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Article

Cone Metric Spaces over Topological Modules and Fixed Point Theorems for Lipschitz Mappings

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Abstract: In this paper, we introduce the concept of cone metric space over a topological left module and we establish some coincidence and common fixed point theorems for self-mappings satisfying a condition of Lipschitz type. The main results of this paper provide extensions as well as substantial generalizations and improvements of several well known results in the recent literature. In addition, the paper contains an example which shows that our main results are applicable on a non-metrizable cone metric space over a topological left module. The article proves that fixed point theorems in the framework of cone metric spaces over a topological left module are more effective and more fertile than standard results presented in cone metric spaces over a Banach algebra.

Keywords: cone metric spaces; topological left modules; fixed point theorems; points of coincidence; weakly compatible self-maps; Lipschitz mappings

1. Introduction

The concept of metric space was defined by the mathematician Fréchet [1,2]. Afterwards, Kurepa [3] introduced more abstract metric spaces, where the metric values are given in an ordered vector space. Nowadays, the metric spaces with vector valued metric are known under different names: vector-valued metric spaces, cone-valued metric spaces, generalized metric spaces, K-metric spaces, pseudometric spaces, cone metric spaces or TVS cone metric space [4–7].

The distance between two elements x and y in a cone metric space X is defined to be a vector in a ordered Banach space E, and a mapping $T : X \to X$ is said to be a contraction if there is a positive constant $k < 1$ such that

$$d(Tx, Ty) \leq k \cdot d(x, y), \text{ for all } x, y \in X. \tag{1}$$

Recently, it was proved that any cone metric space (X, d) is equivalent with the usual metric space (X, d^*), where the real-valued metric d^* is defined by a nonlinear scalarization function ξ_e [4] or by a Minkowski functional q_e [8]. In addition, it was proved that for each linear contractive mapping T in (X, d), which satisfies Equation (1), one has

$$d^*(Tx, Ty) \leq k \cdot d^*(x, y), \text{ for all } x, y \in X. \tag{2}$$

The above results have been extended by Olaru and Secelean [9] to nonlinear contractive condition on TVS-cone metric space. Afterwards, some other generalizations were pointed out. Liu and Xu [10] introduced the concept of cone metric space over a Banach algebra, replacing the Banach space E by a Banach algebra as the underlying space of a cone metric space. They proved some fixed point theorems of generalized Lipschitz mappings, with Lipschitz constant k such that $\rho(k) < 1$, where $\rho(k)$ is the spectral radius of k. Later on, by omitting the assumption of normality, Xu and Radenović [11] extended the results of Liu and Xu [10]. A survey regarding cone metric spaces, which presents the results obtained after 2007, was published by Aleksić et al. [12]. In this paper, we replace the Banach algebra by a topological module over a topological ordered ring. The Lipschitz constant will be considered as a summable element of the ring, with some additional properties.

2. Methods

In the following, we recall some facts related to properties and examples of topological ordered rings and topological modules. More details can be found, for instance, in Arnautov et al. [13], Steinberg [14], and Warner [15].

Definition 1. *Let us consider $(G,+)$ a group and $\preceq$ a partial order on G. G is called a partially ordered group if translations in G are order preserving:*

$$x \preceq y \text{ implies } a+x+b \preceq a+y+b, \text{ for all } x,y,a,b \in G. \tag{3}$$

Definition 2. *Let $(R,+,\cdot)$ be a ring with identity 1 such that $1 \neq 0$ and let $\preceq$ be a partial order on R. R is called a partially ordered ring if:*

(R_1) $(R,+)$ is a partially ordered group;
(R_2) $0 \preceq a$ and $0 \preceq b$ implies $0 \preceq a \cdot b$, for all $a,b \in R$.

The positive cone of R is $R^+ = \{r \in R \mid 0 \preceq r\}$. The set of invertible elements of R will be denoted by $U(R)$ and $U(R) \cap R^+$ will be denoted by $U_+(R)$.

Example 1. *The ring $\mathcal{M}_{n\times n}(\mathbb{R})$ endowed with the partial order $\preceq_{M_{n\times n}(\mathbb{R})}$, defined by*

$$A \preceq_{M_{n\times n}(\mathbb{R})} B \text{ if and only if } a_{ij} \leq b_{ij} \text{ for each } i,j = \overline{1,n},$$

is a partial ordered ring.

Example 2. *Let $(A,\preceq)$ be a partial ordered ring and let S be a nonempty set. The ring A^S of A-valued functions on S can be ordered. A partial order on A^S can be defined as follows:*

$$f \preceq g \;\text{ if and only if } f(s) \leq g(s) \text{ for all } s \in S.$$

The above order can be contracted to any subring of A^S, e.g., the ring of continuous real valued functions on a topological space or, also, the ring $\mathbb{R}[X_1, X_2, \cdots, X_n]$ of polynomials functions.

Definition 3. *Let $(G,+)$ be an abelian group. G is called a topological group if G is endowed with a topology $\mathcal{G}$ and the following conditions are satisfied:*

(AC) the mapping $G \times G \ni (g_1, g_2) \mapsto g_1 + g_2 \in G$ is continuous, where $G \times G$ is considered with the product topology;
(AIC) the mapping $G \ni g \mapsto -g \in G$ is continuous.

We denote $(G,+,\mathcal{G})$ or more simply $(G,\mathcal{G})$ the topological group.

Definition 4. *A ring $(R,+,\cdot)$ is called a topological ring if R is endowed with a topology $\mathcal{R}$ such that the additive group of the ring R becomes a topological group and the following condition is valid:*

(MC) the mapping $R \times R \ni (r_1, r_2) \mapsto r_1 \cdot r_2 \in R$ is continuous, where $R \times R$ is considered with respect to the product topology.

If $\mathcal{R}$ is a Hausdorff topology, then $(R,+,\cdot,\mathcal{R})$ is called a Hausdorff topological ring. We denote $(R,+,\cdot,\mathcal{R})$ or more simple $(R,\mathcal{R})$ the topological ring.

In order to give an example of a topological ring, we need the definition of the norm on a ring.

Definition 5. *A function N from a ring $(R, +, \cdot)$ to $\mathbb{R}_+$ is a norm if the following conditions hold for every $x, y \in R$:*

(N_1) $N(0_R) = 0$;
(N_2) $N(x + y) \leq N(x) + N(y)$;
(N_3) $N(-x) = N(x)$;
(N_4) $N(x \cdot y) \leq N(x) \cdot N(y)$;
(N_5) $N(x) = 0$ only if $x = 0_R$.

Example 3. *The application*

$$N : M_{n\times n}(\mathbb{R}) \to \mathbb{R}, N(A) = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}|$$

is a norm which generates a ring topology on $M_{n\times n}(\mathbb{R})$.

Proof. The ring topology is generated by the metric d defined by $d(x, y) = N(x - y)$ for all $x, y \in R$. □

Remark 1. *([15] (p. 4)) The Cartesian product of a family $(R_i, \mathcal{R}_i)_{i\in I}$ of topological rings, together with the product topology, is a topological ring.*

Remark 2. *Let $(R_i, \mathcal{R}_i)_{i\in I}$ be a family of topological rings, $R = \prod_{i\in I} R_i$ the Cartesian product endowed with the product topology $\mathcal{R}$, $r \in R$, and $(r_n)_{n\in\mathbb{N}}$ a sequence in R. Then, $r_n \xrightarrow{\mathcal{R}} r$ if and only if $pr_i(r_n) \xrightarrow{\mathcal{R}_i} pr_i(r)$ for all $i \in I$. Here, $pr_i : R \to R_i$ is the canonical projection.*

Proof. $\Rightarrow$ Suppose that $r_n \xrightarrow{\mathcal{R}} r$. Then, $pr_i(r_n) \xrightarrow{\mathcal{R}_i} pr_i(r)$ for all $i \in I$, since pr_i are continuous.

$\Leftarrow$ Suppose that $pr_i(r_n) \xrightarrow{\mathcal{R}_i} pr_i(r)$ for all $i \in I$ and let be V an open neighborhood of r in $\mathcal{R}$. Then, there is a finite set $F \subseteq I$ and the open sets $V_i \subseteq R_i$, $i \in I$, such that $V_i = R_i$ for every $i \in I \setminus F$ and $r \in \prod_{i\in I} V_i \subseteq V$. For each $i \in F$, there is $N_i \in \mathbb{N}$ such that $pr_i(r_n) \in V_i$ whenever $n \geq N_i$. Let $N = \max_{i\in F} N_i$. Then, for each $n \geq N$, we have $pr_i(r_n) \in V_i$, $i \in I$, and hence $r_n \in \prod_{i\in I} V_i \subseteq V$. Thus, $r_n \xrightarrow{\mathcal{R}} r$. □

Definition 6. *Let $(R, \mathcal{R})$ be a topological ring. A left R-module $(E, +, \cdot)$ is called a topological R-module if, on E, a topology $\mathcal{E}$ is specified such that the additive group $(E, +)$ is a topological abelian group and the following condition is satisfied:*

$$(RMC) \quad R \times E \ni (r, x) \mapsto r \cdot x \in E \text{ is continuous,}$$

where $R \times E$ is considered with respect to the product topology $\mathcal{R} \times \mathcal{E}$. We denote $(E, +, \cdot, \mathcal{E})$ or more simply $(E, \mathcal{E})$ a topological left R-module.

Remark 3. *([15] (p. 17)) The Cartesian product of a family $(E_i, \mathcal{E}_i)_{i\in I}$ of topological left R-modules, endowed with the product topology, is a topological left R-module.*

In a way similar to the proof of Remark 2, we can prove the following:

Remark 4. *Let $(E_i, \mathcal{E}_i)_{i\in I}$ be a family of topological left R-modules, $E = \prod_{i\in I} E_i$, the Cartesian product endowed with the product topology $\mathcal{E}$, $x \in E$ an element, and $(x_n)_{n\in\mathbb{N}}$ a sequence in E. Then, $x_n \xrightarrow{\mathcal{E}} x$ if and only if $pr_i(x_n) \xrightarrow{\mathcal{E}_i} pr_i(x)$ for all $i \in I$, where $pr_i : E \to E_i$ is the canonical projection.*

Definition 7. *Let $(E, +, \cdot, \mathcal{E})$ be a topological R-module. A subset P of E is called a cone if:*

(P_1) *P is nonempty, closed and $P \neq \{0_E\}$;*
(P_2) *$a, b \in R^+$ and $x, y \in P$ implies $a \cdot x + b \cdot y \in P$;*
(P_3) *$P \cap -P = \{0_E\}$.*

P is called a solid cone if $intP \neq \emptyset$, where $intP$ denotes the interior of P.

For a given cone $P \subset E$, let define on E the partial ordering $\leq_P$ with respect to P by

$$x \leq_P y \text{ if and only if } y - x \in P. \tag{4}$$

We shall write $x <_P y$ to indicate that $x \leq_P y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$.
Let us consider the following hypotheses:

Hypothesis 1 (H1). *$(R, \oplus, \odot, \mathcal{R})$ is a topological Hausdorff ring such that:*

(i) $U_+(R) \neq \emptyset$;
(ii) 0_R *is an accumulation point of* $U_+(R)$;
(iii) *there is a partial order on R, denoted by $\preceq_R$;*

Hypothesis 2 (H2). *$(E, +, \cdot, \mathcal{E})$ is a topological left R-module;*

Hypothesis 3 (H3). *$P \subset E$ is a solid cone of E.*

Proposition 1. *Let us consider $(E, +, \cdot, \mathcal{E})$ a topological left R-module and $P \subset E$ such that the Hypotheses H1, H2, and H3 are fulfilled. Then:*

(i) $intP + intP \subseteq intP$;
(ii) $\lambda \odot intP \subseteq intP$, *where* $\lambda \in U_+(R)$;
(iii) *if $x \leq_P y$ and $\alpha \in R^+$, then $\alpha \odot x \leq_P \alpha \odot y$;*
(iv) *if $u \leq_P v$ and $v \ll w$, then $u \ll w$;*
(v) *if $u \ll v$ and $v \leq_P w$, then $u \ll w$;*
(vi) *if $u \ll v$ and $v \ll w$, then $u \ll w$;*
(vii) *if $0_E \leq_P u \ll c$ for every $c \in intP$, then $u = 0$;*
(viii) *if $0_E \ll c$ and $(a_n)_{n \in \mathbb{N}}$ is a sequence in E such that $a_n \to 0_E$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n \geq n_0$.*

Proof. (i) Let be $x \in intP + intP$. Then, there exist $x_1, x_2 \in intP$ such that $x = x_1 + x_2$. It follows that there exist the neighborhoods V_1, V_2 of x_1, x_2, respectively, such that $x_1 \in V_1 \subset P$ and $x_2 \in V_2 \subset P$. Since, for every $x_0 \in E$, the mapping $x \mapsto x + x_0$ is a homeomorphism of E into itself, it follows that $V_1 + V_2$ is a neighborhood of x with respect to the topology $\mathcal{E}$. Thus, $x \in intP$.

(ii) Choose $\lambda \in U_+(R)$ and put $x = \lambda \odot c$, $c \in intP$. It follows that there exists a neighborhood V of c such that $c \in V \subset P$. Therefore, $x \in \lambda \odot V \subset \lambda \odot P \subset P$. Since the mapping $x \to \lambda \odot x$ is a homeomorphism of E onto itself, $\lambda \odot V$ is a neighborhood of x. Thus, $x \in intP$.

(iii) If $x \leq_P y$, then $y - x \in P$. It follows that, for all $\alpha \in R^+$, we have $\alpha \odot (y - x) \in P$, i.e., $\alpha \odot x \leq_P \alpha \odot y$.

(iv) We have to prove that $w - u \in intP$ if $v - u \in P$ and $w - v \in intP$. However, there exists a neighborhood V of 0_E such that $w - v + V \subset P$. It follows that $w - u + V = (w - v) + V + (v - u) \subset P + P \subset P$. Hence, $w - u \in int\ P$.

(v) Analogous with (iv).

(vi) Follows from (i).

(vii) Let us consider $c \in intP$. Since 0_R is an accumulation point of $U_+(R)$, it follows that there exists a sequence $(\alpha_n)_{n \in \mathbb{N}} \in U_+(R)$, $\alpha_n \neq 0_R$, such that $\alpha_n \to 0_R$. From (ii), we get $\alpha_n \odot c \in intP$. Therefore, $\alpha_n \odot c - u \in intP$, so $\lim\limits_{n \to +\infty} (\alpha_n \odot c - u) = -u \in \overline{P} = P$. Thus, $u \in P \cap -P = \{0_E\}$.

$(viii)$ Let $0_E \ll c$ and $(a_n)_{n\in\mathbb{N}} \subset E$ such that $a_n \overset{\mathcal{E}}{\to} 0_E$. Then, there exists a symmetric neighborhood U of 0_E such that $c + U \subset P$. Since a_n converges to 0_E, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq n_0$. Then, we have $c - a_n \in c + U \subset P$ for all $n \geq n_0$. Finally, $a_n \ll c$ for all $n \geq n_0$. $\square$

In the sequel, we provide some notions and results related to the sequences defined in a topological ring. We define the convergence and the Cauchy property of a sequence defined in a topological ring. Next, we define the summability of a family of elements from a topological ring. The summability is used in order to introduce the Lipschitz constant of mappings defined on a cone metric space over a topological left module. Furthermore, $(R, +, \cdot, \mathcal{R})$ denotes a Hausdorff topological ring.

Definition 8. *By a directed set, we understand a partially ordered set* $(\Gamma, \leq)$ *that satisfies the following condition:*

(D) for every $\gamma_1, \gamma_2 \in \Gamma$, *there is* $\gamma_3 \in \Gamma$ *such that* $\gamma_1 \leq \gamma_3$ *and* $\gamma_2 \leq \gamma_3$.

Definition 9. *A sequence in* R *is a family of elements* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *indexed by a directed set.*

Definition 10. *The sequence* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *converges to* $x \in R$ *if for every neighborhood* V *of* x *there is* $\gamma_0 \in \Gamma$ *such that* $x_\gamma \in V$ *for each* $\gamma \in \Gamma$ *with* $\gamma_0 \leq \gamma$.

Definition 11. *The sequence* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *is called a Cauchy sequence if for every neighborhood* V *of* 0_R *there is* $\gamma_0 \in \Gamma$ *such that* $x_{\gamma_1} - x_{\gamma_2} \in V$ *for all* $\gamma_0 \leq \gamma_1$ *and* $\gamma_0 \leq \gamma_2$.

Remark 5. *Each convergent sequence* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *is a Cauchy sequence.*

Proof. Let V be a neighborhood of 0_R. Then, there exists a symmetric neighborhood W of 0_R such that $W + W \subseteq V$. Since $x_\gamma \overset{\mathcal{R}}{\to} x \in R$, it follows that there is $\gamma_0 \in \Gamma$ such that $x_\gamma \in x + W$ for all $\gamma_0 \leq \gamma$. Then, for $\gamma_0 \leq \gamma_1$ and $\gamma_0 \leq \gamma_2$, we have $x_{\gamma_1} - x_{\gamma_2} \in (x + W) + (-(x + W)) = W - W = W + W \subseteq V$. Therefore, $(x_\gamma)_{\gamma\in\Gamma}$ is a Cauchy sequence. $\square$

In order to define the sumability of a family of elements from a topological ring, we consider the set $\mathcal{F}(\Gamma)$ of all finite subsets of Γ directed by inclusion $\subseteq$.

Definition 12. *An element* $s \in R$ *is the sum of a family* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *if the sequence* $(s_J)_{J\in\mathcal{F}(\Gamma)}$ *converges to* s, *where for every* $J \in \mathcal{F}(\Gamma)$,

$$s_J = \sum_{\gamma\in J} x_\gamma.$$

The family $(x_\gamma)_{\gamma\in\Gamma}$ *is summable if it has a sum* $s \in R$.

Definition 13. *A family* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *satisfies the Cauchy condition if for every neighborhood* V *of* 0_R *there is* $J_V \in \mathcal{F}(\Gamma)$ *such that*

$$\sum_{\gamma\in K} x_\gamma \in V,$$

for every $K \in \mathcal{F}(\Gamma)$ *disjoint with* J_V.

Remark 6. *A family* $(x_\gamma)_{\gamma\in\Gamma} \subset R$ *satisfies the Cauchy condition if and only if* $(s_J)_{J\in\mathcal{F}(\Gamma)}$ *is a Cauchy sequence.*

Proof. Let V be a neighborhood of 0_R and let W be a symmetric neighborhood of 0_R such that $W + W \subseteq V$.

Let us suppose that $(x_\gamma)_{\gamma\in\Gamma} \subset R$ satisfies the Cauchy condition. Then, there exists $J_W \in \mathcal{F}(\Gamma)$ such that for every $K \in \mathcal{F}(\Gamma)$ disjoint with J_W we have $s_K = \sum_{\gamma\in K} x_\gamma \in W$. Let $J_1, J_2 \in \mathcal{F}(\Gamma)$ be such that $J_W \subseteq J_1$ and $J_W \subseteq J_2$. Then,

$$s_{J_1} - s_{J_2} = (s_{J_1} - s_{J_W}) - (s_{J_2} - s_{J_W}) = s_{J_1\setminus J_W} - s_{J_2\setminus J_W} \in W + W \subseteq V.$$

Therefore, $(s_J)_{J\in\mathcal{F}(\Gamma)}$ is a Cauchy sequence.

Conversely, we assume that $(s_J)_{J\in\mathcal{F}(\Gamma)}$ is a Cauchy sequence. Then, for every neighborhood V of 0_R, there is $J_W \in \mathcal{F}(\Gamma)$ such that $s_{J_1} - s_{J_2} \in V$ for all $J_1, J_2 \in \mathcal{F}(\Gamma)$ with $J_W \subseteq J_1$ and $J_W \subseteq J_2$. Let $K \in \mathcal{F}(\Gamma)$ be disjoint with J_W. Then,

$$s_K = s_{K\cup J_W} - s_{J_W} \in V.$$

Therefore, $(x_\gamma)_{\gamma\in\Gamma}$ satisfies the Cauchy condition. $\square$

Remark 7. *Let $(x_\gamma)_{\gamma\in\Gamma}$ be a summable family in R. Then, for every neighborhood V of 0_R, there is $K \in \mathcal{F}(\Gamma)$ such that $x_\gamma \in V$ for all $\gamma \in \Gamma \setminus K$.*

Proof. Let V be a neighborhood of 0_R. Since $(x_\gamma)_{\gamma\in\Gamma}$ is a summable family, it follows that $(s_J)_{J\in\mathcal{F}(\Gamma)}$ converges. Then, via Remark 5, we find that $(s_J)_{J\in\mathcal{F}(\Gamma)}$ is a Cauchy sequence. Remark 6 implies that $(s_J)_{J\in\mathcal{F}(\Gamma)}$ satisfies the Cauchy condition. Thus, there exists $K \in \mathcal{F}(\Gamma)$ such that $x_\gamma \in V$ whenever $\{\gamma\} \cap K = \emptyset$, that is, $\gamma \in \Gamma \setminus K$. $\square$

Definition 14. *Let $(G, +, \mathcal{G})$ be a topological group.*

(a) *A filter $\mathcal{F} = (F_\gamma)_{\gamma\in\Gamma}$ is called a Cauchy filter if for any neighborhood V of 0_R there exists $F_\gamma \in \mathcal{F}$ such that $F_\gamma - F_\gamma \subseteq V$;*
(b) *G is called complete if any Cauchy filter $\mathcal{F}$ of G has its limit in G.*

Definition 15. *A topological ring $(R, +, \cdot, \mathcal{R})$ is called complete if the topological additive group of the ring $(R, +, \mathcal{R})$ is complete.*

Remark 8. *([15]) If R is a complete Hausdorff topological ring and the open additive subgroups constitute fundamental systems of neighborhoods of 0_R, then the family $(x_\gamma)_{\gamma\in\Gamma} \subset R$ is summable if and only if for every neighborhood V of 0_R we have $x_\gamma \in V$ for all but finitely many $\gamma \in \Gamma$.*

3. Results

Definition 16. *Let us consider X a nonempty set, $(E, +, \cdot, \mathcal{E})$ a topological left R-module, and suppose that the mapping $d : X \times X \to E$ satisfies:*

(d_1) *$0_E \leq_P d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0_E$ if and only if $x = y$;*
(d_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(d_3) *$d(x, y) \leq_P d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then, d is called a cone metric on X and (X, d) is said to be a cone metric space over the topological left R-module E.

Example 4. *Every cone metric space over a Banach algebra is a cone metric space over a topological left module.*

Example 5. *Let us consider $M_{n\times n}(\mathbb{R})$ as in the examples given in Section 2, $\mathbb{R}^n$ as a topological left $M_{n\times n}(\mathbb{R})-$module with the standard topology $\mathcal{D}$, $P_{\mathbb{R}^n} = \{x \in \mathbb{R}^n \mid pr_k(x) \geq 0, k = \overline{1,n}\}$ and*

$$d_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$d_{\mathbb{R}^n}(x,y) = (|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|).$$

Then, the following statements are true:

(a) $U_+(M_{n\times n}(\mathbb{R}_+)) = \{A \in M_{n\times n}(\mathbb{R}_+) \mid detA \neq 0\}$*, and* O_n *is an accumulation point of* $U_+(M_{n\times n}(\mathbb{R}_+))$*;*
(b) $P_{\mathbb{R}^n}$ *is a solid cone, with* $intP_{\mathbb{R}^n} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_i > 0, i = \overline{1,n}\}$*;*
(c) $(\mathbb{R}^n, d)$ *is a cone metric space over the topological left* $M_{n\times n}(\mathbb{R})-$*module* $\mathbb{R}^n$*.*

Proof. (a) Since the sequence $A_k = \frac{1}{k} \cdot U_n \in U_+(M_{n\times n}(\mathbb{R}_+))$, where U_n is the unit matrix of $M_{n\times n}(\mathbb{R}_+)$, converges to O_n, we find that O_n is an accumulation point of $U_+(M_{n\times n}(\mathbb{R}_+))$.

Using the operations on coordinates, the statements (b) and (c) are easy to check. □

Example 6. *Let us consider* $\mathbb{R}^n$ *as in the previous example, I an uncountable set,* $R = \prod_{i\in I} M_{n\times n}(\mathbb{R})$*,* $E = \prod_{i\in I} \mathbb{R}^n$ *endowed with the product topology, and:*

$$(A,B) \ni R \times R \mapsto A \oplus_R B \stackrel{def}{=} (pr_i(A) + pr_i(B))_{i\in I},$$

$$(A,B) \ni R \times R \mapsto A \odot_R B \stackrel{def}{=} (pr_i(A) \cdot pr_i(B))_{i\in I},$$

$$(x,y) \ni E \times E \mapsto x \oplus_E y \stackrel{def}{=} (pr_i(x) + pr_i(y))_{i\in I},$$

$$(A,x) \ni R \times E \mapsto A \odot_E x \stackrel{def}{=} (pr_i(A) \cdot pr_i(x))_{i\in I},$$

$$A \preceq_R B \; if \; and \; only \; if \; pr_i(A) \preceq_{M_{n\times n}(\mathbb{R})} pr_i(B) \; for \; all \; i \in I.$$

Then, the following properties are valid:

(a) R *endowed with the product topology is a topological ring;*
(b) $U_+(R) = \{A \in R \mid \det(pr_i(A)) \neq 0,\ for\ all\ i \in I\}$*, and* $O_R = \prod_{i\in I} O_n$*,* O_n *being the null matrix of* $M_{n\times n}(\mathbb{R})$*, is an accumulation point of* $U_+(R)$*;*
(c) E *endowed with the product topology is a non-metrizable topological left R-module;*
(d) the set $P_E = \{x \in E \mid 0_{\mathbb{R}^n} \leq_{P_{\mathbb{R}^n}} pr_i(x),\ for\ all\ i = \overline{1,n}\}$ *is a solid cone in E;*
(e) the map $d_E : E \times E \to E$*, defined by*

$$d_E(x,y) = (d_{\mathbb{R}^n}(pr_i(x), pr_i(y)))_{i\in I} \; for \; all \; x,y \in E,$$

is a cone metric over the topological left R-module E.

Proof. (a) Follows from Remark 1.

(b) Let $A \in R$. Then,

$$A \in U_+(R) \text{ if and only if } (\exists)A' \in R \text{ such that } A \odot_R A' = \prod_{i\in I} U_n,$$

which is equivalent to

$$pr_i(A) \in U_+(M_{n\times n}(\mathbb{R})) \text{ for all } i \in I,$$

i.e.,

$$\det(pr_i(A)) \neq 0 \text{ for all } i \in I,$$

where U_n means the unit matrix from $M_{n\times n}(\mathbb{R})$. Since the sequence $\prod_{i\in I} \frac{1}{k} \cdot U_n$ converges to O_R with respect to the product topology, it follows that O_R is an accumulation point of $U_+(R)$.

(c) It follows from Remark 3 taking into account that I is an uncountable set.

(d) We first check that P_E verifies the conditions from Definition 7.

(P_1) Since $P_{\mathbb{R}^n}$ is a cone, it follows that $P_{\mathbb{R}^n}$ is a nonempty and closed set. Then, there exists $x \in P_{\mathbb{R}^n}$ so that $\prod_{i \in I} x \in P_E$ and $P_E \neq \emptyset$. Let us consider a sequence $(x_n)_{n \in \mathbb{N}} \subset P_E$ such that $x_n \to x$ in the product topology, hence $pr_i(x_n) \to pr_i(x)$ for all $i \in I$. Since $pr_i(x_n) \in P_{\mathbb{R}^n}$ belongs to the cone $P_{\mathbb{R}^n}$, we find that $pr_i(x) \in P_{\mathbb{R}^n}$ for all $i \in I$. Therefore, $x \in P_E$, which means that P_E is closed.

(P_2) Let $A, B \in R$ such that $O_R \preceq_R A$, $O_R \preceq_R B$, and $x, y \in P_E$. Since $pr_i(A) \cdot pr_i(x) + pr_i(B) \cdot pr_i(y) \in P_{\mathbb{R}^n}$ for all $i \in I$, we get $A \odot_E x \oplus_E B \odot_E y \in P_E$.

(P_3) Choose $x \in P_E \cap -P_E$. Then, for each $i \in I$, we have $pr_i(x) \in P_{\mathbb{R}^n} \cap -P_{\mathbb{R}^n} = \{0_{\mathbb{R}^n}\}$, thus $x = 0_E$.

Next, we prove that $intP_E \neq \emptyset$. As $P_{\mathbb{R}^n}$ is a solid cone, there exists $x \in P_{\mathbb{R}^n}$ and $(x_k)_{k \in \mathbb{N}} \subset P_{\mathbb{R}^n}$, $x_k \neq x$, such that $x_k \to x$. Then, from Remark 4, the sequence $\prod_{i \in I} x_k$ converges in the product topology to $\prod_{i \in I} x$. This means that $\prod_{i \in I} x \in intP_E$.

(e) In order to show that d_E is a cone metric, we will check the conditions from Definition 16.

(d_1) Consider $x, y \in E$. Then,

$$d_E(x,y) = 0_E \text{ if and only if } d_{\mathbb{R}^n}(pr_i(x), pr_i(y)) = 0_{\mathbb{R}^n} \text{ for all } i \in I,$$

which is equivalent to

$$x = y.$$

(d_2) $d_{\mathbb{R}^n}(pr_i(x), pr_i(y)) = d_{\mathbb{R}^n}(pr_i(y), pr_i(x))$ implies

$$d_E(x,y) = d_E(y,x) \text{ for all } x, y \in E.$$

(d_3) Let $x, y, z \in E$. Since, for all $i \in I$,

$$d_{\mathbb{R}^n}(pr_i(x), pr_i(y)) \leq_{P_{\mathbb{R}^n}} d_{\mathbb{R}^n}(pr_i(x), pr_i(z)) + d_{\mathbb{R}^n}(pr_i(z), pr_i(y)),$$

we get

$$d_E(x,y) \leq_{P_E} d_E(x,z) \oplus_E d_E(z,y).$$

□

Definition 17. *Let us consider (X, d) a cone metric space over the topological left R-module E, $x \in X$ an element and $(x_n)_{n \in \mathbb{N}} \subset X$ a sequence. We say that:*

(i) the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ converges to x, and we denote by $\lim_{n \to +\infty} x_n = x$, if, for every $0 \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > N$;

(ii) the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is a Cauchy sequence if, for every $0 \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $m, n > N$.

The cone metric space (X, d) is called complete if every Cauchy sequence is convergent.

Example 7. *Assume that the hypotheses of the Example 5 are fulfilled and*

$$d_{C([a,b],\mathbb{R}^n)} : C([a,b],\mathbb{R}^n) \times C([a,b],\mathbb{R}^n) \to \mathbb{R}^n,$$

$$d_{C([a,b],\mathbb{R}^n)}(x,y) = (\|x_1 - y_1\|_\infty, \|x_2 - y_2\|_\infty, \cdots, \|x_n - y_n\|_\infty).$$

Then, $(C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)})$ is a complete cone metric space over the topological left $M_{n \times n}(\mathbb{R})-$module $\mathbb{R}^n$.

Proof. It is obvious that $d_{C([a,b],\mathbb{R}^n)}$ is a cone metric. Next, we will prove that $(C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)})$ is a complete cone metric space. Let $(f_k)_{k \in \mathbb{N}} \subset C([a,b],\mathbb{R}^n)$ be a Cauchy sequence. Then, for each

$c = (c_1, c_2, \cdots, c_n) \in intP_{\mathbb{R}^n}$, there exists $N \in \mathbb{N}$ such that $d_{C([a,b],\mathbb{R}^n)}(f_k, f_l) \ll c$ for all $k, l \geq N$. It follows that $(pr_i \circ f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$. Therefore, there is $f^i \in C([a,b],\mathbb{R})$ such that $\|f^i - pr_i \circ f_k\|_\infty \to 0$ as $k \to +\infty$. Taking $f = (f^1, f^2, \cdots, f^n)$ we find that $d_{C([a,b],\mathbb{R}^n)}(f, f_k) \ll c$ for all $k \geq N$, i.e., f_k converges to f in the cone metric $d_{C([a,b],\mathbb{R}^n)}$. □

Example 8. *Let suppose that the conditions of Example 6 are fulfilled and*

$$d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)} : \prod_{i\in I} C([a,b],\mathbb{R}^n) \times \prod_{i\in I} C([a,b],\mathbb{R}^n) \longrightarrow \prod_{i\in I} \mathbb{R}^n,$$

$$d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)}(x,y) = (d_{C([a,b],\mathbb{R}^n)}(pr_i(x), pr_i(y)))_{i\in I}.$$

Then, $(\prod\limits_{i\in I} C([a,b],\mathbb{R}^n), d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)})$ *is a complete cone metric space, which is not metrizable in the product topology generated by* $\|\cdot\|_\infty$ *on* $C([a,b],\mathbb{R}^n)$.

Proof. It is obvious that $d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)}$ is a cone metric, and, since the index set I is uncountable, we find that $\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)$ is not metrizable. Next, we will prove that $(\prod\limits_{i\in I} C([a,b],\mathbb{R}^n), d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)})$ is a complete metric space. Let $(f_k)_{k\in\mathbb{N}} \subset \prod\limits_{i\in I} C([a,b],\mathbb{R}^n)$ be a Cauchy sequence. Then, for every $c \in intP_{\prod\limits_{i\in I}\mathbb{R}^n}$, there exists $N \in \mathbb{N}$ such that $d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)}(f_k, f_l) \ll c$ for all $k, l \geq N$. It follows that, for all $i \in I$, $(pr_i \circ f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the cone metric space $(C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)})$. From Example 7, there exists $f^i \in C([a,b],\mathbb{R}^n)$ such that for all $i \in I$ we have $pr_i \circ f_k \overset{d_{C([a,b],\mathbb{R}^n)}}{\longrightarrow} f^i$ as $k \to +\infty$, thus $d_{C([a,b],\mathbb{R}^n)}(pr_i(f_k), f^i) \ll pr_i(c)$ for every $i \in I$. Then, the element $f = \prod\limits_{i\in I} f^i \in \prod\limits_{i\in I} C([a,b],\mathbb{R}^n)$ has the property that $d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)}(f, f_k) \ll c$ for all $k \geq N$, i.e. $(f_k)_{k\in\mathbb{N}}$ converges to f in the cone metric $d_{\prod\limits_{i\in I} C([a,b],\mathbb{R}^n)}$. □

Remark 9. *Let us consider* (X, d) *a cone metric space over a topological left* R*-module* E *and* $(x_n)_{n\in\mathbb{N}}$ *a sequence in* X. *If* $(x_n)_{n\in\mathbb{N}}$ *converges to* x *and* $(x_n)_{n\in\mathbb{N}}$ *converges to* y, *then* $x = y$.

Proof. Let us consider $0 \ll c$. Then,

$$d(x,y) \leq_P d(x,x_n) + d(x_n,y) \ll c + c \in intP_E.$$

From Proposition 1 (vii), we get $d(x,y) = 0$, i.e., $x = y$. □

Furthermore, we obtain several coincidences and common fixed point theorems for Lipschitz mappings defined on a cone metric space (X, d) over a topological left R-module E. These results are generalizations of some well known theorems in the recent literature. The last example in this section will show that our results are applicable on a non-metrizable cone metric space over a topological left module.

Definition 18. *Let* f *and* g *be self-maps on a set* X. *If* $w = fx = gx$ *for some* $x \in X$, *then* x *is called a coincidence point of* f *and* g, *and* w *is named a point of coincidence of* f *and* g.

Jungck [16] said that a pair of self-mappings are *weakly compatible* if they commute at their coincidence points.

Proposition 2. *([16]) Let* f *and* g *be weakly compatible self-maps on a set* X. *If* f *and* g *have a unique point of coincidence* $w = fx = gx$, *then* w *is the unique common fixed point of* f *and* g.

The main result can be found in the next section

Theorem 1. *Let us consider (X, d) a cone metric space over a topological left R-module E such that the Hypotheses H1, H2 and H3 are fulfilled, the set*

$$\mathcal{S} \stackrel{def.}{=} \{k \in R^+ \mid (k^n)_{n\in\mathbb{N}} \text{ is a summable family}\}$$

and suppose that the mappings $f, g : X \to X$ satisfy:

(i) *the range of g contains the range of f and $g(X)$ is a complete subspace of X;*
(ii) *there exists $k \in \mathcal{S}$ such that $d(fx, fy) \leq_P k \cdot d(gx, gy)$ for all $x, y \in X$.*

Then, f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. We choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then,

$$d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \leq_P k \cdot d(gx_n, gx_{n-1})$$

$$\leq_P k^2 \cdot d(gx_{n-1}, gx_{n-2}) \leq_P \cdots \leq_P k^n \cdot d(gx_1, gx_0).$$

Based on the previous inequality, for all $p \geq 1$, we get

$$d(gx_n, gx_{n+p}) \leq_P d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+p-1}, gx_{n+p})$$

$$\leq_P k^n \cdot d(gx_1, gx_0) + k^{n+1} \cdot d(gx_1, gx_0) + \cdots + k^{n+p-1} \cdot d(gx_1, gx_0)$$

$$\leq_P (1_R + k + \cdots + k^{p-1}) \cdot k^n \cdot d(gx_1, gx_0) \leq_P (\sum_{j=0}^{+\infty} k^j) \cdot k^n \cdot d(gx_1, gx_0).$$

From Remark 7, one obtains $k^n \stackrel{\mathcal{R}}{\to} 0_R$ as $n \to +\infty$, and taking into account that $(k^j)_{j\in\mathbb{N}}$ is a summable family, from the condition (RMC) and Proposition 1 $(viii)$ we find that, for all $0 \ll c$, there exists $N \in \mathbb{N}$ such that $d(gx_n, gx_{n+p}) \ll c$ for every $n \geq N$ and $p \geq 1$. Thus, $(gx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $g(X)$ is complete, there exists $q \in g(X)$ such that $gx_n \to q$ as $n \to +\infty$. Consequently, we can find $p \in X$ such that $gp = q$. Furthermore, for each $0 \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$d(gx_n, fp) = d(fx_{n-1}, fp) \leq_P k \cdot d(gx_{n-1}, gp) \ll k \cdot c \in intP.$$

It follows that $gx_n \to fp$ as $n \to +\infty$. The uniqueness of the limit implies that $fp = gp = q$. Next, we will show that f and g have a unique point of coincidence. For this, we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. Therefore, for every $c \in intP$, we have

$$d(gp_1, gp) = d(fp_1, fp) \leq_P k \cdot d(gp_1, gp) = k \cdot d(fp_1, fp)$$

$$\leq_P k^2 \cdot d(gp_1, gp) \leq_P \cdots \leq_P k^n \cdot d(gp_1, gp) \ll c,$$

for all $n \geq N(c)$. Thus, $d(gp_1, gp) = 0$, i.e., $gp_1 = gp$. From Proposition 2, it follows that f and g have a unique common fixed point. □

Theorem 2. *Let us consider (X, d) a cone metric space over a topological left R-module E such that the Hypotheses H1, H2 and H3 are fulfilled. We suppose that:*

(i) *the range of g contains the range of f and $g(X)$ is a complete subspace of X;*

(ii) *R is complete and the open additive subgroups constitute fundamental systems of neighborhoods of 0_R;*
(iii) *there is $k \in R^+$ such that $k^n \xrightarrow{\mathcal{R}} 0_R$ as $n \to +\infty$, and $d(fx, fy) \leq_P k \cdot d(gx, gy)$ for all $x, y \in X$.*

Then, f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. From (iii) and according to Remark 8, we find that $(k^n)_{n\in\mathbb{N}}$ is a summable family. The conclusion follows now from Theorem 1. □

Corollary 1. *Let us consider (X, d) a cone metric space over a topological left R-module E such that the Hypotheses H1, H2, and H3 are fulfilled, the set*

$$\mathcal{S} \stackrel{def.}{=} \{k \in R^+ \mid (k^n)_{n\in\mathbb{N}} \text{ is a summable family}\}$$

and we suppose that the mapping $f : X \to X$ satisfies:

(i) *there exists $k \in \mathcal{S}$ such that $d(fx, fy) \leq_P k \cdot d(x, y)$ for all $x, y \in X$.*

Then, f has a unique fixed point in X.

Proof. The conclusion follows from Theorem 1 replacing g with the identity map. □

The next corollaries show that our results generalize the main contributions published in the papers [10,11].

Corollary 2. *Let (X, d) be a cone metric space over a Banach algebra E and the mapping $f : X \to X$ such that:*

(i) *there exists $k \in E$ such that the spectral radius $\rho(k)$ is less than one;*
(ii) *$d(fx, fy) \leq_P k \cdot d(x, y)$ for all $x, y \in X$.*

Then, f has a unique fixed point in X.

Proof. The condition (i) implies that $(k^n)_{n\in\mathbb{N}}$ is a summable family of R. The conclusion comes from Corollary 1 taking into account that every Banach algebra is a topological module. □

Corollary 3. *Let (X, d) be a complete cone metric space over a topological left R-module E and $f : X \to X$ be a mapping. Assume that the following two conditions are satisfied:*

(i) *R is complete and the open additive subgroups constitute fundamental systems of neighborhoods of 0_R;*
(ii) *there is $k \in R^+$ such that $k^n \xrightarrow{\mathcal{R}} 0_R$ as $n \to +\infty$, and $d(fx, fy) \leq_P k \cdot d(x, y)$ for all $x, y \in X$.*

Then, f has a unique fixed point in X.

Proof. Follows from Theorem 2 for $g = 1_X$. □

Example 9. *Let us consider the following integral equation:*

$$x(t) = f(t) + \int_a^t K(t, s, x(s))ds, \ t \in [a, b], \tag{5}$$

such that:

(i) *the conditions of the Example 7 are fulfilled;*
(ii) *$x \in C([a, b], \mathbb{R}^n)$, $f \in C([a, b], \mathbb{R}^n)$ and $K \in C([a, b] \times [a, b] \times \mathbb{R}^n, \mathbb{R}^n)$;*

(iii) *there exists* $A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *such that the spectral radius* $\rho(A) < 1$ *and*

$$|pr_i \circ K(t,s,u) - pr_i \circ K(t,s,v)| \leq \sum_{j=1}^{n} a_{ij} \cdot |pr_j(u) - pr_j(v)|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}^n$, $i = \overline{1,n}$.

Then, Equation (5) has a unique solution in the cone metric space $(C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)})$.

Proof. Let us consider

$$T : (C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)}) \longrightarrow (C([a,b],\mathbb{R}^n), d_{C([a,b],\mathbb{R}^n)}),$$

given by

$$T(x)(t) = f(t) + \int_a^t K(t,s,x(s))ds,\ t \in [a,b].$$

Let $x, y \in C([a,b],\mathbb{R}^n)$. From the condition (iii), it follows that, for each $i = \overline{1,n}$ and every $t \in [a,b]$, we have

$$|pr_i \circ T(x)(t) - pr_i \circ T(y)(t)| \leq \int_a^t |pr_i \circ K(t,s,x(s)) - pr_i \circ K(t,s,y(s))| \leq$$

$$\leq \sum_{j=1}^{n} a_{ij} \cdot \|pr_j(x) - pr_j(y)\|_\infty = pr_i(A \cdot d_{\mathbb{R}^n}(x,y)).$$

The previous inequality leads to

$$\|pr_i \circ T(x) - pr_i \circ T(y)\|_\infty \leq pr_i(A \cdot d_{C([a,b],\mathbb{R}^n)}(x,y)), \text{ for all } i = \overline{1,n},$$

which means that

$$d_{C([a,b],\mathbb{R}^n)}(T(x),T(y)) \leq_{P_{\mathbb{R}^n}} A \cdot d_{C([a,b],\mathbb{R}^n)}(x,y).$$

Since $\rho(A) < 1$, we deduce that $(A^n)_{n\in\mathbb{N}}$ is a summable family. The conclusion follows from Corollary 1. □

Example 10. *Let us consider the following integral equation:*

$$x(t) = f(t) + \int_a^t K(t,s,x(s))ds,\ t \in [a,b], \tag{6}$$

such that:

(i) *the hypotheses of Example 8 are fulfilled;*
(ii) $x \in \prod_{i\in I} C([a,b],\mathbb{R}^n)$, $f \in \prod_{i\in I} C([a,b],\mathbb{R}^n)$ *and* $K \in \prod_{i\in I} C([a,b] \times [a,b] \times \prod_{i\in I} \mathbb{R}^n, \mathbb{R}^n)$;
(iii) *there exists* $A \in \prod_{i\in I} \mathcal{M}_{n\times n}(\mathbb{R})$ *such that the spectral radius* $\rho(pr_i(A)) < 1$ *and*

$$|pr_k \circ pr_i \circ K(t,s,u) - pr_k \circ pr_i \circ K(t,s,v)|$$

$$\leq \sum_{l=1}^{n} (pr_i(A))_{kl} \cdot |pr_l \circ pr_i(u) - pr_l \circ pr_i(v)|,$$

for all $t,s \in [a,b]$, $u,v \in \prod_{i\in I} \mathbb{R}^n$, $k = \overline{1,n}$, $i \in I$.

Then, Equation (6) has a unique solution in the cone metric space $\left(\prod_{i\in I} C([a,b],\mathbb{R}^n), d_{\prod_{i\in I} C([a,b],\mathbb{R}^n)}\right)$.

Proof. Let us consider

$$T : \left(\prod_{i\in I} C([a,b],\mathbb{R}^n), d_{\prod_{i\in I} C([a,b],\mathbb{R}^n)}\right) \longrightarrow \left(\prod_{i\in I} C([a,b],\mathbb{R}^n), d_{\prod_{i\in I} C([a,b],\mathbb{R}^n)}\right),$$

defined by

$$T(x)(t) = f(t) + \int_a^t K(t,s,x(s))ds,\ t \in [a,b].$$

Let $x, y \in \prod_{i\in I} C([a,b],\mathbb{R}^n)$. From assumption (iii), it follows that, for every $i \in I$, we have

$$d_{C([a,b],\mathbb{R}^n)}(pr_i \circ T(x), pr_i \circ T(y)) \leq_{P_{\mathbb{R}^n}} pr_i(A) \cdot d_{C([a,b],\mathbb{R}^n)}(pr_i(x), pr_i(y)).$$

The previous relation leads to

$$d_{\prod_{i\in I} C([a,b],\mathbb{R}^n)}(T(x), T(y)) \leq_{P_{\prod_{i\in I}\mathbb{R}^n}} A \cdot d_{\prod_{i\in I} C([a,b],\mathbb{R}^n)}(x, y).$$

Since $\rho(pr_i(A)) < 1$ for every $i \in I$, and taking into account the definition of the relation $\odot_R$, we deduce that $(A^n)_{n\in\mathbb{N}}$ is a summable family. The conclusion follows from Corollary 1. □

4. Conclusions

This paper introduced the concept of cone metric space over a topological left module and established some coincidence and common fixed point theorems for self-mappings satisfying a condition of Lipschitz type. The article proved that the fixed point results in the framework of cone metric spaces over a topological left module are more powerful than the standard theorems presented in cone metric spaces over a Banach algebra, whereas some recent results presented in the literature can be obtained as particular cases of our theorems. The results obtained in this study were applied to prove the existence and uniqueness of the solution of some integral equations. In addition, an example that showed that the main results are applicable on a non-metrizable cone metric space over a topological left module has been given.

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