big)$, is a contraction, since

$$\begin{aligned}
&\| (\Lambda[u,v] - \Lambda[u^*,v^*], \Gamma[u,v] - \Gamma[u^*,v^*]) \|_{B_{\sigma,\delta,s,b}} \\
&\leq CT^{\epsilon} \| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}} \Big(\| (u,v) \|^2_{B_{\sigma,\delta,s,b}} + \| (u,v) \|_{B_{\sigma,\delta,s,b}} \| (u^*,v^*) \|_{B_{\sigma,\delta,s,b}} \\
&+ \| (v^*,v^*) \|^2_{B_{\sigma,\delta,s,b}} \Big) \qquad (28) \\
&\leq 3CT^{\epsilon}R^2 \| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}} \\
&\leq \tfrac{3}{4} \| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}}, \forall (u,v) \in \mathbb{B}(0,R).
\end{aligned}$$

□

3.2. The Uniqueness

The uniqueness of solution for $(\Lambda[u,v], \Gamma[u,v]) = (u,v)$ in $\mathbb{B}(0,R)$ comes from the argument used above (Fixed point). For the proof of uniqueness in the whole space $B_{\delta,s,b} = X_{\delta,s,b} \times X^{\beta}_{\delta,s,b}$ can be seen in [24].

3.3. Continuous Dependence of the Initial Data

We will need to prove the next Lemma

Lemma 7. *Let $s > -\frac{1}{2}$ and $\sigma \geq 1$, $\delta \in \mathbb{R}^+_*$, $b > \frac{1}{2}$. Then, for all $(u_0,v_0), (u_0^*,v_0^*) \in N^{\sigma,\delta,s}$, if (u,v) and (u^*,v^*) are two solutions to (3) with initial data $(u_0,v_0), (u_0^*,v_0^*)$, we have*

$$\|(u-u^*, v-v^*)\|_{C([0,T],G^{\sigma,\delta,s})^2} \leq 2C_0C \|(u_0-u_0^*, v_0-v_0^*)\|_{N^{\sigma,\delta,s}}. \qquad (29)$$

Proof. Let $(u,v), (u^*,v^*)$ be two solutions to (3), with initial data $(u_0,v_0), (u_0^*,v_0^*)$, we have from Lemma 1

$$\|u-u^*\|_{C([0,T],G^{\sigma,\delta,s})} \leq C_0 \|u-u^*\|_{X_{\sigma,\delta,s,b}},$$

and

$$\|v-v^*\|_{C([0,T],G^{\sigma,\delta,s})} \leq C_0 \|v-v^*\|_{X^{\beta}_{\sigma,\delta,s,b}}.$$

Taking $(u,v), (u^*,v^*) \in \mathbb{B}(0,R)$ and $T^{\epsilon} \leq \frac{1}{4CR}$, we get

$$\| u-u^* \|_{X_{\sigma,\delta,s,b}} \leq C \| (u_0-u_0^*, v_0-v_0^*) \|_{N^{\sigma,\delta,s}} + \tfrac{3}{4} \| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}}, \qquad (30)$$

and

$$\| v - v^* \|_{X^{\beta}_{\sigma,\delta,s,b}} \leq C \| (u_0 - u_0^*, v_0 - v_0^*) \|_{N^{\sigma,\delta,s}} + \frac{3}{4} \| (u - u^*, v - v^*) \|_{B_{\sigma,\delta,s,b}} . \tag{31}$$

Thus

$$\| (u - u^*, v - v^*) \|_{B_{\sigma,\delta,s,b}} \leq 4C \| (u_0 - u_0^*, v_0 - v_0^*) \|_{N^{\sigma,\delta,s}}, \tag{32}$$

then

$$\|(u - u^*, v - v^*)\|_{C([0,T],G^{\sigma,\delta,s})^2} \leq 4C_0 C \|(u_0 - u_0^*, v_0 - v_0^*)\|_{N^{\sigma,\delta,s}}. \tag{33}$$

This completes the prove of Theorem 1. □

4. Regularity of the Solution to Coupled System (3)

4.1. Gevrey-3σ Regularity in Time

We will now prove the temporal regularity of solution on the line .

Lemma 8. *([17]) Let $s \geq -\frac{1}{2}, \sigma \geq 1, \delta \in \mathbb{R}^+_*$ and (u, v) be the solution of (3). Then $(u, v) \in G^\sigma \times G^\sigma$ in $x, \forall t \in [0, T]$, i. e., for some $C > 0$, we have*

$$|\partial_x^l u| \leq C^{l+1}(l!)^\sigma, l \in \{0, 1, ...\}, \quad \forall t \in [0, T], \ x \in \mathbb{R}, \tag{34}$$

and

$$|\partial_x^l v| \leq C^{l+1}(l!)^\sigma, l \in \{0, 1, ...\}, \quad \forall t \in [0, T], \ x \in \mathbb{R}. \tag{35}$$

Let us consider as in [18], for $\epsilon > 0$, the sequences

$$m_p = \frac{c(p!)^\sigma}{(p+1)^2}, (p = 0, 1, 2, ...), \tag{36}$$

and

$$M_p = \epsilon^{1-p} m_p, \epsilon > 0 \ \ and \ (p = 1, 2, 3, ...). \tag{37}$$

The constant c will be chosen as in [25] so that the next inequality holds

$$\sum_{0 \leq l \leq k} \binom{k}{l} m_l m_{k-l} \leq m_k. \tag{38}$$

We remove 0 and k from the left hand side of (38). We use M_p, to get

$$\sum_{0 < l < k} \binom{k}{l} M_l M_{k-l} \leq M_k, \forall \epsilon > 0. \tag{39}$$

Then, for any $\epsilon > 0$, the sequence M_p satisfies

$$M_j \leq \epsilon M_{j+1}, \ for \ j \geq 2. \tag{40}$$

It is checked that for a given $C > 1$, there exists $\epsilon_0 > 0$ such that

$$C^{j+1}(j!)^\sigma \leq M_j, \ for \ j \geq 2, \forall 0 < \epsilon \leq \epsilon_0. \tag{41}$$

By the definition of M_1 and M_2 in (37), we have for $j = 1$, that

$$M_1 = a\epsilon M_2, \quad where \quad a = \frac{9}{4(2!)^\sigma},$$

for some $C > 0$. Now, let us define

$$M_0 = \frac{c}{8} \text{ and } M = max\{\sqrt{2}, \frac{8C}{c}, \frac{4C^2}{c}\}. \tag{42}$$

The next Lemma is the main idea for the proof of Theorem 2.

Lemma 9. *Let (u, v) be the solution of (3) that satisfies (34) and (35), then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$, we have for all $x \in \mathbb{R}$, $t \in [0, T]$*

$$|\partial_t^j \partial_x^l u| \leq M^{2j+1} M_{l+3j}, \ j \in \{0, 1, 2, ...\}, \ l \in \{0, 1, 2, ...\}, \tag{43}$$

and

$$|\partial_t^j \partial_x^l v| \leq M^{2j+1} M_{l+3j}, \ j \in \{0, 1, 2, ...\}, \ l \in \{0, 1, 2, ...\}. \tag{44}$$

For this end, we need the next results.

Lemma 10. *([18]) Let $n, k \in \{0, 1, 2, ...\}$. We have*

$$\sum_{p=0}^{n} \sum_{q=0}^{k} \binom{n}{p} \binom{k}{q} L_{(n-p)+3(k-q)} L_{p+3q} \leq \sum_{r=1}^{m} \binom{m}{r} L_r L_{m-r}, \tag{45}$$

where $L_j > 0, j = 0, 1, ..., m$ with $m = n + 3k$.

Proof. (Of Lemma 9)

We will prove (43) and (44) by induction. For this end, let $j = 0$, for $l = 0$, it follows from (34), (35) and the definition of M in (42) that

$$|u| \leq C \leq MM_0, \quad \forall t \in [0, T], \ x \in \mathbb{R},$$

and

$$|v| \leq C \leq MM_0, \quad \forall t \in [0, T], \ x \in \mathbb{R}.$$

Similarly, for $l = 1$, we have

$$|\partial_x u| \leq C^2 \leq MM_1, \ \forall t \in [0, T], \ x \in \mathbb{R},$$

and

$$|\partial_x v| \leq C^2 \leq MM_1, \ \forall t \in [0, T], \ x \in \mathbb{R}.$$

By (34), (35) and (41), for $l \geq 2$ there exists $\epsilon_0 > 0$, we have

$$|\partial_x^l u| \leq C^{l+1} (l!)^\sigma \leq M_l \leq MM_l, \ \forall t \in [0, T], \ x \in \mathbb{R},$$

and

$$|\partial_x^l v| \leq C^{l+1} (l!)^\sigma \leq M_l \leq MM_l, \ \forall t \in [0, T], \ x \in \mathbb{R},$$

for all $0 < \epsilon \leq \epsilon_0$. This completes the proof of (43) and (44) for $j = 0$ and $l \in \{0, 1, ...\}$.

Now, we assume that (43) and (44) are true for $0 \leq q \leq j, l \in \{0, 1, ...\}$ and we will prove that it is true for $q = j + 1, l \in \{0, 1, ...\}$.

Noting that

$$\begin{aligned} |\partial_t^{j+1}\partial_x^l u| = |\partial_t^j\partial_x^l(\partial_t u)| &= |\partial_t^j\partial_x^{l+3}u + \partial_t^j\partial_x^{l+1}(uv^2)| \\ &\leq |\partial_t^j\partial_x^{l+3}u| + |\partial_t^j\partial_x^{l+1}(uv^2)|, \end{aligned} \tag{46}$$

and

$$\begin{aligned} |\partial_t^{j+1}\partial_x^l v| = |\partial_t^j\partial_x^l(\partial_t v)| &= |\partial_t^j\partial_x^{l+3}v + \partial_t^j\partial_x^{l+1}(u^2v)| \\ &\leq |\partial_t^j\partial_x^{l+3}v| + |\partial_t^j\partial_x^{l+1}(u^2v)|. \end{aligned} \tag{47}$$

Owing to the induction hypotheses and the condition $M > \sqrt{2}$, we estimate the second term $\partial_t^j\partial_x^{l+3}u$ and $\partial_t^j\partial_x^{l+3}v$ as follows

$$\begin{aligned} |\partial_t^j\partial_x^{l+3}u| &\leq M^{2j+1}M_{l+3+3j} = M^{-2}M^{2(j+1)+1}M_{l+3(j+1)} \\ &\leq \tfrac{1}{2}M^{2(j+1)+1}M_{l+3(j+1)}, \end{aligned} \tag{48}$$

and

$$|\partial_t^j\partial_x^{l+3}v| \leq \tfrac{1}{2}M^{2(j+1)+1}M_{l+3(j+1)}. \tag{49}$$

All these estimates are taken for the linear terms. For the nonlinear terms F_1 and F_2, using the induction hypothesis and Leibniz's rule twice, we have

For the nonlinear terms $F_1 = \partial_x(uv^2)$**.**

$$\begin{aligned} |\partial_t^j\partial_x^{l+1}(uv^2)| &= \Bigg| \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2} \\ &\quad \times\partial_t^{j-q_1}\partial_x^{l+1-p_1}u\partial_t^{q_1-q_2}\partial_x^{p_1-p_2}v\partial_t^{q_2}\partial_x^{p_2}v\Bigg| \\ &\leq \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2} \\ &\quad \times|\partial_t^{j-q_1}\partial_x^{l+1-p_1}u||\partial_t^{q_1-q_2}\partial_x^{p_1-p_2}v||\partial_t^{q_2}\partial_x^{p_2}v|. \end{aligned} \tag{50}$$

Then, by the induction hypotheses, we have

$$\begin{aligned} |\partial_t^j\partial_x^{l+1}(uv^2)| &\leq \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2} \\ &\quad M^{2(j-q_1)+1}M_{l+1-p_1+3(j-q_1)}M^{2(q_1-q_2)+1} \\ &\quad M_{p_1-p_2+3(q_1-q_2)}M^{2q_2+1}M_{p_2+3q_2}. \end{aligned} \tag{51}$$

Next, using Lemma 10 with $p = p_2, l = p_1, q = q_2, k = q_1, L_j = M_j, m = p_1 + 3q_1$, we obtain

$$\begin{aligned}
&\sum_{p_2=0}^{p_1}\sum_{q_2=0}^{q_1}\binom{p_1}{p_2}\binom{q_1}{q_2}M_{(p_1-p_2)+3(q_1-q_2)}M_{p_2+3q_2}\\
&\leq \sum_{r=1}^{m}\binom{m}{r}M_rM_{m-r} \leq (M_0+\epsilon)M_m\\
&= (M_0+\epsilon)M_{p_1+3q_1}.
\end{aligned} \tag{52}$$

By Lemma 10 with $p = p_1, l = l+1, q = q_1, k = j, L_j = M_j, m = l+3j+1$, we obtain

$$\begin{aligned}
&\sum_{p_1=0}^{l+1}\sum_{q_1=0}^{j}\binom{l+1}{p_1}\binom{j}{q_1}M_{(l+1-p_1)+3(j-q_1)}M_{p_1+3q_1}\\
&\leq \sum_{r=1}^{m}\binom{m}{r}M_rM_{m-r} \leq (M_0+\epsilon)M_m\\
&= (M_0+\epsilon)M_{l+3j+1}.
\end{aligned} \tag{53}$$

Continuing like this, we get all the inequalities (51)–(53). Combining these inequalities together with (40), we obtain

$$\begin{aligned}
|\partial_t^j\partial_x^{l+1}(uv^2)| \quad &\leq \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2}\\
&M^{2(j-q_1)+1}M_{l+1-p_1+3(j-q_1)}M^{2(q_1-q_2)+1}\\
&M_{p_1-p_2+3(q_1-q_2)}M^{2q_2+1}M_{p_2+3q_2}\\
&\leq M^{2(j+1)+1}(M_0+\epsilon)^2M_{l+3j+1}\\
&\leq M^{2(j+1)+1}\epsilon^2(M_0+\epsilon)^2M_{l+3(j+1)}.
\end{aligned} \tag{54}$$

We choose $\epsilon \leq \epsilon_0 = \left(\frac{1}{2(M_0+1)^2}\right)^{\frac{1}{2}} < 1$, to get

$$\epsilon^2(M_0+\epsilon)^2 \leq \epsilon^2(M_0+1)^2 \leq (M_0+1)^2\left(\frac{1}{2(M_0+1)^2}\right) = \frac{1}{2}.$$

Hence,

$$|\partial_t^j\partial_x^{l+1}(uv^2)| \leq \frac{1}{2}M^{2(j+1)+1}M_{l+3(j+1)}. \tag{55}$$

The proof for $F_2 = \partial_x(u^2v)$ is similar.
This completes the proof. □

Proof. (First part of Theorem 2).

By Lemma 9, we have

$$|\partial_t^j\partial_x^l u| \leq M^{2j+1}M_{l+3j}, \quad j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

and

$$|\partial_t^j \partial_x^l v| \leq M^{2j+1} M_{l+3j}, \quad j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

where

$$M_p = \epsilon^{1-p} \frac{c(p!)^\sigma}{(p+1)^2},\ p = 1,2,....$$

The application of the last inequality with $j \in \{1,2,...\}, l = 0$ gives

$$\begin{aligned} |\partial_t^j u| &\leq M^{2j+1} M_{3j} = M M^{2j} \epsilon^{1-3j} \tfrac{c((3j)!)^\sigma}{(3j+1)^2} \\ &\leq M\epsilon c \left(\tfrac{M^2}{\epsilon^3}\right)^j ((3j)!)^\sigma \\ &\leq L_0 L^j ((3j)!)^\sigma \\ &\leq L_0 L^j A^{3\sigma j} ((j!)^3)^\sigma \\ &\leq A_0^{j+1} (j!)^{3\sigma}, \end{aligned} \tag{56}$$

and

$$|\partial_t^j v| \leq A_0^{j+1} (j!)^{3\sigma}, \tag{57}$$

where $L_0 = M\epsilon c$, $L = \frac{M^2}{\epsilon^3}$ since $(3j)! \leq A^{3j}(j!)^3$ for $A > 0$ and $A_0 = max\{L_0, LA^{3\sigma}\}$. We also have from (43) and (44) for $l = j = 0$, that

$$|u| \leq MM_0 = M\frac{c}{8}, \quad \forall\, t \in [0,T],\ x \in \mathbb{R}, \tag{58}$$

and

$$|v| \leq MM_0 = M\frac{c}{8}, \quad \forall\, t \in [0,T],\ x \in \mathbb{R}. \tag{59}$$

We set $C = max\{M\frac{c}{8}, A_0\}$, by (56) and (58) it follows for $j \in \{0,1,2,...\}$,

$$|\partial_t^j u| \leq C^{j+1} (j!)^{3\sigma}, \quad \forall\, t \in [0,T],\ x \in \mathbb{R}.$$

From (57) and (59), for $j \in \{0,1,2,...\}$, we get

$$|\partial_t^j v| \leq C^{j+1} (j!)^{3\sigma}, \quad \forall\, t \in [0,T],\ x \in \mathbb{R}.$$

Hence, $(u,v) \in G^{3\sigma} \times G^{3\sigma}$ in t.
The proof of first part of Theorem 2 is completed. □

4.2. Failure of Gevrey-D Regularity in Time

We replace t with $-t$, our system can be rewritten as

$$\begin{cases} \partial_t u = \partial_x^3 u + \partial_x(uv^2) \\ \partial_t v = \beta \partial_x^3 v + \partial_x(u^2 v), \quad 0 < \beta < 1 \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x). \end{cases} \tag{60}$$

The following Lemma will be used to estimate the higher-order derivatives of a solution with respect to t.

Lemma 11. *[19] Let (u, v) be solution of (60). Then we have*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} v), \tag{61}$$

and

$$\partial_t^j v = \beta \partial_x^{3j} v + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} v), \tag{62}$$

for every $j \in \{1, 2, ...\}$.

Definition 1. *Let $\{\omega_k\}$ be a sequence of positive numbers. We denote by $\mathcal{C}(\omega_k)$ the class of all functions $g(x)$, infinitely differentiable on $[-1, 1]$, for each of which there is a $C > 0$ such that*

$$|g^{(k)}(x)| \leq C^{k+1} \omega_k, \quad x \in [-1, 1] \text{ and } k = 0, 1, 2, ... \tag{63}$$

Lemma 12. *([18]) For any $\sigma > 1$ and any sequence of complex numbers $\{\varphi_k\}$, satisfying*

$$|\varphi_k| \leq C_1^{k+1} k^{k\sigma}, C_1 > 0, \tag{64}$$

there exists a function $g(x) \in \mathcal{C}(k^{k\sigma})$ for which $g^{(k)}(0) = \varphi_k$.

This result will be used for the sequence of real numbers

$$|g^{(k)}(x)| \leq C^{k+1} k^{k\sigma} \leq C^{k+1} (k!)^\sigma e^{k\sigma}, \quad k = 0, 1, 2, ... \tag{65}$$

where $g(x) \in \mathcal{C}(k^{k\sigma})$ such that $g^{(k)}(0) = \varphi_k = (k!)^\sigma$.

We choose $u_0, v_0 \in G_c^\sigma(-2, 2)$ such that

$$\begin{cases} \theta(x) = 1 \; for \; \mid x \mid \leq 1 \\ and \\ \theta(x) = 0 \; for \; \mid x \mid > 2, \end{cases} \tag{66}$$

by modifying $g(x)$ to have a compact support in $(-1, 1)$.
If u_0 and v_0 are extensions of θg, then we have $u_0, v_0 \in G^\sigma$. We have then the relation by $g(x)$

$$u_0^{(k)}(0) = g^{(k)}(0) = (k!)^\sigma \; and \; v_0^{(k)}(0) = g^{(k)}(0) = (k!)^\sigma. \tag{67}$$

We will show that (u, v) does not need to be $G^d \times G^d$, with $1 \leq d < 3\sigma$ in t.

Theorem 3. *Let $s > -\frac{1}{2}$, $0 < \beta < 1, \sigma \geq 1$ and $\delta \in \mathbb{R}_*^+$. The real-valued solution to (60) with real-valued initial data $(u_0, v_0) \in G^\sigma \times G^\sigma$ may not be in $G^d \times G^d$, with $1 \leq d < 3\sigma$ in t.*

Proof. By using (61) and (67) we get

$$\begin{aligned} \partial_t^j u(0,0) &= \partial_x^{3j} u(0,0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\ &= u_0^{3j}(0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} v_0(0)) \\ &\geq u_0^{3j}(0) = ((3j)!)^\sigma \geq (j!)^{3\sigma}, \end{aligned} \tag{68}$$

and

$$\begin{aligned}\partial_t^j v(0,0) &= \partial_x^{3j} v(0,0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\ &= v_0^{3j}(0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} v_0(0)) \\ &\geq v_0^{3j}(0) = ((3j)!)^\sigma \geq (j!)^{3\sigma},\end{aligned} \tag{69}$$

we have proved that $(u(0,\cdot), v(0,\cdot)) \notin G^d \times G^d$ for $1 \leq d < 3\sigma$ and for t near 0. □

This completes the proof of Theorem 2.

5. Conclusions

In mathematics, the Korteweg-de Vries equation is a mathematical model of waves on a shallow water surface, it was first introduced by Boussines (1877) and rediscovered by Kortewek and Gustav de Vries (1895). This is particularly visible as a prototype of an example of a precisely soluble model, that is to say a nonlinear partial differential equation whose solutions can be determined with precision. The mathematical theory underlying Korteweg-de Vries is the subject of active research. The main results in our paper are the following. In the first part of the manuscript, it is proved that (3) is locally well-posed in analytic Gevrey spaces (Theorem 1). In the second part of the manuscript, it is shown that the solutions of (3) obtained are Gevrey functions of order 3σ in time variable. Then, it is shown that the Gevrey regularity in time is sharp for the periodic case, that is, there exist initial data $(u_0, v_0) \in G^{\sigma,\delta,s}(\mathbb{R}) \times G^{\sigma,\delta,s}(\mathbb{R})$ such that the related solution (u, v) of (3) depending on real-valued initial data (u_0, v_0) is not Gevrey in time of order d for any $1 \leq d < 3\sigma$ (Theorem 2).

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