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Subclasses of Bi-Univalent Functions Defined by Frasin Differential Operator

Ibtisam Aldawish [1], Tariq Al-Hawary [2] and B. A. Frasin [3,*]

[1] Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammed Ibn Saud Islamic University), P.O. Box 90950, Riyadh 11623, Saudi Arabia; imaldawish@imamu.edu.sa
[2] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan; tariq_amh@bau.edu.jo
[3] Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan
* Correspondence: bafrasin@yahoo.com

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Abstract: Let Ω denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belonging to the normalized analytic function class $\mathcal{A}$ in the open unit disk $\mathbb{U}=\{z : |z| < 1\}$, which are bi-univalent in $\mathbb{U}$, that is, both the function f and its inverse f^{-1} are univalent in $\mathbb{U}$. In this paper, we introduce and investigate two new subclasses of the function class Ω of bi-univalent functions defined in the open unit disc $\mathbb{U}$, which are associated with a new differential operator of analytic functions involving binomial series. Furthermore, we find estimates on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Several (known or new) consequences of the results are also pointed out.

Keywords: analytic functions; univalent functions; bi-univalent functions; Taylor–Maclaurin series

MSC: 30C45

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\mathbb{U}=\{z : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike if $f(\mathbb{U})$ is a starlike domain with respect to the origin; i.e., the line segment joining any point of $f(\mathbb{U})$ to the origin lies entirely in $f(\mathbb{U})$ and a function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{U})$ is a convex domain; i.e., the line segment joining any two points in $f(\mathbb{U})$ lies entirely in $f(\mathbb{U})$. Analytically, $f \in \mathcal{A}$ is starlike, denoted by $\mathcal{S}^*$, if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$, whereas $f \in \mathcal{A}$ is convex, denoted by $\mathcal{K}$, if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$. The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$, are respectively characterized by

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}), \tag{2}$$

and

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{3}$$

For a function f in $\mathcal{A}$, and making use of the binomial series

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \quad (m \in \mathbb{N} = \{1, 2, \cdots\},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

Frasin [1] (see also [2–4]) introduced the differential operator $D^{\zeta}_{m,\lambda} f(z)$ defined as follows:

$$D^0 f(z) = f(z), \tag{4}$$

$$D^1_{m,\lambda} f(z) = (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) = D_{m,\lambda} f(z),\ \lambda > 0; m \in \mathbb{N}, \tag{5}$$

$$D^{\zeta}_{m,\lambda} f(z) = D_{m,\lambda}(D^{\zeta-1} f(z)) \quad (\zeta \in \mathbb{N}). \tag{6}$$

If f is given by Equation (1), then from Equations (5) and (6) we see that

$$D^{\zeta}_{m,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n,\ \zeta \in \mathbb{N}_0. \tag{7}$$

Using the relation in Equation (7), it is easily verified that

$$C^m_j(\lambda) z (D^{\zeta}_{m,\lambda} f(z))' = D^{\zeta+1}_{m,\lambda} f(z) - (1 - C^m_j(\lambda)) D^{\zeta}_{m,\lambda} f(z) \tag{8}$$

where $C^m_j(\lambda) := \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

We observe that for $m = 1$, we obtain the differential operator $D^{\zeta}_{1,\lambda}$ defined by Al-Oboudi [5] and for $m = \lambda = 1$, we get Sălăgean differential operator D^{ζ} [6].

In [7], Frasin defined the subclass $\mathcal{S}(\alpha, s, t)$ of analytic functions f satisfying the following condition

$$\operatorname{Re} \left\{ \frac{(s-t) z f'(sz)}{f(sz) - f(tz)} \right\} > \alpha, \tag{9}$$

for some $0 \leq \alpha < 1$ $s, t \in \mathbb{C}$ with $|s| \leq 1;\ |t| \leq 1; s \neq t$ and for all $z \in \mathbb{U}$. We also denote by $\mathcal{T}(\alpha, s, t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $z f'(z) \in \mathcal{S}(\alpha, s, t)$. The class $\mathcal{S}(\alpha, 1, t)$ was introduced and studied by Owa et al. [8]. When $t = -1$, the class $\mathcal{S}(\alpha, 1, -1) \equiv \mathcal{S}_s(\alpha)$ was introduced by Sakaguchi [9] and is called Sakaguchi function of order α (see [10,11]), where as $\mathcal{S}_s(0) = \mathcal{S}_s$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. In addition, we note that $\mathcal{S}(\alpha, 1, 0) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{T}(\alpha, 1, 0) = \mathcal{K}(\alpha)$.

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions in $\mathcal{S}$ gives the growth and distortion bounds as well as covering theorems. It is well known that the n-th coefficient a_n is bounded by n for each $f \in \mathcal{S}$.

In this paper, we estimate the initial coefficients $|a_2|$ and $|a_3|$ coefficient problem for certain subclasses of bi-univalent functions.

The Koebe one-quarter theorem [12] proves that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \qquad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where
$$f^{-1}(w) = h(w) = w + \sum_{n=2}^{\infty} A_n w^n. \tag{10}$$

A simple computation shows that
$$w = f(h(w)) = w + (A_2 + a_2)w^2 + (A_3 - 2a_2^2 + a_3)w^3 + (A_4 + 5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{11}$$

Comparing the initial coefficients in Equation (11), we find that $A_2 = -a_2$, $A_3 = 2a_2^2 - a_3$ and $A_4 = 5a_2^3 + 5a_2 a_3 - a_4$.

By putting these values in the Equation (10), we get
$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in the open unit disk $\mathbb{U}$ if both the function f and its inverse f^{-1} are univalent there. Let Ω denote the class of bi-univalent functions defined in the univalent unit disk $\mathbb{U}$. Examples of functions in the class Ω are
$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Ω. Other common examples of functions in $\mathbb{U}$ such as
$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1-z^2}$$
are not members of Ω either.

Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [13]). Brannan and Taha [14] (see also [15]) introduced certain subclasses of the bi-univalent function class Ω similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ (see [16]). Thus, following Brannan and Taha [14] (see also [15]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Omega[\alpha]$ of strongly bi-starlike functions of order $\alpha (0 < \alpha \leq 1)$ if each of the following conditions are satisfied:
$$f \in \Omega \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$
and
$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathbb{U}),$$
where g is the extension of f^{-1} to $\mathbb{U}$. The classes $\mathcal{S}^*_\Omega(\alpha)$ and $\mathcal{K}_\Omega(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes defined by Equations (2) and (3), were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Omega(\alpha)$ and $\mathcal{K}_\Omega(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [14,15]).

Motivated by the earlier works of Srivastava et al. [17] and Frasin and Aouf [18] (see also [10,12,13,19–33]) in the present paper we introduce two new subclasses $\mathcal{B}^\zeta_\Omega(\lambda, \alpha, s, t)$ and $\mathcal{B}^\zeta_\Omega(\lambda, \beta, s, t)$ of the function class Ω, that generalize the previous defined classes. This subclass is defined with the aid of the new differential operator $D^\zeta_{m,\lambda}$ of analytic functions involving binomial series in the open unit disk $\mathbb{U}$. In addition, upper bounds for the second and third coefficients for functions in this new subclass are derived.

In order to derive our main results, we have to recall the following lemma [34].

Lemma 1. *If $\mathbb{P} \in \mathcal{P}$ then*
$$|c_k| \leq 2 \qquad (k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions $\mathbb{P}$, analytic in $\mathbb{U}$, for which

$$Re(\mathbb{P}(z)) > 0 \qquad (z \in \mathbb{U}),$$

where $\mathbb{P}(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \ (z \in \mathbb{U})$.

Unless otherwise mentioned, we presume throughout this paper that

$$\lambda > 0; m \in \mathbb{N}, s, t \in \mathbb{C} \text{ with } |s| \leq 1;\ |t| \leq 1; s \neq t; \zeta \in \mathbb{N}_0.$$

2. Coefficient Bounds for the Function Class $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \alpha, s, t)$

Definition 1. *A function $f(z)$ given by Equation (1) is said to be in the class $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \alpha, s, t)$ if the following conditions are satisfied:*

$$f \in \Omega \text{ and } \left| \arg \left(\frac{(s-t)z(D_{m,\lambda}^{\zeta} f(z))'}{D_{m,\lambda}^{\zeta} f(sz) - D_{m,\lambda}^{\zeta} f(tz)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U}) \tag{12}$$

and

$$\left| \arg \left(\frac{(s-t)w(D_{m,\lambda}^{\zeta} g(w))'}{D_{m,\lambda}^{\zeta} g(sw) - D_{m,\lambda}^{\zeta} g(tw)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathbb{U}) \tag{13}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{14}$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \alpha, s, t)$.

Theorem 1. *Let $f(z)$ given by (1) be in the class $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \alpha, s, t)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\begin{vmatrix} \alpha(6 - 2s^2 - 2t^2 - 2ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} \\ - \left(1 + C_j^m(\lambda)\right)^{2\zeta} \left[2\alpha(2s + 2t - t^2 - s^2 - 2ts) + (\alpha - 1)(2 - s - t)^2\right] \end{vmatrix}}} \tag{15}$$

and

$$|a_3| \leq \frac{4\alpha^2}{|(2 - s - t)^2| \left(1 + C_j^m(\lambda)\right)^{2\zeta}} + \frac{2\alpha}{|(3 - s^2 - t^2 - ts)| \left(1 + 2C_j^m(\lambda)\right)^{\zeta}}. \tag{16}$$

Proof. From Equations (12) and (13), we have

$$\frac{(s-t)z\left(D_{m,\lambda}^{\zeta} f(z)\right)'}{D_{m,\lambda}^{\zeta} f(sz) - D_{m,\lambda}^{\zeta} f(tz)} = [p(z)]^{\alpha} \tag{17}$$

and

$$\frac{(s-t)w\left(D_{m,\lambda}^{\zeta} g(w)\right)'}{D_{m,\lambda}^{\zeta} g(sw) - D_{m,\lambda}^{\zeta} g(tw)} = [q(w)]^{\alpha}, \tag{18}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{19}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{20}$$

This yields the following relations:

$$(2 - s - t)\left(1 + C_j^m(\lambda)\right)^{\zeta} a_2 = \alpha p_1, \tag{21}$$

$$\begin{aligned} &(3 - s^2 - t^2 - ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} a_3 - (2s + 2t - s^2 - 2ts - t^2)\left(1 + C_j^m(\lambda)\right)^{2\zeta} a_2^2 \\ = \ & \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \end{aligned} \tag{22}$$

$$-(2 - s - t)\left(1 + C_j^m(\lambda)\right)^{\zeta} a_2 = \alpha q_1 \tag{23}$$

and

$$\begin{aligned} &\left[(6 - 2s^2 - 2t^2 - 2ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} - (2s + 2t - s^2 - t^2 - 2ts)\left(1 + C_j^m(\lambda)\right)^{2\zeta}\right] a_2^2 \\ &-(3 - s^2 - t^2 - ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} a_3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \end{aligned} \tag{24}$$

From Equations (21) and (23), we obtain

$$p_1 = -q_1 \tag{25}$$

and

$$2(2 - s - t)^2\left(1 + C_j^m(\lambda)\right)^{2\zeta} a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{26}$$

Now by adding Equation (22) and Equation (24), we deduce that

$$\begin{aligned} &\left[(6 - 2s^2 - 2t^2 - 2ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} - 2(2s + 2t - t^2 - s^2 - 2ts)\left(1 + C_j^m(\lambda)\right)^{2\zeta}\right] a_2^2 \\ = \ & \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2). \end{aligned} \tag{27}$$

From Equations (27) and (26), we have

$$\begin{aligned} &\alpha\left[(6 - 2s^2 - 2t^2 - 2ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} - 2(2s + 2t - t^2 - s^2 - 2ts)\left(1 + C_j^m(\lambda)\right)^{2\zeta}\right] a_2^2 \\ = \ & \alpha^2(p_2 + q_2) + (\alpha - 1)(2 - s - t)^2\left(1 + C_j^m(\lambda)\right)^{2\zeta} a_2^2. \end{aligned} \tag{28}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{\left|\begin{array}{c}\alpha(6 - 2s^2 - 2t^2 - 2ts)\left(1 + 2C_j^m(\lambda)\right)^{\zeta} - 2\alpha(2s + 2t - t^2 - s^2 - 2ts)\left(1 + C_j^m(\lambda)\right)^{2\zeta} \\ -(\alpha - 1)(2 - s - t)^2\left(1 + C_j^m(\lambda)\right)^{2\zeta}\end{array}\right|}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|\begin{array}{l}\alpha(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta} \\ -\left(1+C_j^m(\lambda)\right)^{2\zeta}\left[2\alpha(2s+2t-t^2-s^2-2ts)+(\alpha-1)(2-s-t)^2\right]\end{array}\right|}}$$

which gives us the desired estimate on $|a_2|$ as asserted in Equation (15).

Next in order to find the bound on $|a_3|$, by subtracting Equation (24) from Equation (22), we get

$$\begin{aligned} & 2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta} a_3-(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta} a_2^2 \\ = \; & \alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \end{aligned} \tag{29}$$

From Equations (25), (26) and (29), we obtain

$$\begin{aligned} & 2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta} a_3 \\ = \; & (6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta} \frac{\alpha^2(p_1^2+q_1^2)}{2(2-s-t)^2\left(1+C_j^m(\lambda)\right)^{2\zeta}}+\alpha(p_2-q_2) \end{aligned}$$

or, equivalently,

$$a_3=\frac{\alpha^2(p_1^2+q_1^2)}{2(2-s-t)^2\left(1+C_j^m(\lambda)\right)^{2\zeta}}+\frac{\alpha(p_2-q_2)}{2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}}.$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1 and q_2, we have

$$|a_3| \leq \frac{4\alpha^2}{|(2-s-t)^2|\left(1+C_j^m(\lambda)\right)^{2\zeta}}+\frac{2\alpha}{|(3-s^2-t^2-ts)|\left(1+2C_j^m(\lambda)\right)^{\zeta}}.$$

We get desired estimate on $|a_3|$ as asserted in Equation (16). □

Putting $\zeta=0$ in Theorem 1, we get the following consequence.

Corollary 1. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{B}_{\Omega}^{0}(\alpha,s,t)$, $0<\alpha\leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{|\alpha(6-2s^2-2t^2-2ts)-[2\alpha(2s+2t-t^2-s^2-2ts)+(\alpha-1)(2-s-t)^2]|}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{|(2-s-t)^2|}+\frac{2\alpha}{|(3-s^2-t^2-ts)|}.$$

Putting $s=1$ and $t=-1$ in Corollary 1, we immediately have the following result.

Corollary 2. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{B}_{\Omega}^{0}(\alpha,1,-1)$, $0<\alpha\leq 1$. Then*

$$|a_2| \leq \alpha$$

and

$$|a_3| \leq \alpha(\alpha+1).$$

If we put $s = 1$ and $t = 0$ in Corollary 1, we obtain well-known the class $\mathcal{S}^*_\Omega[\alpha]$ of strongly bi-starlike functions of order α and get the following corollary.

Corollary 3. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{S}^*_\Omega[\alpha]$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$|a_3| \leq \alpha(4\alpha+1).$$

3. Coefficient Bounds for the Function Class $\mathcal{B}^\zeta_\Omega(\lambda, \beta, s, t)$

Definition 2. *A function $f(z)$ given by Equation (1) is said to be in the class $\mathcal{B}^\zeta_\Omega(\lambda, \beta, s, t)$ if the following conditions are satisfied:*

$$f \in \Omega \text{ and } Re\left(\frac{(s-t)z(D^\zeta_{m,\lambda}f(z))'}{D^\zeta_{m,\lambda}f(sz) - D^\zeta_{m,\lambda}f(tz)}\right) > \beta \qquad (0 \leq \beta < 1,\ z \in \mathbb{U}) \tag{30}$$

and

$$Re\left(\frac{(s-t)w(D^\zeta_{m,\lambda}g(w))'}{D^\zeta_{m,\lambda}g(sw) - D^\zeta_{m,\lambda}g(tw)}\right) > \beta \qquad (0 \leq \beta < 1,\ w \in \mathbb{U}) \tag{31}$$

where the function g is given by Equation (14).

Theorem 2. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{B}^\zeta_\Omega(\lambda, \beta, s, t)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|(3-s^2-t^2-ts)\left(1+2C^m_j(\lambda)\right)^\zeta - (2s+2t-t^2-s^2-2ts)\left(1+C^m_j(\lambda)\right)^{2\zeta}|}} \tag{32}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{|(2-s-t)^2|\left(1+C^m_j(\lambda)\right)^{2\zeta}} + \frac{2(1-\beta)}{|(3-s^2-t^2-ts)|\left(1+2C^m_j(\lambda)\right)^\zeta}. \tag{33}$$

Proof. It follows from Equations (30) and (31) that there exist p and $q \in \mathcal{P}$ such that

$$\frac{(s-t)z\left(D^\zeta_{m,\lambda}f(z)\right)'}{D^\zeta_{m,\lambda}f(sz) - D^\zeta_{m,\lambda}f(tz)} = \beta + (1-\beta)p(z) \tag{34}$$

and

$$\frac{(s-t)w\left(D^\zeta_{m,\lambda}g(w)\right)'}{D^\zeta_{m,\lambda}g(sw) - D^\zeta_{m,\lambda}g(tw)} = \beta + (1-\beta)q(w) \tag{35}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ given by Equations (19) and (20).

This yields the following relations:

$$(2-s-t)\left(1+C^m_j(\lambda)\right)^\zeta a_2 = (1-\beta)p_1, \tag{36}$$

$$\begin{aligned}&(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}a_3-(2s+2t-s^2-2ts-t^2)\left(1+C_j^m(\lambda)\right)^{2\zeta}a_2^2\\=\;&(1-\beta)p_2,\end{aligned} \tag{37}$$

$$-(2-s-t)\left(1+C_j^m(\lambda)\right)^{\zeta}a_2=(1-\beta)q_1 \tag{38}$$

and

$$\begin{aligned}&\left[(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}-(2s+2t-s^2-t^2-2ts)\left(1+C_j^m(\lambda)\right)^{2\zeta}\right]a_2^2\\&-(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}a_3=(1-\beta)q_2.\end{aligned} \tag{39}$$

From Equations (36) and (38), we obtain

$$p_1=-q_1 \tag{40}$$

and

$$2(2-s-t)^2\left(1+C_j^m(\lambda)\right)^{2\zeta}a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{41}$$

Now by adding Equation (37) and Equation (39), we deduce that

$$\begin{aligned}&\left[(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}-2(2s+2t-s^2-t^2-2ts)\left(1+C_j^m(\lambda)\right)^{2\zeta}\right]a_2^2\\=\;&(1-\beta)(p_2+q_2).\end{aligned} \tag{42}$$

Thus, we have

$$\begin{aligned}|a_2^2|&\leq\frac{(1-\beta)(|p_2|+|q_2|)}{|(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}-2(2s+2t-t^2-s^2-2ts)\left(1+C_j^m(\lambda)\right)^{2\zeta}|}\\&=\frac{2(1-\beta)}{|(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}-(2s+2t-t^2-s^2-2ts)\left(1+C_j^m(\lambda)\right)^{2\zeta}|}\end{aligned}$$

which gives us the desired estimate on $|a_2|$ as asserted in Equation (32). Next in order to find the bound on $|a_3|$, by subtracting Equation (39) from Equation (37), we get

$$\begin{aligned}&2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}a_3-(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}a_2^2\\=\;&(1-\beta)(p_2-q_2).\end{aligned} \tag{43}$$

From Equations (40), (41) and (43), we obtain

$$\begin{aligned}&2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}a_3\\=\;&(1-\beta)(p_2-q_2)+(6-2s^2-2t^2-2ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}\frac{(1-\beta)^2(p_1^2+q_1^2)}{2(2-s-t)^2\left(1+C_j^m(\lambda)\right)^{2\zeta}}\end{aligned}$$

or, equivalently,

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{2(2-s-t)^2\left(1+C_j^m(\lambda)\right)^{2\zeta}}+\frac{(1-\beta)(p_2-q_2)}{2(3-s^2-t^2-ts)\left(1+2C_j^m(\lambda)\right)^{\zeta}}.$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1 and q_2, we have

$$|a_3| \leq \frac{4(1-\beta)^2}{|(2-s-t)^2|\left(1+C_j^m(\lambda)\right)^{2\zeta}} + \frac{2(1-\beta)}{|(3-s^2-t^2-ts)|\left(1+2C_j^m(\lambda)\right)^{\zeta}}.$$

We get desired estimate on $|a_3|$ as asserted in Equation (33). □

It is worth to mention that a similar technique in the real space has been used in the study of random environments, see [35].

Putting $\zeta = 0$ in Theorem 2, we have the following corollary.

Corollary 4. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{B}_{\Omega}^{0}(\beta, s, t)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|3+st-2(s+t)|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{|(2-s-t)^2|} + \frac{2(1-\beta)}{|(3-s^2-t^2-ts)|}.$$

Putting $s = 1$ and $t = -1$ in Corollary 4, we immediately have the following result.

Corollary 5. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{B}_{\Omega}^{0}(\beta, 1, -1)$, $0 \leq \beta < 1$. Then*

$$|a_2| \leq \sqrt{1-\beta}$$

and

$$|a_3| \leq (1-\beta)(2-\beta).$$

If we take $s = 1$ and $t = 0$ in Corollary 4, we obtain well-known the class $\mathcal{S}_{\Omega}^{*}(\beta)$ of strongly bi-starlike functions of order β and get the following corollary.

Corollary 6. *Let $f(z)$ given by Equation (1) be in the class $\mathcal{S}_{\Omega}^{*}(\beta)$, $0 \leq \beta < 1$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq (1-\beta)(5-4\beta).$$

4. Conclusions

In this paper, two new subclasses of bi-univalent functions related to a new differential operator $D_{m,\lambda}^{\zeta}$ of analytic functions involving binomial series in the open unit disk $\mathbb{U}$ were introduced and investigated. Furthermore, we obtained the second and third Taylor–Maclaurin coefficients of functions in these classes. The novelty of our paper consists of the fact that the operator used by defining the new subclasses of Ω is a very general operator that generalizes two important differential operators, Sãlãgean differential operator D^{ζ} and Al-Oboudi differential operator $D_{1,\lambda}^{\zeta}$. These operators are playing an important role in geometric function theory to define new generalized subclasses of analytic univalent functions and then study their properties. The special cases taken from the main results confirm the validity of these results. We mentioned that all the above estimates for the coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \alpha, s, t)$ and $\mathcal{B}_{\Omega}^{\zeta}(\lambda, \beta, s, t)$ are not sharp. To find the sharp upper bounds for the above estimations, it is still an interesting open problem, as well as for $|a_n|$, $n \geq 4$.

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