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Quantum Codes from Constacyclic Codes over the Ring $F_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$

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Abstract: In this paper, we study the structural properties of $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic codes over $\mathcal{R} = F_q[u_1, u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ where $q = p^m$ for odd prime p and $m \geq 1$. We derive the generators of constacyclic and dual constacyclic codes. We have shown that Gray image of a constacyclic code of length n is a quasi constacyclic code of length $4n$. Also we have classified all possible self dual linear codes over this ring $\mathcal{R}$. We have given the applications by computing non-binary quantum codes over this ring $\mathcal{R}$.

Keywords: cyclic code; constacyclic code; quantum codes; negacyclic code; Gray map

1. Introduction

The class of constacyclic codes consists of an algebraically rich family of error-correcting codes and are the generalizations of cyclic and negacyclic codes. These codes can be easily encoded using shift registers and can be easily decoded due to their rich algebraic structure, which justify their preferred role from engineering perspective.

The study of cyclic codes over the semi-local ring $F_2 + vF_2$, $v^2 = v$ was carried by Zhu et al. [1], which was generalized to $F_3 + vF_3$ with $v^2 = 1$ by Cengellenmis [2]. Later on, Zhu and Wang [3], considered $(1 - 2v)$ -constacyclic codes over $F_p + vF_p$ with $v^2 = v$ and proved that the image of a $(1 - 2v)$ -constacyclic code of length n over $F_p + vF_p$ under the Gray map is a distance-invariant linear cyclic code of length $2n$ over F_p . Since then, a lot of research has been done on the study of constacyclic codes over semi-local rings.

Quantum error-correcting codes have a significant role in both quantum communication as well as in quantum computation. The investigations on quantum error-correcting codes has accomplished huge growth after it has been discovered that there exist a class of codes known as quantum error-correcting codes which preserve quantum information as classical error-correcting codes preserve classical information. Quantum error-correcting codes give an efficient technique to overcome decoherence. In [4], Shor obtained the first quantum error-correcting code. After that, a technique was given by Calderbank et al. in [5] to find quantum error-correcting codes from classical error-correcting codes. Gottesman [6] studied quantum error-correcting codes that saturate the quantum Hamming bound. Currently, the research on quantum error-correcting codes has been developed quickly. Using self-orthogonal or dual containing properties of classical cyclic codes over finite field F_q (q is a power of prime number), many new quantum error-correcting codes have been obtained (for references see [7–11]). Further, Ouyang shown that ‘good codes’ can be attained from MDS codes using the construction given in [12].

Quantum codes over different rings is studied by many authors (for references see [9,13–18]). In these researches, researchers have used the orthogonal properties of cyclic codes to construct quantum codes over these ring. From the last few years computation of quantum codes has been going on using constacyclic codes over finite rings. Gao et al. [19] have studied non-binary quantum codes using u -constacyclic codes over $F_p + uF_p$ with $u^2 = 1$. Quantum MDS codes have been studied [7,8,11] using constacyclic codes and BCH like constacyclic codes.

Recently, Ashraf and Mohammad [20] computed quantum codes from cyclic codes over the ring $F_q[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ where $q = p^m$ for odd prime. In this paper we generalize their work over classes of constacyclic codes. Here we have studied the structural properties of $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic codes over the ring $\mathcal{R} = F_q[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ where $q = p^m$ for odd prime p and $m \geq 1$. As an application, we have constructed the quantum codes over F_q .

2. Preliminaries

Let $\mathcal{R} = F_q + u_1F_q + u_2F_q + u_1u_2F_q$ be the ring such that $u_1^2 = u_1$, $u_2^2 = u_2$ and $u_1u_2 = u_2u_1 - 1$, here F_q is the finite field with q elements, where $q = p^m$ for odd prime p and $m \geq 1$. It is commutative semi-local ring with four maximal ideals. This is a principal ideal non chain ring.

Let $\mathcal{C}$ be a linear code of length n over $\mathcal{R}$. Then, it is a $\mathcal{R}$ -submodule of $\mathcal{R}^n$, where elements of $\mathcal{C}$ are called codeword and it is said to be λ -constacyclic code if and only if $\mathcal{C}$ is invariant under the λ -constacyclic shift operator $\Psi_\lambda : \mathcal{R}^n \rightarrow \mathcal{R}^n$ defined by $\Psi_\lambda(d_0, d_1, \dots, d_{n-1}) = (\lambda d_{n-1}, d_0, \dots, d_{n-2})$ which for $\lambda = 1$ becomes the cyclic code. By identifying each codeword $d = (d_0, d_1, \dots, d_{n-1}) \in \mathcal{R}^n$ to a polynomial $d(x) = d_0 + d_1x + \dots + d_{n-1}x^{n-1}$ in $\mathcal{R}[x] / \langle x^n - \lambda \rangle$, a linear code $\mathcal{C}$ is a λ -constacyclic code of length n over $\mathcal{R}$ if and only if it is an ideal of the ring $\mathcal{R}[x] / \langle x^n - \lambda \rangle$.

For any two elements $r = (r_0, r_1, \dots, r_{n-1})$, $s = (s_0, s_1, \dots, s_{n-1}) \in \mathcal{R}^n$ the inner product is defined as $r \cdot s = \sum_{i=0}^{n-1} r_i s_i$. If $r \cdot s = 0$ then r and s are said to be orthogonal. For a linear code $\mathcal{C}$ of length n over $\mathcal{R}$, the dual code is defined by $\mathcal{C}^\perp = \{r \mid \forall s \in \mathcal{C}, r \cdot s = 0\}$, which is also linear code of length n over $\mathcal{R}$. A code $\mathcal{C}$ is called self orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and self dual if $\mathcal{C}^\perp = \mathcal{C}$.

Any element of $\mathcal{R}$ can be expressed as

$$a + u_1b + u_2c + u_1u_2d = (1 - u_1 - u_2 + u_1u_2)a + (u_1 - u_1u_2)(a + b) + (u_2 - u_1u_2)(a + c) + u_1u_2(a + b + c + d)$$

Let $e_1 = 1 - u_1 - u_2 + u_1u_2$, $e_2 = u_1 - u_1u_2$, $e_3 = u_2 - u_1u_2$ and $e_4 = u_1u_2$. Here $e_i^2 = e_i$, $e_i e_j = 0$ and $\sum_{i=1}^4 e_i = 1$, where $i, j = 1, 2, 3, 4$ and $i \neq j$. It is easy to see that $e_i \mathcal{R} \cong F_q$, for $i = 1, 2, 3, 4$. We can express $\mathcal{R} = e_1 \mathcal{R} \oplus e_2 \mathcal{R} \oplus e_3 \mathcal{R} \oplus e_4 \mathcal{R} = e_1 F_q \oplus e_2 F_q \oplus e_3 F_q \oplus e_4 F_q$. Therefore, for any $r \in \mathcal{R}$ can be expressed uniquely as $r = e_1x + e_2y + e_3w + e_4z$, where $x, y, w, z \in F_q$ for $i = 1, 2, 3, 4$.

We define the Gray map as follows:

$$\psi : \mathcal{R} \longrightarrow F_q^4$$

$$\psi(r) = (x, y, w, z)$$

where $x, y, w, z \in F_q$.

This is a linear map and can be extended component-wise in following way:

$$\psi : \mathcal{R}^n \longrightarrow F_q^{4n}$$

$$r' = (r_0, r_1, \dots, r_{n-1}) \longrightarrow (x_0, x_1, \dots, x_{n-1}, y_0, \dots, y_{n-1}, w_0, \dots, w_{n-1}, z_0, \dots, z_{n-1})$$

where $r_i = e_1x_i + e_2y_i + e_3w_i + e_4z_i$, and $x_i, y_i, w_i, z_i \in F_q$ for $i = 0, 1, \dots, n-1$.

For any element $r = e_1x + e_2y + e_3w + e_4z \in \mathcal{R}$, we define the Lee weight of r as $w_L(r) = w_H(\psi(r))$, where $w_H(\psi(r))$ denotes the Hamming weight of $\psi(r)$ over F_q , where the Hamming weight of any element is defined as the number of nonzero components. The Lee distance between r_1 and $r_2 \in \mathcal{C}$ is defined by $d_L(r_1, r_2) = w_L(r_1 - r_2) = w_H(\psi(r_1 - r_2))$. The Lee distance of $\mathcal{C}$ is defined as $d_L(\mathcal{C}) = \min \{d_L(r_1, r_2) \mid r_1 \neq r_2\}$.

Let $B_i, i = 1, 2, 3, 4$ be code over $\mathcal{R}$. We denote that $B_1 \oplus B_2 \oplus B_3 \oplus B_4 = \{b_1 + b_2 + b_3 + b_4 \mid b_i \in A_i, i = 1, 2, 3, 4\}$ and $B_1 \otimes B_2 \otimes B_3 \otimes B_4 = \{(b_1, b_2, b_3, b_4) \mid b_i \in B_i, i = 1, 2, 3, 4\}$. For a linear code $\mathcal{C}$ of length n over $\mathcal{R}$, we define

$$\mathcal{C}_1 = \{x \in F_q^n \mid e_1x + e_2y + e_3w + e_4z \in \mathcal{C}, \text{ and } y, w, z \in F_q^n\},$$

$$\mathcal{C}_2 = \{y \in F_q^n \mid e_1x + e_2y + e_3w + e_4z \in \mathcal{C}, \text{ and } x, w, z \in F_q^n\},$$

$$\mathcal{C}_3 = \{w \in F_q^n \mid e_1x + e_2y + e_3w + e_4z \in \mathcal{C}, \text{ and } x, y, z \in F_q^n\},$$

$$\mathcal{C}_4 = \{z \in F_q^n \mid e_1x + e_2y + e_3w + e_4z \in \mathcal{C}, \text{ and } x, y, w \in F_q^n\}.$$

Here $\mathcal{C}_i$ are linear codes over F_q^n for $i = 1, 2, 3, 4$. Then $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ are q -ary linear codes of length n . So a linear code $\mathcal{C}$ of length n over $\mathcal{R}$ can be uniquely expressed as $\mathcal{C} = e_1\mathcal{C}_1 \oplus e_2\mathcal{C}_2 \oplus e_3\mathcal{C}_3 \oplus e_4\mathcal{C}_4$ and $|\mathcal{C}| = |\mathcal{C}_1||\mathcal{C}_2||\mathcal{C}_3||\mathcal{C}_4|$ and $d_L(\mathcal{C}) = \min\{d_H(\mathcal{C}_i), i = 1, 2, 3, 4\}$.

A matrix is called generator matrix of $\mathcal{C}$ if the rows of the matrix generates $\mathcal{C}$. If G_i are the generator matrices of q -ary linear codes $\mathcal{C}_i, i = 1, 2, 3, 4$ respectively, then the generator matrix of $\mathcal{C}$ is

$$G = \begin{pmatrix} e_1G_1 \\ e_2G_2 \\ e_3G_3 \\ e_4G_4 \end{pmatrix}$$

and the generator matrix of $\psi(\mathcal{C})$ is

$$\psi(G) = \begin{pmatrix} \psi(e_1G_1) \\ \psi(e_2G_2) \\ \psi(e_3G_3) \\ \psi(e_4G_4) \end{pmatrix}.$$

3. Gray Map and Linear Codes over the Ring $\mathcal{R}$

Proposition 1. The Gray map ψ is a distance preserving F_q -linear map from $\mathcal{R}^n$ (Lee distance) to F_q^{4n} (Hamming distance).

Proof. Let $r_1 = x_1e_1 + y_1e_2 + w_1e_3 + z_1e_4$ and $r_2 = x_2e_1 + y_2e_2 + w_2e_3 + z_2e_4 \in \mathcal{R}^n$, where $x_1, y_1, w_1, z_1, x_2, y_2, w_2, z_2 \in F_q^n$. Then

$$\psi(r_1 + r_2) = (x_1 + x_2, y_1 + y_2, w_1 + w_2, z_1 + z_2) = (x_1, y_1, w_1, z_1) + (x_2, y_2, w_2, z_2) = \psi(r_1) + \psi(r_2).$$

Also, for any $r = e_1x + e_2y + e_3w + e_4z \in \mathcal{R}^n$ and $\alpha \in F_q$, we have

$$\psi(\alpha r) = \psi(\alpha e_1x + \alpha e_2y + \alpha e_3w + \alpha e_4z) = \alpha \psi(r)$$

So ψ is F_q -linear.

For the other part, $d_L(r_1, r_2) = w_L(r_1 - r_2) = w_H(\psi(r_1 - r_2)) = w_H(\psi(r_1) - \psi(r_2)) = d_H(\psi(r_1), \psi(r_2))$. Therefore the Gray map ψ is a distance preserving F_q -linear map. $\square$

Proposition 2. Let $\mathcal{C}$ be a $[n, k, d_L]$ linear code over $\mathcal{R}$. Then $\psi(\mathcal{C})$ is a $[4n, k, d_H]$ linear code over F_q^{4n} , where $d_L = d_H$.

Proof. By the above result, ψ is F_q -linear distance preserving map, therefore $d_L = d_H$. As ψ is bijection therefore $|\mathcal{C}| = |\psi(\mathcal{C})| = q^k$ also $\psi(\mathcal{C})$ has length $4n$. Therefore, $\psi(\mathcal{C})$ is a $[4n, k, d_H]$ linear code over F_q^{4n} , where $d_L = d_H$. $\square$

Theorem 1. If $\mathcal{C}$ is a linear code of length n over $\mathcal{R}$, then $\psi(\mathcal{C}^\perp) = \psi(\mathcal{C})^\perp$. Moreover, if $\mathcal{C}$ is self orthogonal, then $\psi(\mathcal{C})$ is also self orthogonal.

Proof. Let $r_1 = x_1e_1 + y_1e_2 + w_1e_3 + z_1e_4 \in \mathcal{C}$ and $r_2 = x_2e_1 + y_2e_2 + w_2e_3 + z_2e_4 \in \mathcal{C}^\perp$. Then by inner product of r_1 and r_2 , we have

$$r_1 \cdot r_2 = e_1x_1x_2 + e_2y_1y_2 + e_3w_1w_2 + e_4z_1z_2 = 0.$$

which implies, $x_1x_2 = y_1y_2 = w_1w_2 = z_1z_2 = 0$. Also

$$\psi(r_1) \cdot \psi(r_2) = (x_1, y_1, w_1, z_1) \cdot (x_2, y_2, w_2, z_2) = x_1x_2 + y_1y_2 + w_1w_2 + z_1z_2 = 0$$

As $\psi(r_1) \in \psi(\mathcal{C})$ so $\psi(r_2) \in \psi(\mathcal{C})^\perp$. Thus, $\psi(\mathcal{C}^\perp) \subseteq \psi(\mathcal{C})^\perp$. Since $|\psi(\mathcal{C}^\perp)| = |\psi(\mathcal{C})^\perp|$, we get $\psi(\mathcal{C}^\perp) = \psi(\mathcal{C})^\perp$.

Similarly we can show the other part. $\square$

Proposition 3. If $\mathcal{C}$ is a linear code of length n over $\mathcal{R}$, then $\psi(\mathcal{C}) = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \otimes \mathcal{C}_4$. Moreover, $|\psi(\mathcal{C})| = |\mathcal{C}_1||\mathcal{C}_2||\mathcal{C}_3||\mathcal{C}_4|$.

Proof. Since $\mathcal{C} = e_1\mathcal{C}_1 \oplus e_2\mathcal{C}_2 \oplus e_3\mathcal{C}_3 \oplus e_4\mathcal{C}_4$ is a code of length n over $\mathcal{R}$, where $\mathcal{C}_i$, $i = 1..4$ are codes of length n over F_q , each $r \in \mathcal{C}$ can be written as $r = e_1x + e_2y + e_3z + e_4w$, where $x \in \mathcal{C}_1, y \in \mathcal{C}_2, z \in \mathcal{C}_3$ and $w \in \mathcal{C}_4$. Now by definition of the Gray map, we have $\psi(r) = (x, y, z, w) \in \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \otimes \mathcal{C}_4$. Hence $\psi(\mathcal{C}) = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \otimes \mathcal{C}_4$. $\square$

Proposition 4. Let $\mathcal{C} = e_1\mathcal{C}_1 \oplus e_2\mathcal{C}_2 \oplus e_3\mathcal{C}_3 \oplus e_4\mathcal{C}_4$ be linear code over $\mathcal{R}$ of length n . Then $\mathcal{C}^\perp = e_1\mathcal{C}_1^\perp \oplus e_2\mathcal{C}_2^\perp \oplus e_3\mathcal{C}_3^\perp \oplus e_4\mathcal{C}_4^\perp$ is also a linear code over $\mathcal{R}$ of length n .

4. Properties of $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -Constacyclic Codes over $\mathcal{R}$

Theorem 2. If $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ is a unit in $\mathcal{R}$ and $\mathcal{C} = e_1\mathcal{C}_1 \oplus e_2\mathcal{C}_2 \oplus e_3\mathcal{C}_3 \oplus e_4\mathcal{C}_4$ is a linear code of length n over $\mathcal{R}$, then $\mathcal{C}$ is a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$ if and only if $\mathcal{C}_1$ is α -constacyclic, $\mathcal{C}_2$ is $(\alpha + \beta)$ -constacyclic, $\mathcal{C}_3$ is $(\alpha + \gamma)$ -constacyclic and $\mathcal{C}_4$ is $(\alpha + \beta + \gamma + \delta)$ -constacyclic codes of length n over F_q respectively.

Proof. Suppose $\mathcal{C}$ is a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$ and $x = (x_0, x_1, \dots, x_{n-1}) \in \mathcal{C}_1$, $y = (y_0, y_1, \dots, y_{n-1}) \in \mathcal{C}_2$, $w = (w_0, w_1, \dots, w_{n-1}) \in \mathcal{C}_3$ and $z = (z_0, z_1, \dots, z_{n-1}) \in \mathcal{C}_4$ where $x_i, y_i, w_i, z_i \in F_q$ for $i = 0, 1, \dots, n-1$. Let $d = (d_0, d_1, \dots, d_{n-1}) \in \mathcal{C}$, where $d_i = e_1x_i + e_2y_i + e_3w_i + e_4z_i$ for $i = 0, 1, \dots, n-1$. Since $\mathcal{C}$ is a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$, so

$$\Psi_{(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)}(c) = ((\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)c_{n-1}, c_0, \dots, c_{n-2}) \in \mathcal{C}$$

As, $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)c_{n-1} = e_1\alpha x_{n-1} + e_2(\alpha + \beta)y_{n-1} + e_3(\alpha + \gamma)w_{n-1} + e_4(\alpha + \beta + \gamma + \delta)z_{n-1}$

Therefore,

$$\begin{aligned}\Psi_{(\alpha+u_1\beta+u_2\gamma+u_1u_2\delta)}(c) &= e_1(\alpha x_{n-1}, x_0, \dots, x_{n-2}) + e_2((\alpha + \beta)y_{n-1}, y_0, \dots, y_{n-2}) \\ &+ e_3((\alpha + \gamma)w_{n-1}, w_0, \dots, w_{n-2}) + e_4((\alpha + \beta + \gamma + \delta)z_{n-1}, z_0, \dots, z_{n-2}) \in \mathcal{C}\end{aligned}$$

Therefore,

$$\Psi_{(\alpha)}x = (\alpha x_{n-1}, x_0, \dots, x_{n-2}) \in \mathcal{C}_1$$

$$\Psi_{(\alpha+\beta)}y = ((\alpha + \beta)y_{n-1}, y_0, \dots, y_{n-2}) \in \mathcal{C}_2$$

$$\Psi_{(\alpha+\gamma)}w = ((\alpha + \gamma)w_{n-1}, w_0, \dots, w_{n-2}) \in \mathcal{C}_3$$

and

$$\Psi_{(\alpha+\beta+\gamma+\delta)}z = ((\alpha + \beta + \gamma + \delta)z_{n-1}, z_0, \dots, z_{n-2}) \in \mathcal{C}_4$$

Hence $\mathcal{C}_1$ is α -constacyclic, $\mathcal{C}_2$ is $(\alpha + \beta)$ -constacyclic, $\mathcal{C}_3$ is $(\alpha + \gamma)$ -constacyclic and $\mathcal{C}_4$ is $(\alpha + \beta + \gamma + \delta)$ -constacyclic codes of length n over F_q respectively.

Conversely, considering the above notations let $\mathcal{C}_1$ is α -constacyclic, $\mathcal{C}_2$ is $(\alpha + \beta)$ -constacyclic, $\mathcal{C}_3$ is $(\alpha + \gamma)$ -constacyclic and $\mathcal{C}_4$ is $(\alpha + \beta + \gamma + \delta)$ -constacyclic codes of length n over F_q . Therefore,

$$\Psi_{(\alpha)}x = (\alpha x_{n-1}, x_0, \dots, x_{n-2}) \in \mathcal{C}_1$$

$$\Psi_{(\alpha+\beta)}y = ((\alpha + \beta)y_{n-1}, y_0, \dots, y_{n-2}) \in \mathcal{C}_2$$

$$\Psi_{(\alpha+\gamma)}w = ((\alpha + \gamma)w_{n-1}, w_0, \dots, w_{n-2}) \in \mathcal{C}_3$$

and

$$\Psi_{(\alpha+\beta+\gamma+\delta)}z = ((\alpha + \beta + \gamma + \delta)z_{n-1}, z_0, \dots, z_{n-2}) \in \mathcal{C}_4$$

Now as

$$\begin{aligned}&e_1(\alpha x_{n-1}, x_0, \dots, x_{n-2}) + e_2((\alpha + \beta)y_{n-1}, y_0, \dots, y_{n-2}) \\ &+ e_3((\alpha + \gamma)w_{n-1}, w_0, \dots, w_{n-2}) + e_4((\alpha + \beta + \gamma + \delta)z_{n-1}, z_0, \dots, z_{n-2}) \\ &= ((\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)c_{n-1}, c_0, \dots, c_{n-2}) = \Psi_{(\alpha+u_1\beta+u_2\gamma+u_1u_2\delta)}(c)\end{aligned}$$

Therefore, $\mathcal{C}$ is a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. $\square$

Definition 1. Let $a \in F_q^{mn}$ with $a = (a^1|a^2|\dots|a^{r-1}|a^m)$, where $a^i \in F_q^n$ for $i = 1, \dots, m$. Let η_m^λ be a map from F_q^{mn} to F_q^{mn} defined $\eta_m^\lambda(a) = (\Psi_{\lambda_1}(a^1)|\dots|\Psi_{\lambda_m}(a^m))$, where Ψ_{λ_i} is the λ_i -constacyclic shift from F_q^n to F_q^n . A code of length mn over F_q is called a quasi constacyclic code of index m if $\eta_m^\lambda(\mathcal{C}) = \mathcal{C}$.

Theorem 3. Let $\mathcal{C}$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then $\psi(\mathcal{C})$ is a quasi constacyclic code of index 4 and of length $4n$.

Proof. Let $\mathcal{C}$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then by Theorem 2, $\mathcal{C}_1$ is α -constacyclic, $\mathcal{C}_2$ is $(\alpha + \beta)$ -constacyclic, $\mathcal{C}_3$ is $(\alpha + \gamma)$ -constacyclic and $\mathcal{C}_4$ is $(\alpha + \beta + \gamma + \delta)$ -constacyclic codes of length n over F_q . Then by the above definition, $\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \otimes \mathcal{C}_4$ is quasi

constacyclic code of index 4 and of length $4n$. As by the Proposition 3, $\psi(\mathcal{C}) = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \mathcal{C}_3 \otimes \mathcal{C}_4$. Hence, $\psi(\mathcal{C})$ is a quasi constacyclic code of index 4 and of length $4n$. $\square$

Corollary 1. Let $\mathcal{C}$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then its dual $\mathcal{C}^\perp = e_1\mathcal{C}_1^\perp \oplus e_2\mathcal{C}_2^\perp \oplus e_3\mathcal{C}_3^\perp \oplus e_4\mathcal{C}_4^\perp$ is a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)^{-1}$ -constacyclic code of length n over $\mathcal{R}$ if and only if $\mathcal{C}_1^\perp$ is α^{-1} -constacyclic, $\mathcal{C}_2^\perp$ is $(\alpha + \beta)^{-1}$ -constacyclic, $\mathcal{C}_3^\perp$ is $(\alpha + \gamma)^{-1}$ -constacyclic and $\mathcal{C}_4^\perp$ is $(\alpha + \beta + \gamma + \delta)^{-1}$ -constacyclic codes of length n over F_q respectively.

Proof. Using Proposition 4 and arguing similar like Theorem 2, we can prove this result. $\square$

Corollary 2. If $\mathcal{C}$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$, then $\mathcal{C}$ is self dual if and only if $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta) = 1, -1, 1 - 2u_1, 1 - 2u_2, 1 - 2u_1u_2, -1 + 2u_1, -1 + 2u_2, -1 + 2u_1u_2, 1 - 2u_1 + 2u_1u_2, 1 - 2u_2 + 2u_1u_2, -1 + 2u_1 - 2u_1u_2, -1 + 2u_2 - 2u_1u_2, 1 - 2u_1 - 2u_2 + 2u_1u_2, -1 + 2u_1 + 2u_2 - 2u_1u_2, 1 - 2u_1 - 2u_2 + 4u_1u_2, -1 + 2u_1 + 2u_2 - 4u_1u_2$.

Proof. Proof follows from the fact that, $\alpha = \pm 1, (\alpha + \beta) = \pm 1, (\alpha + \gamma) = \pm 1$ and $(\alpha + \beta + \gamma + \delta) = \pm 1$. $\square$

Now using the values $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ from Corollary 2, here we classify all possible types of linear codes over F_q depending upon the values of units. Using the Theorem 2 and Corollary 2 we give the classification of codes over F_q in Table below.

Table 1. Behaviour of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ depending on $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$.

$(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
1	Cyclic	Cyclic	Cyclic	Cyclic
-1	Negacyclic	Negacyclic	Negacyclic	Negacyclic
$1 - 2u_1$	Cyclic	Negacyclic	Cyclic	Cyclic
$-1 + 2u_1$	Negacyclic	Cyclic	Negacyclic	Cyclic
$1 - 2u_2$	Cyclic	Negacyclic	Cyclic	Negacyclic
$-1 + 2u_1$	Negacyclic	Cyclic	Negacyclic	Cyclic
$1 - 2u_1u_2$	Cyclic	Cyclic	Cyclic	Negacyclic
$-1 + 2u_1u_2$	Negacyclic	Negacyclic	Negacyclic	Cyclic
$1 - 2u_1 + 2u_1u_2$	Cyclic	Negacyclic	Cyclic	Cyclic
$-1 + 2u_1 - 2u_1u_2$	Negacyclic	Cyclic	Negacyclic	Negacyclic
$1 - 2u_2 + 2u_1u_2$	Cyclic	Cyclic	Negacyclic	Cyclic
$-1 + 2u_2 - 2u_1u_2$	Negacyclic	Negacyclic	Cyclic	Negacyclic
$1 - 2u_1 - 2u_2 + 2u_1u_2$	Cyclic	Negacyclic	Negacyclic	Negacyclic
$-1 + 2u_1 + 2u_2 - 2u_1u_2$	Negacyclic	Cyclic	Cyclic	Cyclic
$1 - 2u_1 - 2u_2 + 4u_1u_2$	Cyclic	Negacyclic	Negacyclic	Cyclic
$-1 + 2u_1 + 2u_2 - 4u_1u_2$	Negacyclic	Cyclic	Cyclic	Negacyclic

Each row of the Table 1, demonstrates the structure of codes for certain value of a unit. For example, if we take the last row of the above Table 1, then $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta) = -1 + 2u_1 + 2u_2 - 4u_1u_2$. As in this case $\alpha = -1, \beta = 2, \gamma = 2, \delta = -4$. Then by Theorem 2, we have the following result.

Proposition 5. $\mathcal{C}$ is a $(-1 + 2u_1 + 2u_2 - 4u_1u_2)$ -constacyclic code of length n over $\mathcal{R}$ if and only if $\mathcal{C}_1, \mathcal{C}_4$ are negacyclic and $\mathcal{C}_2, \mathcal{C}_3$ are cyclic codes of length n over the F_q .

Remark 1. For each different values of $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ from the first column of the Table 1, we can show similar type of results as in Proposition 5.

Proposition 6. Let $C = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then

1. $C = \langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle$ and $|C| = q^{4n - (\deg(f_1) + \deg(f_2) + \deg(f_3) + \deg(f_4))}$.
2. $C = \langle f(x) \rangle$ and $f(x) | (x^n - (\alpha + u_1\beta + u_2\gamma + u_1u_2\delta))$, such that $f(x) = e_1f_1(x) + e_2f_2(x) + e_3f_3(x) + e_4f_4(x)$.

where f_1, f_2, f_3 and f_4 are the generator polynomials of C_1, C_2, C_3 and C_4 respectively.

Proof. (1.) Let C be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$, then by Theorem 2 C_1 is α -constacyclic, C_2 is $(\alpha + \beta)$ -constacyclic, C_3 is $(\alpha + \gamma)$ -constacyclic code and C_4 is $(\alpha + \beta + \gamma + \delta)$ -constacyclic codes of length n over F_q respectively. Then we can write, $C_1 = (f_1(x)) \subseteq F_q[x]/(x^n - \alpha)$, $C_2 = (f_2(x)) \subseteq F_q[x]/(x^n - (\alpha + \beta))$, $C_3 = (f_3(x)) \subseteq F_q[x]/(x^n - (\alpha + \gamma))$ and $C_4 = (f_4(x)) \subseteq F_q[x]/(x^n - (\alpha + \beta + \gamma + \delta))$. Also as $C = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$, we can write C as,

$$C = \{f(x) \mid f(x) = e_1f_1(x) + e_2f_2(x) + e_3f_3(x) + e_4f_4(x) \text{ where } f_i(x) \in C_i \text{ for } i = 1, 2, 3, 4\}$$

This implies $C \subseteq \langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle \subseteq \mathcal{R}[x]/(x^n - (\alpha + u\beta + v\gamma + uv\delta))$.

On the other hand, let $e_1f_1(x)g_1(x) + e_2f_2(x)g_2(x) + e_3f_3(x)g_3(x) + e_4f_4(x)g_4(x) \in \langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle$, where $g_1(x), g_2(x), g_3(x)$ and $g_4(x)$ are elements of $\mathcal{R}[x]/(x^n - (\alpha + u_1\beta + u_2\gamma + u_1u_2\delta))$. Then there exists $k_1(x) \in F_q[x]/(x^n - \alpha)$, $k_2(x) \in F_q[x]/(x^n - (\alpha + \beta))$, $k_3(x) \in F_q[x]/(x^n - (\alpha + \gamma))$ and $k_4(x) \in F_q[x]/(x^n - (\alpha + \beta + \gamma + \delta))$ such that $e_1g_1(x) = e_1k_1(x)$, $e_2g_2(x) = e_2k_2(x)$, $e_3g_3(x) = e_3k_3(x)$ and $e_4g_4(x) = e_4k_4(x)$. Therefore, $\langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle \subseteq C$. Hence, $C = \langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle$.

As $|\psi(C)| = |C_1||C_2||C_3||C_4|$ and $|\psi(C)| = |C|$, Therefore,

$$|C| = |C_1||C_2||C_3||C_4| = q^{n-\deg(f_1)}q^{n-\deg(f_2)}q^{n-\deg(f_3)}q^{n-\deg(f_4)} = q^{4n - (\sum_{i=1}^4 \deg(f_i))}$$

(2.) Let C be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$ and f_i be the generator polynomial of C_i for $i = 1, 2, 3, 4$. Then by the above result $C = \langle e_1f_1, e_2f_2, e_3f_3, e_4f_4 \rangle$. Suppose $C = \langle e_1f_1 + e_2f_2 + e_3f_3 + e_4f_4 \rangle$, then it is obvious that, $C \subseteq C$. As $e_i^2 = e_i$, $e_ie_j = 0$ where $i, j = 1, 2, 3, 4$ and $i \neq j$ then, $e_i(e_1f_1 + e_2f_2 + e_3f_3 + e_4f_4) = e_if_i$ for $i = 1, 2, 3, 4$, this implies $C \subseteq C$. Therefore $C = C$, and $C = \langle f(x) \rangle$, where $f(x) = e_1f_1 + e_2f_2 + e_3f_3 + e_4f_4$.

Now suppose f_i are the generator polynomials of C_i , for $i = 1, 2, 3, 4$. Then f_1 divides $x^n - \alpha$, f_2 divides $x^n - (\alpha + \beta)$, f_3 divides $x^n - (\alpha + \gamma)$ and f_4 divides $x^n - (\alpha + \beta + \gamma + \delta)$ such that, $x^n - \alpha = g_1f_1$, $x^n - (\alpha + \beta) = g_2f_2$, $x^n - (\alpha + \gamma) = g_3f_3$ and $x^n - (\alpha + \beta + \gamma + \delta) = g_4f_4$, for some $g_i \in \mathcal{C}_i$. Then

$$\begin{aligned} & x^n - (\alpha + u_1\beta + u_2\gamma + u_1u_2\delta) \\ &= x^n - (e_1\alpha + e_2(\alpha + \beta) + e_3(\alpha + \gamma) + e_4(\alpha + \beta + \gamma + \delta)) \\ &= (e_1g_1 + e_2g_2 + e_3g_3 + e_4g_4)(e_1f_1 + e_2f_2 + e_3f_3 + e_4f_4). \end{aligned}$$

Therefore, $x^n - (\alpha + u_1\beta + u_2\gamma + u_1u_2\delta) = f(x)(e_1g_1 + e_2g_2 + e_3g_3 + e_4g_4)$. Hence, $f(x) | (x^n - (\alpha + u_1\beta + u_2\gamma + u_1u_2\delta))$. $\square$

Corollary 3. Let $C = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$ and f_i are the generator polynomials of C_i for $i = 1, 2, 3, 4$. Then

1. $C^\perp = \langle e_1h_1^*(x) + e_2h_2^*(x) + e_3h_3^*(x) + e_4h_4^*(x) \rangle$ and $|C^\perp| = q^{\sum_{i=1}^4 (\deg(f_i(x)))}$.

$$2. \quad C^\perp = \langle h^*(x) \rangle \text{ where } h^*(x) = e_1 h_1^*(x) + e_2 h_2^*(x) + e_3 h_3^*(x) + e_4 h_4^*(x)$$

where $h_i^*(x)$ are the reciprocal polynomials of $h_i(x)$ for $i = 1, 2, 3, 4$, such that $h_1 f_1 = x^n - \alpha$, $h_2 f_2 = x^n - (\alpha + \beta)$, $h_3 f_3 = x^n - (\alpha + \gamma)$, $h_4 f_4 = x^n - (\alpha + \beta + \gamma + \delta)$.

5. Quantum Codes from $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -Constacyclic Codes over $\mathcal{R}$

Quantum error-correcting codes (QECC) are a key ingredient to implement information processing based on quantum mechanics. For quantum systems composed of n subsystems of dimension $q \geq 2$, so-called qudits, a quantum code $\mathcal{C} = ((N; K))_q$ is a k -dimensional subspace of the Hilbert space $(\mathbb{C}^q)^{\otimes n}$. If the dimension of the code $\mathcal{C}$ is q^k , it will be denoted by $\mathcal{C} = [[n; k; d]]_q$, where d is the minimum distance.

The following theorem is an important construction of quantum error-correcting codes which is used to construct quantum codes and is called CSS construction:

Theorem 4 ([5]). (CSS Construction) Suppose C_1 and C_2 are $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$ linear codes over $GF(q)$ respectively such that $C_2^\perp \subseteq C_1$. Also, if $d = \min\{d_1, d_2\}$, then there exists a quantum error-correcting code $\mathcal{C}$ with parameters $[[n, k_1 + k_2 - n, d]]_q$. In particular, if $C_1^\perp \subseteq C_1$, then there exists a quantum error-correcting code $\mathcal{C}$ with parameters $[[n, 2k_1 - n, d_1]]_q$.

Lemma 1 ([5]). If $\mathcal{C}$ is a q -ary linear cyclic or negacyclic code with generator polynomial $f(x)$, then $\mathcal{C}$ contains its dual code if and only if

$$x^n - \kappa \equiv 0 \pmod{ff^*}$$

where f^* is the reciprocal polynomial of f and $\kappa = \pm 1$

Theorem 5. Let $\mathcal{C} = e_1 C_1 \oplus e_2 C_2 \oplus e_3 C_3 \oplus e_4 C_4$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then $\mathcal{C}^\perp \subseteq \mathcal{C}$ if and only if $x^n - \alpha \equiv 0 \pmod{f_1 f_1^*}$, $x^n - (\alpha + \beta) \equiv 0 \pmod{f_2 f_2^*}$, $x^n - (\alpha + \gamma) \equiv 0 \pmod{f_3 f_3^*}$ and $x^n - (\alpha + \beta + \gamma + \delta) \equiv 0 \pmod{f_4 f_4^*}$, where f_i^* is the reciprocal polynomial of f_i for $i = 1, 2, 3, 4$ and $\alpha = \alpha + \beta = \alpha + \gamma = \alpha + \beta + \gamma + \delta = \pm 1$.

Proof. Let $x^n - \alpha \equiv 0 \pmod{f_1 f_1^*}$, $x^n - (\alpha + \beta) \equiv 0 \pmod{f_2 f_2^*}$, $x^n - (\alpha + \gamma) \equiv 0 \pmod{f_3 f_3^*}$ and $x^n - (\alpha + \beta + \gamma + \delta) \equiv 0 \pmod{f_4 f_4^*}$. Then by Lemma 1, we have $C_i^\perp \subseteq C_i$, $i = 1, 2, 3, 4$. This implies that $e_i C_i^\perp \subseteq e_i C_i$, $i = 1, 2, 3, 4$. Therefore, $e_1 C_1^\perp \oplus e_2 C_2^\perp \oplus e_3 C_3^\perp \oplus e_4 C_4^\perp \subseteq e_1 C_1 \oplus e_2 C_2 \oplus e_3 C_3 \oplus e_4 C_4$. Hence, $\mathcal{C}^\perp \subseteq \mathcal{C}$.

Conversely, if $\mathcal{C}^\perp \subseteq \mathcal{C}$, then $e_1 C_1^\perp \oplus e_2 C_2^\perp \oplus e_3 C_3^\perp \oplus e_4 C_4^\perp \subseteq e_1 C_1 \oplus e_2 C_2 \oplus e_3 C_3 \oplus e_4 C_4$. Since C_i are the linear codes over F_q such that $e_i C_i = \mathcal{C} \pmod{e_i}$, for $i = 1, 2, 3, 4$. So $C_i^\perp \subseteq C_i$, $i = 1, 2, 3, 4$. Therefore, $x^n - \alpha \equiv 0 \pmod{f_1 f_1^*}$, $x^n - (\alpha + \beta) \equiv 0 \pmod{f_2 f_2^*}$, $x^n - (\alpha + \gamma) \equiv 0 \pmod{f_3 f_3^*}$ and $x^n - (\alpha + \beta + \gamma + \delta) \equiv 0 \pmod{f_4 f_4^*}$. $\square$

Corollary 4. Let $\mathcal{C} = e_1 C_1 \oplus e_2 C_2 \oplus e_3 C_3 \oplus e_4 C_4$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. Then $\mathcal{C}^\perp \subseteq \mathcal{C}$ if and only if $C_i^\perp \subseteq C_i$, $i = 1, 2, 3, 4$.

Theorem 6. Let $\mathcal{C} = e_1 C_1 \oplus e_2 C_2 \oplus e_3 C_3 \oplus e_4 C_4$ be a $(\alpha + u_1\beta + u_2\gamma + u_1u_2\delta)$ -constacyclic code of length n over $\mathcal{R}$. If $C_i^\perp \subseteq C_i$, $i = 1, 2, 3, 4$, then $\mathcal{C}^\perp \subseteq \mathcal{C}$ and there exists a quantum error-correcting code with parameters $[[4n, 2k - 4n, d_L]]_q$, where d_L denotes the minimum Lee weight of the code $\mathcal{C}$ and k denotes the dimension of the code $\psi(\mathcal{C})$.

Proof. Let $C_i^\perp \subseteq C_i$ for $i = 1, 2, 3, 4$. Then by the Corollary 4, $\mathcal{C}^\perp \subseteq \mathcal{C}$. Now let $x \in \psi(\mathcal{C}^\perp) = \psi(\mathcal{C})^\perp$ then there exists $y \in \mathcal{C}^\perp$ such that $x = \psi(y)$ where $y \cdot y' = 0$ for all $y' \in \mathcal{C}$. As $\mathcal{C}^\perp \subseteq \mathcal{C}$ and $y \in \mathcal{C}^\perp$ therefore $y \in \mathcal{C}$. Hence, $x = \psi(y) \in \psi(\mathcal{C})$. Therefore, $\psi(\mathcal{C})^\perp \subseteq \psi(\mathcal{C})$. As $\psi(\mathcal{C})$ is a $[4n, k, d_L]$ linear

code over F_q^{4n} . Then by Theorem 4, there exists a quantum error-correcting code with parameters $[[4n, 2k - 4n, d_L]]_q$. $\square$

6. Example

Example 1. Let $\mathcal{R} = F_3[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ and $n = 12$. We have

$$x^{12} - 1 = (1 + x)^3(2 + x)^3(1 + x^2)^3 \in F_3[x],$$

$$x^{12} + 1 = (2 + x + x^2)^3(2 + 2x + x^2)^3 \in F_3[x]$$

Let $f_1(x) = f_2(x) = f_3(x) = (1 + x^2)$ and $f_4(x) = (2 + x + x^2)$. Then $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = \langle 1 + x^2 \rangle$ are the cyclic codes over F_3 having the same parameters $[12, 10, 2]$ and $\mathcal{C}_4 = \langle 2 + x + x^2 \rangle$ is a negacyclic codes over F_3 with parameters $[12, 10, 3]$. Thus,

$$\mathcal{C} = \langle e_1f_1(x), e_2f_2(x), e_3f_3(x), e_4f_4(x) \rangle$$

is a $(1 - 2u_1u_2)$ -constacyclic code of length 12 over $\mathcal{R}$. Since all $f_i(x)f_i^*(x)$ divide $x^{12} - 1$ for $i = 1, 2, 3$ and $f_4(x)f_4^*(x)$ divides $x^{12} + 1$, $\mathcal{C}^\perp \subseteq \mathcal{C}$. Also, $\psi(\mathcal{C})$ is a linear quasi constacyclic code over F_3 of index 4 with parameters $[48, 40, 2]$. Now, using Theorem 6, we get a quantum code with parameters $[[48, 32, 2]]_3$.

Example 2. Let $\mathcal{R} = F_5[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ and $n = 15$. We have

$$x^{15} - 1 = (4 + x)^5(1 + x + x^2)^5 \in F_5[x],$$

$$x^{15} + 1 = (1 + x)^5(1 + 4x + x^2)^5 \in F_5[x]$$

Let $f_1(x) = f_3(x) = f_4(x) = (1 + x + x^2)$ and $f_2(x) = (1 + 4x + x^2)$. Then $\mathcal{C}_1 = \mathcal{C}_3 = \mathcal{C}_4 = \langle 1 + x + x^2 \rangle$ are the cyclic codes over F_5 having the same parameters $[15, 13, 3]$ and $\mathcal{C}_2 = \langle 1 + 4x + x^2 \rangle$ is a negacyclic codes over F_5 with parameters $[15, 13, 3]$. Thus,

$$\mathcal{C} = \langle e_1f_1(x), e_2f_2(x), e_3f_3(x), e_4f_4(x) \rangle$$

is a $(1 - 2u_1 + 2u_1u_2)$ -constacyclic code of length 15 over $\mathcal{R}$. Since all $f_i(x)f_i^*(x)$ divide $x^{15} - 1$ for $i = 1, 3, 4$ and $f_2(x)f_2^*(x)$ divides $x^{15} + 1$, $\mathcal{C}^\perp \subseteq \mathcal{C}$. Also, $\psi(\mathcal{C})$ is a linear quasi constacyclic code over F_5 of index 4 with parameters $[60, 52, 3]$. Now, using Theorem 6, we get a quantum code with parameters $[[60, 44, 3]]_5$.

Now, we find some new quantum error correcting codes over the field F_5 using the Gray images of $(1 - 2u_1u_2)$ -constacyclic codes over the ring $\mathcal{R} = F_5[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$. First column of the following table denotes the length of cyclic codes over $\mathcal{R}$, $f_i(x)$ are generator polynomials of codes $\mathcal{C}_i$ for $i = 1, 2, 3, 4$, parameters of the Gray images of $(1 - 2u_1u_2)$ -constacyclic codes over $\mathcal{R}$ are represented by column four and the parameters of the corresponding quantum codes are denoted by the last column.

Comparison Compared to previously known quantum error-correcting codes, some of our quantum error-correcting codes are new, e.g., the codes $[[60, 52, 2]]_5$ and $[[180, 164, 3]]_5$ have better parameters than the known quantum codes $[[60, 48, 2]]_5$ and $[[180, 156, 3]]_5$, respectively in [21]. Also, the codes $[[88, 68, 6]]_5$ and $[[208, 180, 9]]_5$ have better parameters than the known quantum codes $[[93, 69, 6]]_5$ and $[[208, 178, 9]]_5$, respectively in (<https://www.mathi.uni-heidelberg.de/~yves/Matritzen/QTBC/QTBCTab5.html>).

In [22], minimum Hamming distance is restricted to $d \leq q + 1$, but in our case we can find the codes that have distances greater than $q + 1$, for example, the codes $[[200, 172, 7]]_5$ and $[[208, 180, 9]]_5$ have the minimum Hamming distances greater than 6.

7. Conclusions

In this paper, we have studied properties of constacyclic codes over the ring $F_q[u_1, u_2] / \langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$. We have classified all self dual linear codes over F_q using self duality condition of the units. As an application, we have obtained non binary quantum codes from these classes of codes using self duality condition. For future work, it would be interesting to compute quantum codes from other classes of constacyclic codes given in Table 2.

Table 2. Parameters of Quantum Codes.

n	$f_1(x) = f_2(x) = f_3(x)$	$f_4(x)$	$\psi(\mathcal{C})$	$[[n, k, d]]_5$
15	$4 + x$	$1 + x$	$[60, 56, 2]$	$[[60, 52, 2]]_5$
22	$4 + x + x^2 + 4x^3 + 2x^4 + x^5$	$2 + 3x + 2x^2 + x^3 + 2x^4 + x^5$	$[88, 68, 6]$	$[[88, 48, 6]]_5$
25	$4 + x$	$1 + x$	$[100, 96, 2]$	$[[100, 92, 2]]_5$
25	$4 + x$	$4 + 2x + x^2$	$[100, 95, 2]$	$[[100, 90, 2]]_5$
30	$1 + x + x^2$	$1 + x$	$[120, 113, 2]$	$[[120, 106, 2]]_5$
35	$4 + x$	$1 + 4x + x^2 + 4x^3 + x^4 + 4x^5 + x^6$	$[140, 131, 2]$	$[[140, 122, 2]]_5$
40	$1 + x$	$2 + x^4$	$[160, 153, 2]$	$[[160, 146, 2]]_5$
45	$1 + x + x^2$	$1 + 4x + x^2$	$[180, 172, 3]$	$[[180, 164, 3]]_5$
45	$1 + x + x^2$	$1 + x$	$[180, 173, 2]$	$[[180, 166, 2]]_5$
50	$1 + x$	$2 + x$	$[200, 196, 2]$	$[[200, 192, 2]]_5$
50	$(1 + x)^4(4 + x)^2$	$(2 + x)^4(3 + x)^2$	$[200, 186, 7]$	$[[200, 172, 7]]_5$
52	$(1 + 4x + x^3 + x^4) \times (1 + 2x + 2x^3 + x^4)$	$(4 + 2x + x^3 + x^4) \times (4 + x + 2x^3 + x^4)$	$[208, 194, 9]$	$[[208, 180, 9]]_5$
55	$(4 + x + x^2 + 4x^3 + 2x^4 + x^5)$	$(1 + 3x + 4x^2 + 4x^3 + x^4 + x^5)$	$[220, 210, 6]$	$[[220, 200, 6]]_5$

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