Article

Approximation by Shifts of Compositions of Dirichlet L -Functions with the Gram Function

Artūras Dubickas [†], Ramūnas Garunkštis ^{*,†} and Antanas Laurinčikas [†]

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania; arturas.dubickas@mif.vu.lt (A.D.); antanas.laurincikas@mif.vu.lt (A.L.)

* Correspondence: ramunas.garunkstis@mif.vu.lt

† These authors contributed equally to this work.

Received: 22 March 2020; Accepted: 3 May 2020; Published: 9 May 2020

Abstract: In this paper, a joint approximation of analytic functions by shifts of Dirichlet L -functions $L(s + ia_1 t_\tau, \chi_1), \dots, L(s + ia_r t_\tau, \chi_r)$, where $a_1, \dots, a_r$ are non-zero real algebraic numbers linearly independent over the field $\mathbb{Q}$ and t_τ is the Gram function, is considered. It is proved that the set of their shifts has a positive lower density.

Keywords: Dirichlet L -function; Gram function; joint universality

1. Introduction

Let $\chi : \mathbb{N} \rightarrow \mathbb{C}$ be a Dirichlet character modulo $q \in \mathbb{N}$. Note that $\chi(m)$ is periodic with period q , completely multiplicative (i.e., $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$ and $\chi(1) = 1$), $\chi(m) = 0$ for $(m, q) \neq 1$ and $\chi(m) \neq 0$ for $(m, q) = 1$. Let $s = \sigma + it$. In [1], L. Dirichlet introduced a function

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad (\sigma > 1), \quad (1)$$

which is now called the Dirichlet L -function. In virtue of the complete multiplicativity of $\chi(m)$, the function (1) can be written as an Euler product

$$L(s, \chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers and has a meromorphic continuation to the whole complex plane with a unique simple pole at the point $s = 1$ (if χ is the principal character modulo q) with residue $\prod_{p|q} (1 - 1/p)$. Since then, the function (1) has become a subject of intensive investigation. See, for instance, References [2–4] for some very recent papers on its zeros and moments. For $q = 1$, the function $L(s, \chi)$ becomes the Riemann zeta-function $\zeta(s)$.

In Reference [5], S. M. Voronin established the universality of Dirichlet L -functions. He proved that if $f(s)$ is a continuous non-vanishing function on the disc $|s| \leq r$ with any fixed r , $0 < r < 1/4$, and analytic in the interior of that disc, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leq r} |L(s + 3/4 + i\tau, \chi) - f(s)| < \varepsilon.$$

The Voronin theorem was extended to more general compact sets independently in References [6–8]. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements, and by $H_0(K)$, where $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that

are analytic in the interior of K . Then the modern version of the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |L(s + i\tau, \chi) - f(s)| < \varepsilon \right\} > 0,$$

where $\text{meas} A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ (see, for example, Reference [9]). The latter inequality shows that there are infinitely many shifts $L(s + i\tau, \chi)$ approximating a given function from the class $H_0(K)$.

In Reference [10], Voronin considered the joint functional independence of Dirichlet L -functions using the joint universality. We recall that two Dirichlet characters are called non-equivalent if they are not generated by the same primitive character. Thus, the following statement is valid [10,11]; see also References [9,12,13].

Theorem 1. Let $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

The non-equivalence of the characters $\chi_1, \dots, \chi_r$ ensures a certain independence of the functions $L(s, \chi_1), \dots, L(s, \chi_r)$ which is necessary for a simultaneous approximation of the collection $f_1(s), \dots, f_r(s)$. Later, it turned out that, in place of non-equivalent characters, different shifts can be used. This was observed by Nakamura [14]. More precisely, he proved the following theorem.

Theorem 2. Let $a_1 = 1, a_2, \dots, a_r$ be real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$ and $\chi_1, \dots, \chi_r$ be arbitrary Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $a \in \mathbb{R} \setminus \{0\}$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia a_j \tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In Reference [15], Pańkowski obtained the joint universality of Dirichlet L -functions using the shifts $L(s + i\alpha_j \tau^{a_j} \log^{b_j} \tau, \chi_j)$, $j = 1, \dots, r$, where $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, $a_1, \dots, a_r \in \mathbb{R}^+$ are distinct, $b_1, \dots, b_r$ are distinct and satisfy

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{N}, \\ (-\infty, 0] \cup (1 + \infty) & \text{if } a_j \in \mathbb{N}. \end{cases}$$

The aim of this paper is to introduce new shifts of Dirichlet L -functions that approximate collections of analytic functions from the class $H_0(K)$. Let, as usual, $\Gamma(s)$ be the Euler gamma-function. For $t > 0$, denote the increment $\theta(t)$ of the argument of the function $\pi^{-s/2} \Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. Then it is known (see, for example, Reference [16] [Lemma 1.1]) that, for $\tau \geq 0$, the equation

$$\theta(t) = (\tau - 1)\pi$$

has the unique solution t_τ satisfying $\theta'(t_\tau) > 0$. For $n \in \mathbb{N}$, the numbers t_n are called the Gram points. They were introduced and studied in Reference [17]. Therefore, we call t_τ the Gram function. A very interesting property of the Gram points is the relation $t_n \sim \gamma_n$ as $n \rightarrow \infty$, where $\gamma_n > 0$ are imaginary parts of non-trivial zeros of the Riemann zeta-function. In the paper, we will consider the

joint approximation of analytic functions by shifts of Dirichlet L -functions involving the Gram function. More precisely, we will prove the following joint universality theorem.

Theorem 3. Suppose that $a_1, \dots, a_r$ are real non-zero algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1, \dots, \chi_r$ are arbitrary Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t_\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ia_j t_\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of Theorem 3, we will use the probabilistic approach based on weakly convergent probability measures in the space of analytic functions.

2. Lemmas

We start with a lemma on the functional properties of the function t_τ . (Its proof can be found in Reference [16] [Lemma 1.1].)

Lemma 1. Suppose that $\tau \rightarrow \infty$. Then

$$t_\tau = \frac{2\pi\tau}{\log \tau} \left(1 + \frac{\log \log \tau}{\log \tau} (1 + o(1)) \right),$$

$$t'_\tau = \frac{2\pi}{\log \tau} \left(1 + \frac{\log \log \tau}{\log \tau} (1 + o(1)) \right)$$

and

$$t''_\tau = -\frac{\pi}{\tau(\log \tau)^2} \left(1 + \frac{\log \log \tau}{\log \tau} (2 + o(1)) \right).$$

The next lemma provides an estimate for certain trigonometric integral.

Lemma 2. Suppose that $F(x)$ is a real differentiable function, the derivative $F'(x)$ is monotonic and $F'(x) \geq \lambda > 0$ or $F'(x) \leq -\lambda < 0$ on the interval (a, b) . Then

$$\left| \int_a^b \exp\{iF(x)\} dx \right| \leq \frac{4}{\lambda}.$$

The proof of the lemma is given, for example, in Reference [11].

We will also use Baker's theorem on linear forms in logarithms of algebraic numbers (see, for example, Reference [18]).

Lemma 3. Suppose that $\lambda_1, \dots, \lambda_r \in \overline{\mathbb{Q}}$ are such that their logarithms $\log \lambda_1, \dots, \log \lambda_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \dots, \beta_r$, not all zero, we have

$$|\beta_0 + \beta_1 \log \lambda_1 + \dots + \beta_r \log \lambda_r| > H^{-C},$$

where H is the maximum of the heights of $\beta_0, \beta_1, \dots, \beta_r$, and C is an effectively computable constant depending on $r, \lambda_1, \dots, \lambda_r$ and the maximum of the degrees of $\beta_0, \beta_1, \dots, \beta_r$.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Define

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then Ω^r is also a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(\mathbb{X})$ the Borel σ -field of the space $\mathbb{X}$, we see that, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r exists. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_T(A) = \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \left((p^{-ia_1 t_\tau} : p \in \mathbb{P}), \dots, (p^{-ia_r t_\tau} : p \in \mathbb{P}) \right) \in A \right\}.$$

Then the following limit theorem holds.

Lemma 4. Under hypotheses of Theorem 2 on the numbers $a_1, \dots, a_r$, Q_T converges weakly to the Haar measure m_H^r as $T \rightarrow \infty$.

Proof. We apply the Fourier transform method. It is well known that the dual group of Ω^r is isomorphic to the group

$$\bigoplus_{j=1}^r \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $j = 1, \dots, r$, $p \in \mathbb{P}$. Hence it follows that characters of the group Ω^r are of the form

$$\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_{jp}}(p),$$

where $\omega_j(p)$ is the p th component of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$, and the sign “ $*$ ” means that only a finite number of integers k_{jp} are distinct from zero. Therefore

$$\int_{\Omega^r} \left(\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_{jp}}(p) \right) d\mu \quad (2)$$

is the Fourier transform of a measure μ on $(\Omega^r, \mathcal{B}(\Omega^r))$.

Let $g_{Q_T}(k)$, $k = (k_1, \dots, k_r)$, $k_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, be the Fourier transform of Q_T . In view of (2) we have

$$g_{Q_T}(k) = \int_{\Omega^r} \left(\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_{jp}}(p) \right) dQ_T.$$

Thus, by the definition of Q_T ,

$$\begin{aligned} g_{Q_T}(k) &= \frac{1}{T-2} \int_2^T \prod_{j=1}^r \prod_{p \in \mathbb{P}}^* p^{-ik_{jp} a_j t_\tau} d\tau \\ &= \frac{1}{T-2} \int_2^T \exp \left\{ -it_\tau \sum_{j=1}^r \sum_{p \in \mathbb{P}}^* a_j k_{jp} \log p \right\} d\tau. \end{aligned} \quad (3)$$

Obviously, if $\underline{k} = (\underline{0}, \dots, \underline{0})$, then

$$g_{Q_T}(\underline{k}) = 1. \quad (4)$$

Now suppose that $\underline{k} = (k_1, \dots, k_r) \neq (\underline{0}, \dots, \underline{0})$. Note that

$$A_{\underline{k}} \stackrel{\text{def}}{=} \sum_{j=1}^r \sum_{p \in \mathbb{P}}^* a_j k_{jp} \log p = \sum_{p \in \mathbb{P}}^* \log p \sum_{j=1}^r a_j k_{jp}.$$

Since $\underline{k}_j \neq \underline{0}$ for some $j \in \{1, 2, \dots, r\}$, there is a prime number p such that $k_{jp} \neq 0$. For this p , the sum $\beta_p \stackrel{\text{def}}{=} \sum_{j=1}^r a_j k_{jp}$ is non-zero, because the numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, in view of Lemma 3,

$$A_{\underline{k}} = \sum_{p \in \mathbb{P}}^* \beta_p \log p \neq 0. \quad (5)$$

Now, (3) and Lemmas 1 and 2 show that, in the case $\underline{k} \neq (\underline{0}, \dots, \underline{0})$,

$$g_{Q_T}(\underline{k}) \ll \frac{\log T}{TA_{\underline{k}}}.$$

This together with (4) and (5) give

$$\lim_{T \rightarrow \infty} g_{Q_T}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } \underline{k} \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r , the lemma follows by a continuity theorem for probability measures on compact groups. $\square$

$H(D)$ denotes the space of analytic functions on the strip D endowed with the topology of uniform convergence on compacta. Lemma 4 implies a limit theorem for probability measures on $(H(D), \mathcal{B}(H(D)))$ defined by means of absolutely convergent Dirichlet series.

For a fixed number $\theta > 1/2$ and $m, n \in \mathbb{N}$, set

$$v_n(m) = \exp \left\{ - \left(\frac{m}{n} \right)^\theta \right\}. \quad (6)$$

Then we define the series

$$L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s}$$

and

$$L_n(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) \omega_j(m) v_n(m)}{m^s},$$

$j = 1, \dots, r$, where the functions $\omega_j(p)$ are extended to the set $\mathbb{N}$ by the formula

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N}.$$

Denote the elements of Ω^r by $\omega = (\omega_1, \dots, \omega_r)$. Put $\underline{\chi} = (\chi_1, \dots, \chi_r)$, and set

$$\underline{L}_n(s, \underline{\chi}) = (L_n(s, \chi_1), \dots, L_n(s, \chi_r)) \quad (7)$$

and

$$\underline{L}_n(s, \omega, \underline{\chi}) = (L_n(s, \omega_1, \chi_1), \dots, L_n(s, \omega_r, \chi_r)).$$

Moreover, let $u_n : \Omega^r \rightarrow H^r(D)$ be given by the formula

$$u_n(\omega) = \underline{L}_n(s, \omega, \underline{\chi}).$$

The absolute convergence of the series for $\underline{L}_n(s, \omega_j, \chi_j)$ implies the continuity of the mapping u_n . Let $V_n = m_H^r u_n^{-1}$, where, for $A \in \mathcal{B}(H^r(D))$,

$$V_n(A) = m_H^r u_n^{-1}(A) = m_H^r(u_n^{-1}A). \quad (8)$$

In view of (7) and (8) we conclude that Lemma 4, the continuity of u_n and the well-known property on preservation of weak convergence under mapping lead to the following statement.

Lemma 5. Under hypothesis of Theorem 3 on the numbers $\underline{a} = (a_1, \dots, a_r)$, we have

$$P_{T,n}(A) \stackrel{\text{def}}{=} \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \underline{L}_n(s + i \underline{a} \tau, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)),$$

converges weakly to the measure V_n as $T \rightarrow \infty$.

The probability measure V_n is very important for the proof of Theorem 3. Let

$$\underline{L}(s, \omega, \underline{\chi}) = (L(s, \omega_1, \chi_1), \dots, L(s, \omega_r, \chi_r)),$$

where

$$\underline{L}(s, \omega, \underline{\chi}) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p) \chi_j(p)}{p^s} \right)^{-1}, \quad j = 1, \dots, r. \quad (9)$$

Note that the latter products are uniformly convergent on compact subsets of the strip D for almost all $\omega_j \in \Omega_j$, and define the $H(D)$ -valued random elements on the probability space $(\Omega_j, \mathcal{B}(\Omega_j), m_{jH})$, where m_{jH} is the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$. Therefore, $\underline{L}(s, \omega, \underline{\chi})$ is the $H^r(D)$ -valued random element on $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(s, \omega, \underline{\chi})$, that is,

$$P_{\underline{L}}(A) = m_H^r \left\{ \omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

We recall that the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, where the space $\mathbb{X}$ is separable, is a minimal closed set $S_P \subset \mathbb{X}$ such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$ such that, for every open neighbourhood G of x , the inequality $P(G) > 0$ is satisfied.

The measure V_n is independent on any hypothesis. Therefore, from Reference [19] it follows that:

Lemma 6. The measure V_n converges weakly to $P_{\underline{L}}$ as $n \rightarrow \infty$. Moreover, the support of $P_{\underline{L}}$ is the set S^r , where

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Proof. To be precise, in Reference [19] it was proved that a certain measure P_N converges weakly to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ (as $N \rightarrow \infty$), and the measure P is the limit measure of V_n as $n \rightarrow \infty$. Moreover, it was proved that $P = P_{\underline{L}}$.

It remains to prove that the support of $P_{\underline{L}}$ is the set S^r . It is well known that the support of the random element

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p) \chi(p)}{p^s} \right)^{-1}, \quad \omega \in \Omega, \quad (10)$$

is the set S for every Dirichlet character χ . Since the space $H^r(D)$ is separable, we have

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_r$$

(see [20]). Therefore, it suffices to consider the measure $P_{\underline{L}}$ on the sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in \mathcal{B}(H(D)).$$

Since the Haar measure m_H^r is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$, we deduce that

$$m_H^r\{\omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A\} = \prod_{j=1}^r m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j\}.$$

This equality and the minimality of the support together with remark on the support of the element (10) show that the support of $P_{\underline{L}}$ is the set S^r . $\square$

3. Mean Square Estimates

Define

$$\underline{L}(s, \underline{\chi}) = (L(s, \chi_1), \dots, L(s, \chi_r)). \quad (11)$$

To pass from $\underline{L}_n(s + iat_\tau, \underline{\chi})$ (defined by (7)) to $\underline{L}(s + iat_\tau, \underline{\chi})$, certain mean square estimates for Dirichlet L -functions are necessary. Let χ be an arbitrary character modulo q .

Lemma 7. Suppose that $\sigma, 1/2 < \sigma < 1$, and $a \in \mathbb{R} \setminus \{0\}$ are fixed. Then, for $t \in \mathbb{R}$,

$$\int_2^T |L(\sigma + it + iat_\tau, \chi)|^2 d\tau \ll T(1 + |t|).$$

Proof. It is well known that, for fixed $\sigma > 1/2$,

$$\int_2^T |L(\sigma + it, \chi)|^2 dt \ll_\sigma T.$$

Therefore, in view of Lemma 1, for $1/2 < \sigma < 1$,

$$\begin{aligned} \int_2^T |L(\sigma + it + iat_\tau, \chi)|^2 d\tau &= \frac{1}{a} \int_2^T \frac{1}{t'_\tau} |L(\sigma + it + iat_\tau, \chi)|^2 d(at_\tau) \\ &= \frac{1}{a} \int_2^T \frac{1}{t'_\tau} d\left(\int_2^{t+at_\tau} |L(\sigma + iu, \chi)|^2 du\right) \\ &\ll \frac{\log T}{a} \int_2^{|t|+|a|t_T} |L(\sigma + iu, \chi)|^2 du \\ &\ll_{\sigma,a} \log T \left(|t| + |a| \frac{T}{\log T}\right) \ll_{\sigma,a} T(1 + |t|), \end{aligned}$$

which is the required estimate. $\square$

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad (12)$$

where $\{K_l\} \subset D$ is a sequence of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric in the space $H(D)$ inducing the topology of uniform convergence on compacta. Now, putting, for $\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$,

$$\rho(\underline{g}_1, \underline{g}_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j}) \quad (13)$$

gives a metric in $H^r(D)$ inducing the product topology. The next lemma provides a certain approximation of $\underline{L}(s, \chi)$ (see definition (11)) by $\underline{L}_n(s, \chi)$.

Lemma 8. Suppose that $\underline{a} \neq (0, \dots, 0)$. Then

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T-2} \int_2^T \rho(\underline{L}(s + iat_\tau, \underline{\chi}), \underline{L}_n(s + iat_\tau, \underline{\chi})) \, d\tau = 0.$$

Proof. From the definition (13) of the metric ρ , it follows that it suffices to prove that, for $a \neq 0$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T-2} \int_2^T \rho(L(s + iat_\tau, \chi_j), L_n(s + iat_\tau, \chi)) \, d\tau = 0 \quad (14)$$

for every $j = 1, \dots, r$. We will prove the above equality for the character χ modulo q .

Let θ be from the definition (6) of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^\varepsilon. \quad (15)$$

Then the representation

$$L_n(s, \chi) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} L(s+z, \chi) l_n(z) \frac{dz}{z},$$

is true. Its proof is the same as in Section 5.4 of [21] for the Riemann zeta-function. Hence, taking $\theta_1 > 0$, by the residue theorem, we obtain

$$L_n(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} L(s+z, \chi) l_n(z) \frac{dz}{z} + R_n(s, \chi), \quad (16)$$

where

$$R_n(s, \chi) = \begin{cases} 0 & \text{if } \chi \text{ is a non-principal character,} \\ \prod_{p|q} \left(1 - \frac{1}{p}\right) \frac{l_n(1-s)}{1-s} & \text{otherwise.} \end{cases}$$

Let $K \subset D$ be an arbitrary compact set. Denote by $s = \sigma + iv$ the points of K , and suppose that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ with fixed $\varepsilon > 0$ for $s \in K$. More precisely, we select $\theta_1 = \sigma - \varepsilon - 1/2 \geq \varepsilon > 0$. Then, in view of (16),

$$\begin{aligned} & |L_n(s + iat_\tau, \chi) - L(s + iat_\tau, \chi)| \\ & \ll \int_{-\infty}^{\infty} |L(s + iat_\tau - \theta_1 + it, \chi)| \frac{|l_n(-\theta_1 + it)|}{|-\theta_1 + it|} \, dt + |R_n(s + iat_\tau, \chi)|. \end{aligned}$$

Now, taking t in place of $t + v$, we get that, for $s \in K$,

$$\begin{aligned} & |L_n(s + iat_\tau, \chi) - L(s + iat_\tau, \chi)| \\ & \ll \int_{-\infty}^{\infty} |L(1/2 + \varepsilon + i(t + at_\tau), \chi)| \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt \\ & + |R_n(s + iat_\tau, \chi)|. \end{aligned}$$

This implies the estimate

$$\begin{aligned} & \frac{1}{T-2} \int_2^T \sup_{s \in K} |L(s + iat_\tau, \chi) - L_n(s + iat_\tau, \chi)| d\tau \\ & \ll \frac{1}{T-2} \int_2^T \int_{-\infty}^{\infty} |L(1/2 + \varepsilon + i(t + at_\tau), \chi)| \sup_{s \in K} \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt d\tau \\ & + \frac{1}{T-2} \int_2^T \sup_{s \in K} |R_n(s + iat_\tau, \chi)| d\tau \\ & \ll J_1 + J_2, \end{aligned} \tag{17}$$

where

$$J_1 = \int_{-\infty}^{\infty} \frac{1}{T-2} \int_2^T (|L(1/2 + \varepsilon + i(t + at_\tau), \chi)| d\tau) \sup_{s \in K} \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt$$

and

$$J_2 = \frac{1}{T-2} \int_2^T \sup_{s \in K} |R_n(s + iat_\tau, \chi)| d\tau. \tag{18}$$

It is well known that uniformly in σ , $\sigma_1 \leq \sigma \leq \sigma_2$, with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0.$$

Therefore, by the definition (15) of the function $l_n(s)$, we find that, for $s \in K$,

$$\begin{aligned} \left| \frac{l_n(1/2 + \varepsilon - s + it)}{1/2 + \varepsilon - s + it} \right| &= \frac{n^{1/2 + \varepsilon - \sigma}}{\theta} \left| \Gamma \left(\frac{1/2 + \varepsilon - \sigma}{\theta} + \frac{i(t - v)}{\theta} \right) \right| \\ &\ll_{\theta, K} n^{-\varepsilon} \exp \left\{ -\frac{c_1}{\theta} |t| \right\}, \quad c_1 > 0. \end{aligned} \tag{19}$$

In the same way, for $s \in K$, we obtain

$$R_n(s + iat_\tau, \chi) \ll_{\theta, q, K} n^{1-\sigma} \exp \left\{ -\frac{c_2}{\theta} |a|t_\tau \right\}. \tag{20}$$

Suppose that $\theta = 1/2 + \varepsilon$. Then (17), (19) and Lemma 7 lead to the bound

$$J_1 \ll_{\varepsilon, K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |t|) \exp\{-c_3|t|\} dt \ll_{\varepsilon, K, a} n^{-\varepsilon}, \quad c_3 > 0. \tag{21}$$

Moreover, by (18), Lemma 1 and (20),

$$\begin{aligned}
J_2 &\ll_{\varepsilon, K, q} n^{1/2-2\varepsilon} \frac{1}{T-2} \int_2^T \exp \left\{ -c_4 |a| \frac{\tau}{\log \tau} \right\} d\tau \\
&\ll_{\varepsilon, K, q} n^{1/2-2\varepsilon} \frac{\log T}{T-2} + \frac{n^{1/2-2\varepsilon}}{T-2} \int_{\log T}^T \exp \left\{ -c_4 |a| \frac{\tau}{\log \tau} \right\} d\tau \\
&\ll_{\varepsilon, K, q, a} n^{1/2-2\varepsilon} \frac{\log T}{T-2}.
\end{aligned}$$

Thus, in view of (17) and (21),

$$\frac{1}{T-2} \int_2^T \sup_{s \in K} |L(s + iat_\tau, \chi) - L_n(s + iat_\tau, \chi)| d\tau \ll_{\varepsilon, K, q, a} n^{-\varepsilon} + n^{1/2-2\varepsilon} \frac{\log T}{T-2}.$$

From this, it follows that

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T-2} \int_2^T \sup_{s \in K} |L(s + iat_\tau, \chi) - L_n(s + iat_\tau, \chi)| d\tau = 0. \quad (22)$$

Now, the definition (12) of the metric ρ implies (14), which completes the proof of Lemma 8. $\square$

4. A Limit Theorem

For $A \in \mathcal{B}(H^r(D))$, define

$$P_T(A) = \frac{1}{T-2} \text{meas} \left\{ \tau \in [2, T] : \underline{L}(s + iat_\tau, \underline{\chi}) \in A \right\}. \quad (23)$$

In this section, we will prove the following statement.

Theorem 4. Suppose that $a_1, \dots, a_r$ are non-zero real algebraic numbers linearly independent over $\mathbb{Q}$, and $\chi_1, \dots, \chi_r$ are arbitrary Dirichlet characters. Then P_T converges weakly to $P_{\underline{L}}$ as $T \rightarrow \infty$. The support of $P_{\underline{L}}$ is the set S^r .

First we recall a useful property of convergence in distribution ($\xrightarrow{\mathcal{D}}$) (see Theorem 4.2 in Reference [20]).

Lemma 9. Suppose that the space $(\mathbb{X}, d)$ is separable, the random elements X_{kn} and Y_n , $k \in \mathbb{N}$, $n \in \mathbb{N}$, are defined on the same probability space with measure μ ,

$$X_{kn} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} X_k,$$

for every $k \in \mathbb{N}$,

$$X_k \xrightarrow[k \rightarrow \infty]{\mathcal{D}} X,$$

and, for every $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \{d(X_{kn}, Y_n) \geq \varepsilon\} = 0.$$

Then $Y_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} X$.

In the theory of weak convergence of probability measures, the notions of relative compactness and tightness of families of probability measures are very useful. We recall that the family $\{P\}$ of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a weakly convergent subsequence to a certain measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and this family is called tight, if for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$. By the direct Prokhorov theorem (see Theorem 5.1 in Billingsley [20]), every tight family $\{P\}$ is relatively compact. We apply the above remarks to the sequence $\{V_n : n \in \mathbb{N}\}$, where V_n (defined by (8)) is the limit measure in Lemma 5.

Lemma 10. *The sequence $\{V_n\}$ is relatively compact.*

Proof. By the above mentioned Prokhorov theorem, it suffices to prove that the sequence $\{V_n\}$ is tight.

Suppose θ_T is a random variable defined on a certain probability space with measure μ and uniformly distributed on $[2, T]$. Define the $H^r(D)$ -valued random element

$$\underline{X}_{T,n} = \underline{X}_{T,n}(s) = (X_{T,n,1}(s), \dots, X_{T,n,r}(s)) = L_n(s + ia_t \theta_T, \underline{\chi}).$$

Moreover, let

$$\underline{X}_n = \underline{X}_n(s) = (X_{n,1}(s), \dots, X_{n,r}(s)) \quad (24)$$

be the $H^r(D)$ -valued random element with the distribution V_n . Then Lemma 5 implies the relation

$$\underline{X}_{T,n} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \underline{X}_n. \quad (25)$$

By Lemma 7 with $t = 0$, we have, for $1/2 < \sigma < 1$,

$$\int_2^T |L(\sigma + ia_j t_\tau, \chi_j)|^2 d\tau \ll_{\sigma, a_j} T, \quad j = 1, \dots, r. \quad (26)$$

Let K_l be a compact set from the definition of the metric ρ . Then (26) together with the Cauchy integral formula show that

$$\int_2^T \sup_{s \in K_l} |L(s + ia_j t_\tau, \chi_j)| d\tau \ll_{l, a_j} T, \quad j = 1, \dots, r.$$

This combined with (22) implies the inequality

$$\sup_{n \in \mathbb{N}} \limsup_{T \rightarrow \infty} \frac{1}{T-2} \int_2^T \sup_{s \in K_l} |L_n(s + ia_j t_\tau, \chi_j)| d\tau \ll R_{lj}, \quad j = 1, \dots, r. \quad (27)$$

Fix $\varepsilon > 0$, and define $M_{lj} = M_{lj}(s) = 2^l r R_{lj} \varepsilon^{-1}$. Then, in view of (27), we find that, for each $n \in \mathbb{N}$,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \mu \left\{ \exists j : \sup_{s \in K_l} |X_{T,n,j}(s)| > M_{lj} \right\} \\ & \leq \sum_{j=1}^r \limsup_{T \rightarrow \infty} \mu \left\{ \sup_{s \in K_l} |X_{T,n,j}(s)| > M_{lj} \right\} \\ & \leq \sum_{j=1}^r \limsup_{T \rightarrow \infty} \frac{1}{(T-2)M_{lj}} \int_2^T \sup_{s \in K_l} |L_n(s + ia_j t_\tau, \chi_j)| d\tau \leq \sum_{j=1}^r \frac{R_{lj}}{M_{lj}} = \frac{\varepsilon}{2^l}. \end{aligned}$$

This together with (25) shows that, for all $l, n \in \mathbb{N}$,

$$\mu \left\{ \exists j : \sup_{s \in K_l} |X_{n,j}(s)| > M_{lj} \right\} \leq \frac{\varepsilon}{2^l}. \quad (28)$$

Define the set

$$K_j = K_j(s) = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leq M_{lj}, l \in \mathbb{N} \right\}.$$

Then K_j is a compact set in $H(D)$, and, in virtue of (24) and (28),

$$\mu\{\underline{X}_n \in K\} \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. In other words, we have

$$V_n(K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. $\square$

Proof of Theorem 4. By Lemma 10, there exists a subsequence $\{V_{n_k}\}$ of the sequence $\{V_n\}$ that is weakly convergent to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \rightarrow \infty$. This can be written as

$$\underline{X}_{n_k} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} P. \quad (29)$$

Define one more $H^r(D)$ -valued random element

$$\underline{X}_T = \underline{X}_T(s) = \underline{L}(s + iat_{\theta_T}, \underline{\chi}).$$

Then Lemma 8 implies that, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mu \left\{ \underline{\rho}(\underline{X}_T, \underline{X}_{T,n}) \geq \varepsilon \right\} \\ & \leq \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{(T-2)\varepsilon} \int_2^T \underline{\rho} \left(\underline{L}(s + iat_{\tau}, \underline{\chi}), \underline{L}_n(s + iat_{\tau}, \underline{\chi}) \right) d\tau = 0. \end{aligned}$$

The latter equality together with (25), (29), and Lemma 9 shows that

$$\underline{X}_T \xrightarrow[T \rightarrow \infty]{\mathcal{D}} P, \quad (30)$$

or, in other words, P_T converges weakly to P as $T \rightarrow \infty$. Moreover, by the relation (30), the measure P is independent of the subsequence $\{V_{n_k}\}$. Thus, we deduce that

$$\underline{X}_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P,$$

or V_n converges weakly to P as $n \rightarrow \infty$. Therefore, the theorem follows by Lemma 6. $\square$

5. Proof of Universality

The proof of Theorem 3 is based on Mergelyan's theorem on the approximation of analytic functions by polynomials [22], Theorem 4, and the properties of weak convergence. For convenience, we state them as lemmas.

Lemma 11 (Mergelyan theorem). *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $f(s)$ be a continuous function on K and analytic in the interior of K . Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

We recall that $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of the measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where ∂A is a boundary of A .

Lemma 12. Let P_n , $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent:

- 1° P_n converges weakly to P as $n \rightarrow \infty$;
 2° For every open set $G \subset \mathbb{X}$,

$$\liminf_{n \rightarrow \infty} P_n(G) \geq P(G);$$

- 3° For every continuity set A of P ,

$$\lim_{n \rightarrow \infty} P_n(A) = P(A).$$

The above lemma is a part of Theorem 2.1 from Reference [20]. Now, we can give the proof of Theorem 3.

Proof of Theorem 3. First part. In view of Lemma 11, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}. \quad (31)$$

The set

$$G_\varepsilon^r = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \right\} \quad (32)$$

is an open neighbourhood of the element $(e^{p_1(s)}, \dots, e^{p_r(s)}) \in S^r$. Thus, by Theorem 4, $P_{\underline{L}}(G_\varepsilon^r) > 0$, where the distribution $P_{\underline{L}}$ is defined by (9). Hence, from Theorem 4 again and Lemma 12,

$$\liminf_{T \rightarrow \infty} P_T(G_\varepsilon^r) \geq P_{\underline{L}}(G_\varepsilon^r) > 0,$$

and the definitions (23) and (32) of P_T and G_ε^r together with (31) prove the first part of the theorem.

Second part. Introduce one more set

$$A_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}. \quad (33)$$

Then the boundary of A_ε lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\},$$

thus, $\partial A_{\varepsilon_1} \cap \partial A_{\varepsilon_2} = \emptyset$ for different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. This shows that the set A_ε is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemma 12,

$$\lim_{T \rightarrow \infty} P_T(A_\varepsilon) = P_{\underline{L}}(A_\varepsilon) \quad (34)$$

for all but at most countably many $\varepsilon > 0$. Moreover, (31) shows the inclusion $G_\varepsilon^r \subset A_\varepsilon$. This, (34) and the definitions (23) and (33) of P_T and A_ε prove the second assertion of the theorem. $\square$

Author Contributions: Conceptualization, A.D., R.G. and A.L.; Investigation, A.D., R.G. and A.L.; Writing—original draft, A.D., R.G. and A.L.; Writing—review and editing, A.D., R.G. and A.L. All authors contributed equally to the manuscript and typed, read, and approved final manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by European Social Fund (project No 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMTLT).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dirichlet, P.G.L. Beweis des Satzes, Dass jede Unbegrenzte Arithmetische Progression, Deren Erstes Glied und Differenz Ganze Zahlen Ohne Gemeinschaftlichen Factor Sind, Unendlich viele Primzahlen Enthält. In *Abhandlungen der Königlich Preussischen Akademie der Wissenschaften von*; 1837; pp. 45–81.
2. Andrade, J.; Baluyot, S. Small zeros of Dirichlet L -functions of quadratic characters of prime modulus. *Res. Number Theory* **2020**, *6*, 1–20.
3. Sono, K. The second moment of quadratic Dirichlet L -functions. *J. Number Theory* **2020**, *206*, 194–230. [\[CrossRef\]](#)
4. Zacharias, R. Mollification of the fourth moment of Dirichlet L -functions. *Acta Arith.* **2019**, *191*, 201–257. [\[CrossRef\]](#)
5. Voronin, S.M. Theorem on the “universality” of the Riemann zeta-function. *Izv. Akad. Nauk SSSR Ser. Matem.* **1975**, *39*, 475–486. (In Russian) [\[CrossRef\]](#)
6. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph.D. Thesis, Indian Statistical Institute, Calcutta, India, 1981. [\[CrossRef\]](#)
7. Bagchi, B. A joint universality theorem for Dirichlet L -functions. *Math. Z.* **1982**, *181*, 319–334.
8. Gonek, S.M. Analytic Properties of Zeta and L -Functions. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, 1975. [\[CrossRef\]](#)
9. Steuding, J. *Value-Distribution of L -Functions*; Lecture Notes Math.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007; Volume 1877.
10. Voronin, S.M. On the functional independence of Dirichlet L -functions. *Acta Arith.* **1975**, *27*, 493–503. (In Russian)
11. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; Walter de Gruyter: Berlin, Germany, 1992.
12. Laurinćikas, A. On joint universality of Dirichlet L -functions. *Chebysh. Sbornik* **2011**, *12*, 124–139.
13. Matsumoto, K. A survey of the theory of universality for zeta and L -functions. In *Number Theory: Plowing and Staring Through High Wave Forms, Proceedings of the 7th China-Japan Semin., Fukuoka, Japan, 28 October–1 November 2013*; Ser. Number Theory Appl.; Kaneko, M., Kanemitsu, S., Liu, J., Eds.; World Scientific Publishing Co: Singapore, 2015; Volume 11, pp. 95–144.
14. Nakamura, T. The joint universality and the generalized strong recurrence for Dirichlet L -functions. *Acta Arith.* **2009**, *138*, 357–362.
15. Pańkowski, Ł. Joint universality for dependent L -functions. *Ramanujan J.* **2018**, *45*, 181–195. [\[CrossRef\]](#)
16. Korolev, M.A. Gram’s law in the theory of the Riemann zeta-function, Part 1. *Proc. Steklov Inst. Math.* **2016**, *292*, 1–146. [\[CrossRef\]](#)
17. Gram, J.-P. Sur les zéros de la fonction $\zeta(s)$ de Riemann. *Acta Math.* **1903**, *27*, 289–304. [\[CrossRef\]](#)
18. Baker, A. The theory of linear forms in logarithms. In *Transcendence Theory: Advances and Applications*; Academic Press: Boston, MA, USA, 1977; pp. 1–27. [\[CrossRef\]](#)
19. Dubickas, A.; Laurinćikas, A. Joint discrete universality of Dirichlet L -functions. *Arch. Math.* **2015**, *104*, 25–35.
20. Billingsley, P. *Convergence of Probability Measures*, 2nd ed.; Wiley: Chichester, UK, 1999. [\[CrossRef\]](#)
21. Laurinćikas, A. *Limit Theorems for the Riemann Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
22. Mergelyan, S.N. Uniform approximations to functions of a complex variable. *Uspekhi Matem. Nauk* **1952**, *7*, 31–122. (In Russian)

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).