

On Distance Signless Laplacian Spectral Radius and Distance Signless Laplacian Energy

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Abstract: In this article, we find sharp lower bounds for the spectral radius of the distance signless Laplacian matrix of a simple undirected connected graph and we apply these results to obtain sharp upper bounds for the distance signless Laplacian energy graph. The graphs for which those bounds are attained are characterized.

Keywords: distance signless Laplacian matrix; spectral radius; distance signless Laplacian energy

MSC: 05C50; 15A18; 15A42; 35P15

1. Introduction and Preliminaries on Distance Matrix

Let $G = (V(G), E(G))$ be a connected simple undirected graph with vertices set $V(G)$ and edges set $E(G)$. The distance $d(v_i, v_j)$ between the vertices v_i and v_j of G is equal to the length of (number of edges in) the shortest path that connects v_i and v_j . The distance matrix of graph G is an $n \times n$ matrix such that the entry (i, j) is equal to the distance between vertices v_i and v_j of the graph G . This is $D_{i,j}(G) = d_{i,j} = d(v_i, v_j)$.

Definition 1. Let G be a simple connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance degree of a vertex v , denoted by $Tr_G(v)$ is defined to be the sum of the distances from v to all other vertices in G , that is,

$$Tr_G(v) = \sum_{u \in V(G)} d(u, v).$$

The distance degree is also called the first distance degree or Transmission of a vertex v .

Let $Tr(G)$ be the $n \times n$ diagonal matrix defined by $Tr_{i,i}(G) = Tr_G(v_i)$.

For abbreviation, for $i = 1, \dots, n$, we write Tr_i instead of $Tr_G(v_i)$ when no confusion can arise.

Definition 2. A connected graph G is called a k -distance degree regular graph if $Tr_i = k$ for all $i = 1, 2, \dots, n$.

The Wiener index of a graph G , denoted by $W(G)$, was introduced by Wiener [1] and is defined by

$$W(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_{i,j}(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v).$$

It is clear,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v).$$

In [2], Aouchiche and Hansen introduced the distance signless Laplacian matrix of a connected graph G as the $n \times n$ matrix defined by $DQ(G) = Tr(G) + D(G)$, where $D(G)$ is the distance matrix of G and $Tr(G)$ is the diagonal matrix of vertex transmissions of G . Furthermore, they prove the equivalence between the distance Laplacian matrix, distance signless Laplacian matrix and the distance spectra for the class of transmission regular graphs.

Since $D(G)$ and $DQ(G)$ are real symmetric matrices, we can denote

$$\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \dots \geq \lambda_{n-1}(D(G)) \geq \lambda_n(D(G))$$

and

$$\lambda_1(DQ(G)) \geq \lambda_2(DQ(G)) \geq \dots \geq \lambda_{n-1}(DQ(G)) \geq \lambda_n(DQ(G))$$

to the eigenvalues of $D(G)$ and $DQ(G)$, respectively.

We recall that the spectral radius of a matrix A is $\rho(A) = \max_{1 \leq i \leq n} \{|\lambda_i(A)|\}$ where, for $i = 1, \dots, n$, $\lambda_i(A)$ are the eigenvalues of matrix A .

Theorem 1 ([3]). *If A is a nonnegative matrix then its spectral radius $\rho(A)$ is an eigenvalue of A and it has an associated nonnegative eigenvector. Furthermore, if A is irreducible, then $\rho(A)$ is a simple eigenvalue of A with an associated positive eigenvector.*

Clearly, $D(G)$ and $DQ(G)$ are irreducible nonnegative matrices; then, $\rho(D(G))$ and $\rho(DQ(G))$ are a simple eigenvalues of $D(G)$ and $DQ(G)$, respectively. Moreover, $DQ(G)$ is a positive semidefinite matrix, i.e., all its eigenvalues are nonnegative.

In several articles, different authors find upper and lower bounds for the distance signless Laplacian spectral radius. In [4], Hong and You gave a lower bound on the distance signless Laplacian spectral radius in terms of the sum row of matrix. In [5], the authors determined the graphs with minimum distance signless Laplacian spectral radius among the the trees, unicyclic graphs and bipartite graphs with fixed numbers of vertices, respectively, and determined the graphs with minimum distance signless Laplacian spectral radius among the connected graphs with fixed numbers of vertices and pendant vertices and the connected graphs with fixed number of vertices and connectivity, respectively. In [6], the authors gave a lower bound on the distance Laplacian spectral radius in terms of largest sum row of the distance matrix and they characterized that the extremal graph attains the maximum distance spectral radius in terms of order and the clique number of graph. In [7], Alhevaz et al. determined some upper and lower bounds on the distance signless Laplacian spectral radius of G based on its order and independence number and characterized the extremal graphs. In [8], the authors gaive bounds of the spectral radius of distance Laplacian and distance signless Laplacian matrices.

Connected simple undirected graphs are of great interest in molecular topology, a field of chemistry which reduces the molecule to a connected simple undirected graph. Eigenvalues and characteristic polynomial have found their use in the characterization of chemical compounds. On applications of eigenvalues, characteristic polynomial, distance matrix and graph invariants to chemistry, we mention the works in [9–19].

On the other hand, the energy of a graph is a concept defined in 1978 by Ivan Gutman [20] and originating from theoretical chemistry. Let $A(G)$ be an adjacency matrix of a graph G of order n ; then, the energy of the graph G is $E(G) = \sum_{i=1}^n |\lambda_i(A(G))|$. Several authors study the energy of bipartite graphs, cyclic and acyclic graphs, regular graph (see, e.g., [21–25], and for more details on graph energy, see [26,27]). The concept energy of a graph has been extended to different matrices associated

with a graph: let M be a matrix associated with a graph G ; then, the energy of matrix M is defined in [28] by

$$E_M(G) = \sum_{i=1}^n |\lambda_i(M(G)) - \bar{\lambda}(M(G))|, \quad (1)$$

where $\bar{\lambda}(M(G))$ is the average of eigenvalues of M . In addition, some important applications of the Laplacian spectra and Laplacian energy are covered in Sections 2.2 and 3.6 in [19].

About distance signless Laplacian energy, in [7], Alhevaz et al. gave a description of the distance signless Laplacian energy of the join of regular graphs in terms of their adjacency spectrum. In [29], Das et al. proved that the complete graph and the star give the smallest distance signless Laplacian energy among all the graphs and trees of order n , respectively. In [30], the authors gave upper bounds on the distance energy, distance Laplacian energy and distance signless Laplacian energy. Moreover, they characterized the graphs attaining the corresponding upper bound.

In this work, we find new sharp lower bounds on the distance signless Laplacian spectral radius and we find new upper bounds on the distance signless Laplacian energy.

To finish this section, we recall some results that are used in this article. The next theorem can be found, in simple form, in A. Cauchy's "Cours d'Analyse" of 1821, which were the lecture notes used at the Ecole Polytechnique of Paris [31]; the version to be presented and its proof can be found in [32].

Theorem 2. (Cauchy's Interlacing Theorem) [32] Let A be a symmetric matrix of order n . Let B be a principal submatrix of order m obtained by deleting both i th row and i th column of A , for some values of i . Suppose A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and B has eigenvalues $\beta_1 \geq \beta_2 \geq \dots \geq \beta_m$. Then,

$$\lambda_{n-m+k} \leq \beta_k \leq \lambda_k,$$

for $k = 1, 2, \dots, m$.

We recall that the Frobenius norm of an $n \times n$ matrix $M = (m_{i,j})$ is

$$\|M\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |m_{i,j}|^2}.$$

Moreover, if M is a normal matrix, then $\|M\|_F^2 = \sum_{i=1}^n |\lambda_i(M)|^2$ where $\lambda_1(M), \dots, \lambda_n(M)$ are the eigenvalues of M . In particular,

$$\|D(G)\|_F^2 = \sum_{i=1}^n (\lambda_i(D(G)))^2 \text{ and } \|DQ(G)\|_F^2 = \sum_{i=1}^n (\lambda_i(DQ(G)))^2.$$

It is clear that,

$$\|DQ(G)\|_F \geq \rho(DQ(G)). \quad (2)$$

Definition 3 ([33]). Let G be a simple connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, distance matrix $D_{i,j} = (d_{i,j})$ and the distance degree sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$, such that $Tr_1 \geq Tr_2 \geq \dots \geq Tr_n$. Then, the second distance degree of a vertex v_i , denoted by T_i , is defined to be

$$T_i = \sum_{j=1}^n d_{i,j} Tr_j.$$

Lemma 1 ([33]). Let G be a simple connected graph of order n with distance degree sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$ and second distance degree sequence $\{T_1, T_2, \dots, T_n\}$. Then,

$$Tr_1^2 + Tr_2^2 + \dots + Tr_n^2 = T_1 + T_2 + \dots + T_n.$$

2. Lower Bounds on the Distance Signless Laplacian Spectral Radius

In this section, we obtain new bounds on the distance signless Laplacian radius in terms of parameters that depend on the distance between the vertices and the order of the graph. Before that, we give some known results.

Lemma 2 ([2]). A connected graph G has only two distinct distance signless Laplacian eigenvalues if and only if G is a complete graph.

Theorem 3 ([2]). Let G be a connected graph on n vertices and m edges, with $m \geq n$. Consider the connected graph \tilde{G} obtained from G by the deletion of an edge. Let

$$\rho(DQ(G)), \lambda_2(DQ(G)), \dots, \lambda_n(DQ(G))$$

and

$$\rho(DQ(\tilde{G})), \lambda_2(DQ(\tilde{G})), \dots, \lambda_n(DQ(\tilde{G}))$$

be the distance signless Laplacian spectra of G and \tilde{G} , respectively. Then, for all $i = 1, \dots, n$,

$$\lambda_i(DQ(G)) \leq \lambda_i(DQ(\tilde{G})).$$

Corollary 1. Let G be a connected graph on n vertices. Then,

$$2(n-1) \leq \rho(DQ(G)).$$

Let \mathbf{e} be the n -dimensional vector of ones.

Theorem 4 ([5]). Let G be a connected graph of order $n > 2$. Then,

$$\rho(DQ(G)) \geq \frac{4W(G)}{n}.$$

The equality holds if and only if G is a distance degree regular graph.

Theorem 5. Let G be a simple connected graph, such that, for $i = 1, \dots, n$, Tr_i is the distance degree of vertex v_i . Then,

$$\rho(DQ(G)) \geq 2 \sqrt{\frac{Tr_1^2 + Tr_2^2 + \dots + Tr_n^2}{n}}.$$

The equality holds if and only if G is a distance degree regular graph.

Proof. Let $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ be the unit positive Perron eigenvector of $DQ(G)$ corresponding to $\rho(DQ(G))$.

We take the unit positive vector $\mathbf{y} = \frac{1}{\sqrt{n}}\mathbf{e}$. Then, we have

$$\rho(DQ(G)) = \sqrt{\rho(DQ(G))^2} = \sqrt{\mathbf{x}^T(DQ(G))^2\mathbf{x}} \geq \sqrt{\mathbf{y}^T(DQ(G))^2\mathbf{y}}. \quad (3)$$

Since $(DQ(G))\mathbf{y} = \frac{1}{\sqrt{n}}[2Tr_1, 2Tr_2, \dots, 2Tr_n]^T$, we obtain

$$\mathbf{y}^T(DQ(G))^2\mathbf{y} = \frac{4(Tr_1^2 + Tr_2^2 + \dots + Tr_n^2)}{n}.$$

Therefore,

$$\rho(DQ(G)) \geq 2\sqrt{\frac{Tr_1^2 + Tr_2^2 + \dots + Tr_n^2}{n}}.$$

Now, we assume that the equality holds. By Equation (3), we have that \mathbf{y} is the positive eigenvector corresponding to $\rho(DQ(G))$. From $DQ(G)\mathbf{y} = \rho(DQ(G))\mathbf{y}$, we obtain that $2Tr_i = \rho(DQ(G))$, for $i = 1, \dots, n$. Therefore, the graph G is a $\frac{\rho(DQ(G))}{2}$ -distance degree regular graph.

Conversely, if G is a distance degree regular graph, then the matrix $DQ(G)$ has constant row sum, $2Tr_1 = 2Tr_2 = \dots = 2Tr_n = k$. From Theorem 1, $k = \rho(DQ(G))$. However,

$$\rho(DQ(G)) = k = \sqrt{\frac{nk^2}{n}} = \sqrt{\frac{4Tr_1^2 + 4Tr_2^2 + \dots + 4Tr_n^2}{n}}.$$

Therefore, the equality holds. \square

Theorem 6. Let G be a simple connected graph, such that, for $i = 1, \dots, n$, Tr_i is the distance degree of vertex v_i and T_i is the second distance degree of vertex v_i . Then,

$$\rho(DQ(G)) \geq \sqrt{\frac{(Tr_1^2 + T_1)^2 + (Tr_2^2 + T_2)^2 + \dots + (Tr_n^2 + T_n)^2}{Tr_1^2 + Tr_2^2 + \dots + Tr_n^2}}.$$

The equality holds if and only if $(Tr_i + \frac{T_i}{Tr_i})$ is constant for all i .

Proof. Using $\mathbf{y} = \frac{1}{\sqrt{Tr_1^2 + Tr_2^2 + \dots + Tr_n^2}}[Tr_1, Tr_2, \dots, Tr_n]^T$, the proof is similar to the previous theorem. \square

Remark 1. Notice that the lower bound given in Theorem 6 improves the lower bound given in Theorem 5 and the lower bound given in Theorem 5 improves the lower bound given in Theorem 4.

In fact, we observe that

$$n \sum_{i=1}^n (Tr_i^2 + T_i)^2 \geq \left(\sum_{i=1}^n (Tr_i^2 + T_i) \right)^2 = \left(\sum_{i=1}^n Tr_i^2 + \sum_{i=1}^n T_i \right)^2.$$

From Lemma 1, we have $\sum_{i=1}^n T_i = \sum_{i=1}^n Tr_i^2$. Then,

$$n \sum_{i=1}^n (Tr_i^2 + T_i)^2 \geq 4 \left(\sum_{i=1}^n Tr_i^2 \right)^2.$$

Moreover, we recall that, $\left(\sum_{i=1}^n Tr_i \right)^2 \leq n \sum_{i=1}^n Tr_i^2$. Thus,

$$\sqrt{\frac{\sum_{i=1}^n (Tr_i^2 + T_i)^2}{\sum_{i=1}^n Tr_i^2}} \geq \sqrt{\frac{4 \left(\sum_{i=1}^n Tr_i^2 \right)^2}{n \sum_{i=1}^n Tr_i^2}} = 2\sqrt{\frac{\left(\sum_{i=1}^n Tr_i^2 \right)}{n}}$$

and

$$2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} \geq 2\sqrt{\frac{\left(\sum_{i=1}^n Tr_i\right)^2}{n^2}} = \frac{4W(G)}{n}.$$

Definition 4 ([34]). For $i = 1, \dots, n$, the sequence $M_1^{(1)}, M_2^{(2)}, \dots, M_i^{(t)}, \dots$ is defined as follows: fix $\alpha \in \mathbb{R}$, let $M_i^{(1)} = (Tr_i)^\alpha$ and for each $t \geq 2$, let $M_i^{(t)} = \sum_{j=1}^n d_{i,j} M_j^{(t-1)}$.

Remark 2. For $\alpha = 1$ and $i = 1, \dots, n$: $M_i^{(1)} = Tr_i$ and $M_i^{(2)} = T_i$.

A generalization of Theorem 6 is the next theorem.

Theorem 7. Let G be a connected graph, α be a real number and t be an integer. Then,

$$\rho(DQ(G)) \geq \sqrt{\frac{\sum_{i=1}^n \left(Tr_i M_i^{(t)} + M_i^{(t+1)}\right)^2}{\sum_{i=1}^n \left(M_i^{(t)}\right)^2}}.$$

The equality holds (for particular values of α and t) if and only if $Tr_i + \frac{M_i^{(t+1)}}{M_i^{(t)}}$ is constant for all i .

Proof. Let $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ be the unit positive Perron eigenvector of $DQ(G)$ corresponding to $\rho(DQ(G))$. Let \mathbf{y} be the unit positive vector defined by

$$\mathbf{y} = \frac{1}{\sqrt{\sum_{i=1}^n \left(M_i^{(t)}\right)^2}} [M_1^{(t)}, M_2^{(t)}, \dots, M_n^{(t)}]^T.$$

Since

$$\rho(DQ(G)) = \sqrt{\rho(DQ(G))^2} = \sqrt{\mathbf{x}^T (DQ(G))^2 \mathbf{x}} \geq \sqrt{\mathbf{y}^T (DQ(G))^2 \mathbf{y}} \quad (4)$$

and

$$(DQ(G))\mathbf{y} = \frac{1}{\sqrt{\sum_{i=1}^n \left(M_i^{(t)}\right)^2}} \left[Tr_1 M_1^{(t)} + M_1^{(t+1)}, Tr_2 M_1^{(t)} + M_2^{(t+1)}, \dots, Tr_n M_n^{(t)} + M_n^{(t+1)}\right]^T,$$

we obtain

$$\mathbf{y}^T (DQ(G))^2 \mathbf{y} = \frac{\sum_{i=1}^n \left(Tr_i M_i^{(t)} + M_i^{(t+1)}\right)^2}{\sum_{i=1}^n \left(M_i^{(t)}\right)^2}.$$

Therefore,

$$\rho(DQ(G)) \geq \sqrt{\frac{\sum_{i=1}^n \left(Tr_i M_i^{(t)} + M_i^{(t+1)}\right)^2}{\sum_{i=1}^n \left(M_i^{(t)}\right)^2}}.$$

Now, we assume that the equality holds. By Equation (4), we have that \mathbf{y} is the positive eigenvector corresponding to $\rho(DQ(G))$. From $DQ(G)\mathbf{y} = \rho(DQ(G))\mathbf{y}$, we obtain that $Tr_i + \frac{M_i^{(t+1)}}{M_i^{(t)}} = \rho(DQ(G))$, for $i = 1, \dots, n$.

Conversely if $Tr_i + \frac{M_i^{(t+1)}}{M_i^{(t)}} = k$, then $Tr_i M_i^{(t)} + M_i^{(t+1)} = k M_i^{(t)}$ for all $i \in \{1, 2, \dots, n\}$. Hence,

$$(DQ(G))\mathbf{y} = \frac{1}{\sqrt{\sum_{i=1}^n (M_i^{(t)})^2}} [kM_1^{(t)}, kM_2^{(t)}, \dots, kM_n^{(t)}]^T = k\mathbf{y}.$$

Therefore, k is an eigenvalue of $DQ(G)$ and \mathbf{y} its corresponding eigenvector. We recall that \mathbf{y} is a positive vector, applying the Theorem 1, we obtain $k = \rho(DQ(G))$. To finish, we show that

$$\rho(DQ(G)) = k = \sqrt{\frac{\sum_{i=1}^n k^2 (M_i^{(t)})^2}{\sum_{i=1}^n (M_i^{(t)})^2}} = \sqrt{\frac{\sum_{i=1}^n (Tr_i M_i^{(t)} + M_i^{(t+1)})^2}{\sum_{i=1}^n (M_i^{(t)})^2}}.$$

□

Example 1. For $i = 1, \dots, 7$, let G_i be the graphs given in Figure 1. In particular, G_5 , G_6 and G_7 are the star, path and cycle on seven vertices, denoted by S_7 , P_7 and C_7 , respectively.

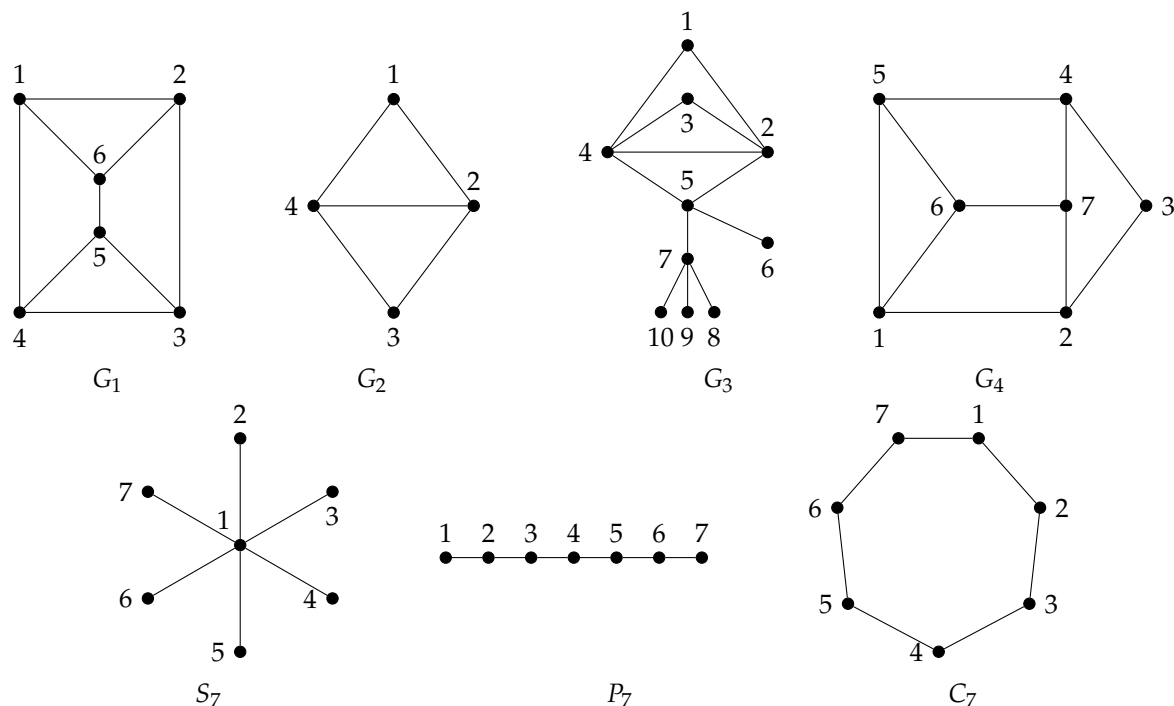


Figure 1. Examples of seven connected simple undirected graphs.

We observe that G_1 is a 7-distance degree regular graph and C_7 is a 12-distance degree regular graph. In Table 1 we show the lower bounds for $\rho(DQ(G_i))$, using six decimal places.

Table 1. Lower bounds for $r(DQ(G_i))$, using six decimal places, where G_i is given in Figure 1

	G_1	G_2	G_3	G_4	S_7	P_7	C_7
$\rho(DQ(G))$	14	7.236067	43.543838	19.080481	21.389866	34.508309	24
Theorem 4	14	7	41.2	18.857142	20.571428	32	24
Theorem 5	14	7.071067	41.923740	18.913336	20.866925	32.741411	24
Theorem 6	14	7.211102	43.318868	19.039390	21.352497	34.223366	24
Theorem 7 $\alpha = 1 \wedge t = 1$	14	7.211102	43.318868	19.039390	21.352497	34.223366	24
Theorem 7 $\alpha = 1 \wedge t = 2$	14	7.197090	43.254373	19.030900	21.340541	34.042264	24
Theorem 7 $\alpha = 1 \wedge t = 3$	14	7.199505	43.264033	19.033222	21.341225	34.081419	24
Theorem 7 $\alpha = 1 \wedge t = 4$	14	7.199129	43.261693	19.032408	21.341188	34.073923	24
Theorem 7 $\alpha = 2 \wedge t = 1$	14	7.230827	43.507702	19.078663	21.376257	34.443494	24
Theorem 7 $\alpha = 2 \wedge t = 2$	14	7.181322	43.190400	19.018253	21.332838	33.830677	24
Theorem 7 $\alpha = 2 \wedge t = 3$	14	7.201677	43.271151	19.035480	21.341620	34.115449	24
Theorem 7 $\alpha = 2 \wedge t = 4$	14	7.198779	43.259397	19.031369	21.341167	34.067028	24
Theorem 7 $\alpha = -1 \wedge t = 1$	14	6.811754	38.9303735	18.725749	19.105982	30.246953	24
Theorem 7 $\alpha = -1 \wedge t = 2$	14	7.222337	43.377491	19.044974	21.368542	34.352322	24
Theorem 7 $\alpha = -1 \wedge t = 3$	14	7.194507	43.242240	19.028687	21.339406	34.000209	24
Theorem 7 $\alpha = -1 \wedge t = 4$	14	7.199892	43.266253	19.033852	21.341285	34.089018	24
Theorem 7 $\alpha = -2 \wedge t = 1$	14	6.472952	34.761644	18.501026	15.666680	27.454188	24
Theorem 7 $\alpha = -2 \wedge t = 2$	14	7.230194	43.423747	19.048631	21.384377	34.421688	24
Theorem 7 $\alpha = -2 \wedge t = 3$	14	7.191858	43.228397	19.026696	21.337653	33.958092	24
Theorem 7 $\alpha = -2 \wedge t = 4$	14	7.200275	43.268215	19.034333	21.341377	34.096220	24

3. Upper Bounds on the Distance Signless Laplacian Energy

In this section, we obtain new bounds on the distance signless Laplacian energy in terms of spectral radius and parameters that depend on the distance between the vertices and the order of the graph. Previously, we extend the ideas given by Koolen and Moulton [35] to the distance signless Laplacian matrix and we use the lower bounds of spectral radius obtained in the previous section.

Theorem 8. Let G be a connected graph on n vertices. Then,

$$E_{DQ} \leq \rho(DQ(G)) - \frac{2W(G)}{n} + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - \left(\rho(DQ(G)) - \frac{2W(G)}{n} \right)^2 \right]}. \quad (5)$$

Moreover,

1. The function

$F(x) = x - \frac{2W(G)}{n} + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - \left(x - \frac{2W(G)}{n} \right)^2 \right]}$ is strictly decreasing in the interval

$$\sqrt{\frac{\|DQ\|^2}{n} - \frac{4W(G)^2}{n^2}} + \frac{2W(G)}{n} < x \leq \sqrt{\|DQ\|^2 - \frac{4W(G)^2}{n}} + \frac{2W(G)}{n}.$$

2. Equality holds if and only if either G is a complete graph or G has at most three distinct distance signless Laplacian eigenvalues. In particular, if G has three distinct distance signless Laplacian eigenvalues, these are $\frac{2W(G)}{n}$, $-\frac{2W(G)}{n}$ and $\rho(DQ(G)) \neq \pm \frac{2W(G)}{n}$.

Proof. From Equation (1), we have that distance signless Laplacian energy of a graph G is

$$E_{DQ}(G) = \sum_{i=1}^n |\lambda_i(DQ(G)) - \bar{\lambda}(DQ(G))|.$$

We observe that

$$\bar{\lambda}(DQ(G)) = \frac{\text{trace}(DQ(G))}{n} = \frac{\sum_{i=1}^n \text{Tr}_i(G)}{n} = \frac{2W(G)}{n}. \quad (6)$$

On the other hand,

$$E_{DQ}(G) = \rho(DQ(G)) - \bar{\lambda}(DQ(G)) + \sum_{i=2}^n |\lambda_i(DQ(G)) - \bar{\lambda}(DQ(G))|.$$

Using the Cauchy–Schwarz inequality, we obtain

$$(E_{DQ}(G) - \rho(DQ(G)) + \bar{\lambda}(DQ(G)))^2 \leq (n-1) \sum_{i=2}^n (\lambda_i(DQ(G)) - \bar{\lambda}(DQ(G)))^2. \quad (7)$$

where

$$\sum_{i=2}^n (\lambda_i(DQ(G)) - \bar{\lambda}(DQ(G)))^2 = \sum_{i=2}^n (\lambda_i(DQ(G))^2 - 2\lambda_i(DQ(G))\bar{\lambda}(DQ(G)) + \bar{\lambda}(DQ(G))^2).$$

For simplicity, we omit the $(DQ(G))$ notation; then,

$$\begin{aligned} \sum_{i=2}^n (\lambda_i(DQ(G)) - \bar{\lambda}(DQ(G)))^2 &= \sum_{i=2}^n \lambda_i^2 - 2\bar{\lambda} \sum_{i=2}^n \lambda_i + (n-1)\bar{\lambda}^2 \\ &= \|DQ\|_F^2 - \rho(G)^2 - 2\bar{\lambda} \sum_{i=1}^n \lambda_i + 2\bar{\lambda}\rho(G) + (n-1)\bar{\lambda}^2 \\ &= \|DQ\|_F^2 - 2\bar{\lambda} \sum_{i=1}^n \lambda_i + n\bar{\lambda}^2 - (\rho(G) - \bar{\lambda})^2 \end{aligned}$$

Using the equalities given in Equation (6), we have

$$\begin{aligned} \sum_{i=2}^n (\lambda_i(DQ) - \bar{\lambda}(DQ))^2 &= \|DQ\|_F^2 - \frac{4W(G)}{n} 2W(G) + n \frac{4W(G)^2}{n^2} - (\rho(G) - \bar{\lambda})^2 \\ &= \|DQ\|_F^2 - \frac{4W(G)^2}{n} - \left(\rho(G) - \frac{2W(G)}{n} \right)^2. \end{aligned}$$

Then, from the inequality in Equation (7), we conclude that

$$E_{DQ} \leq \rho(G) - \frac{2W(G)}{n} + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - \left(\rho(G) - \frac{2W(G)}{n} \right)^2 \right]}.$$

Moreover, let

$$F(x) = \left(x - \frac{2W(G)}{n} \right) + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - \left(x - \frac{2W(G)}{n} \right)^2 \right]}$$

be a real function. We observe that the function F is a strictly decreasing on the interval

$$\sqrt{\frac{\|DQ(G)\|^2}{n} - \frac{4W(G)^2}{n^2}} + \frac{2W(G)}{n} < x \leq \sqrt{\|DQ\|^2 - \frac{4W(G)^2}{n}} + \frac{2W(G)}{n}.$$

Now, the equality in Equation (5) holds if and only if the equality in Equation (7) holds, which is, for $i = 2, \dots, n$,

$$\left| \lambda_2(DQ(G)) - \frac{2W(G)}{n} \right| = \dots = \left| \lambda_n(DQ(G)) - \frac{2W(G)}{n} \right|$$

Therefore, only three cases are possible: (i) the graph G has only one distance signless Laplacian eigenvalues; (ii) the graph G has only two distinct distance signless Laplacian eigenvalues; or (iii) G is a graph with three distinct distance signless Laplacian eigenvalues.

In Case (i), we conclude that $G = K_1$.

In Case (ii), from Lemma 2, we conclude that $G = K_n$.

In Case (iii), we have that $\rho(DQ(G)) \neq \frac{2W(G)}{n}$ and there is r such that $\lambda_2 = \dots = \lambda_r = -\lambda_{r+1} = \dots = -\lambda_n = \frac{2W(G)}{n}$, we conclude that $DQ(G)$ have exactly three distinct eigenvalues.

Conversely, the result is immediate. \square

Lemma 3. Let G be a connected graph. Then,

$$\|DQ(G)\|^2 > \frac{4(W(G))^2}{n}.$$

Proof. Since $\sum_{i=1}^n \left[\lambda_i(DQ(G)) - \frac{2W(G)}{n} \right]^2 > 0$, the result is immediate. \square

Lemma 4 ([30]). Let m and n be natural numbers such that $m \geq n > 2$. Let a_1, a_2, \dots, a_m be positive real numbers. Then,

$$\frac{2}{n} \left(\sum_{i=1}^m a_i \right)^2 > \sum_{i=1}^m a_i^2.$$

Lemma 5. Let G be a connected graph. Then,

$$\sqrt{\frac{\|DQ(G)\|^2}{n} - \frac{4(W(G))^2}{n^2}} < \frac{2W(G)}{n}. \quad (8)$$

Proof. From Lemma 3, $\frac{\|DQ(G)\|^2}{n} - \frac{4(W(G))^2}{n^2} > 0$. Thus, the inequality in Equation (8) is equivalent to

$$\|DQ(G)\|^2 < \frac{2(2W(G))^2}{n}.$$

We recall that

$$\|DQ(G)\|^2 = \sum_{i=1}^n (\lambda_i(DQ(G)))^2 \text{ and } 2W(G) = \sum_{i=1}^n \lambda_i(DQ(G)).$$

Now, using Lemma 4, we have

$$\frac{2(2W(G))^2}{n} = \frac{2 \left(\sum_{i=1}^n \lambda_i(DQ(G)) \right)^2}{n} > \sum_{i=1}^n (\lambda_i(DQ(G)))^2 = \|DQ(G)\|^2.$$

\square

Lemma 6.

$$\rho(DQ(G)) \leq \sqrt{\|DQ(G)\|^2 - \frac{4W(G)^2}{n}} + \frac{2W(G)}{n}.$$

Proof. In fact, using $\|DQ(G)\|^2 - \frac{4W(G)^2}{n} > \|DQ(G)\|^2$ and the inequality in Equation (2) we obtain the result. \square

Then, we obtain the next three theorems.

Theorem 9. Let G be a connected graph on n vertices. Then,

$$E_{DQ}(G) \leq \frac{2W(G)}{n} + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4(n+1)W(G)^2}{n^2} \right]}.$$

Moreover, the equality holds if and only if either G is a complete graph or G is a distance degree regular graph with at most three distinct distance signless Laplacian eigenvalues: $\frac{2W(G)}{n}$, $\frac{-2W(G)}{n}$ and $\rho(DQ(G)) = \frac{4W(G)}{n}$.

Proof. From Lemma 5, we obtain

$$\sqrt{\frac{\|DQ(G)\|^2}{n} - \frac{4W(G)^2}{n^2}} + \frac{2W(G)}{n} < \frac{4W(G)}{n} \quad (9)$$

and, from Theorem 4 and Lemma 6, we have

$$\frac{4W(G)}{n} \leq \sqrt{\|DQ(G)\|^2 - \frac{4W(G)^2}{n}} + \frac{2W(G)}{n}.$$

Then, applying Theorem 8, we conclude that

$$E_{DQ}(G) \leq F\left(\frac{4W(G)}{n}\right) = \frac{2W(G)}{n} + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4(n+1)W(G)^2}{n^2} \right]}.$$

Moreover, the equality holds if and only if the conditions of equality given in Theorems 5 and 8 are satisfied and the fact that $E_{DQ}(G) = F(\rho(G))$. \square

Theorem 10. Let G be a connected graph on n vertices. Then,

$$E_{DQ}(G) \leq 2 \left(\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} - \frac{W(G)}{n} \right) + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - 4 \left(\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} - \frac{W(G)}{n} \right)^2 \right]}.$$

Moreover, the bound is achieved if and only if either G is a complete graph or G is a distance degree regular graph with at most three distinct distance signless Laplacian eigenvalues: $\frac{2W(G)}{n}$, $\frac{-2W(G)}{n}$ and

$$\rho(DQ(G)) = 2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}}.$$

Proof. From Remark 1,

$$\frac{4W(G)}{n} \leq 2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} \leq \rho(DQ(G)).$$

Now, using Equation (9) and Lemma 6, we have

$$\sqrt{\frac{\|DQ(G)\|^2}{n} - \frac{4W(G)^2}{n^2}} + \frac{2W(G)}{n} < 2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}}$$

and

$$2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} \leq \sqrt{\frac{\|DQ(G)\|^2}{n} - \frac{4W(G)^2}{n^2}} + \frac{2W(G)}{n}.$$

Applying Theorem 8, the bound is obtained.

To show that the bound is achieved, we use the conditions of equality given in Theorems 5 and 8 and the fact that $E_{DQ}(G) = F(\rho(G))$. \square

Theorem 11. Let G be a connected graph on n vertices. Then,

$$E_{DQ}(G) \leq \left(\sqrt{\frac{\sum_{i=1}^n (Tr_i^2 + T_i)^2}{\sum_{i=1}^n Tr_i^2} - \frac{2W(G)}{n}} \right) + \sqrt{(n-1) \left[\|DQ(G)\|^2 - \frac{4W(G)^2}{n} - \left(\sqrt{\frac{\sum_{i=1}^n (Tr_i^2 + T_i)^2}{\sum_{i=1}^n Tr_i^2} - \frac{2W(G)}{n}} \right)^2 \right]}.$$

Moreover, the equality holds if and only if either G is a complete graph or (a) $(Tr_i + \frac{T_i}{Tr_i})$ is constant for all i on G and (b) G has at most three distinct distance signless Laplacian eigenvalues $\frac{2W(G)}{n}$, $\frac{-2W(G)}{n}$ and

$$\rho(DQ(G)) = \sqrt{\frac{\sum_{i=1}^n (Tr_i^2 + T_i)^2}{\sum_{i=1}^n Tr_i^2}}.$$

Proof. The proof is similar to given in Theorem 10. \square

Remark 3. From Remark 1 and Theorem 8, we observe that the upper bound given in Theorem 10 improves the upper bound given in Theorem 9 and the upper bound given in Theorem 11 improves the upper bound given in Theorem 10.

Example 2. We consider the graphs $G_1, G_2, G_3, G_4, G_5 = S_7, G_6 = P_7$ and $G_7 = C_7$ given in Example 1 and $F(x)$ given in Theorem 8. Using six decimal places, we obtain the upper bounds for distance signless Laplacian energy, as shown in the Table 2.

Table 2. Upper bounds for EDQ(G_i), using six decimal places, where G_i is given in Figure 1.

	G ₁	G ₂	G ₃	G ₄	S ₇	P ₇	C ₇
$E_{DQ}(G)$	14	7.472200	54.287676	19.796514	22.208371	41.857600	24
$F\left(\frac{4W(G)}{n}\right)$	16.219544	8	71.515224	23.460734	27.191289	52.3318	29.663521
$F\left(2\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}}\right)$	16.219544	7.900272	69.482675	23.287817	26.354659	51.010236	29.663521
$F\left(\sqrt{\frac{\sum_{i=1}^n (Tr_i^2 + T_i)^2}{\sum_{i=1}^n Tr_i^2}}\right)$	16.219544	7.671303	64.745970	22.880250	24.695026	47.607854	29.663521

4. Conclusions

In this paper, we find sharp lower bounds for the spectral radius of the distance signless Laplacian matrix of a simple undirected connected graph G and we apply these results to obtain sharp upper bounds for the distance signless Laplacian energy of G . The graphs for which those bounds are attained are characterized.

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