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Fixed Point Results for a Selected Class of Multi-Valued Mappings under $(\theta, \mathcal{R})$-Contractions with an Application

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Abstract: In this article, we introduce a relatively new concept of multi-valued $(\theta, \mathcal{R})$-contractions and utilize the same to prove some fixed point results for a special class of multi-valued mappings in metric spaces endowed with an amorphous binary relation. Illustrative examples are also provided to exhibit the utility of our results proved herein. Finally, we utilize some of our results to investigate the existence and uniqueness of a positive solution for the integral equation of Volterra type.

Keywords: fixed point; monotone type mappings; multi-valued θ-contractions; binary relations; integral equations

1. Introduction

The classical Banach contraction principle continues to be the soul of metric fixed point theory, which states that every contraction mapping S defined on a complete metric space (M, ρ) has a unique fixed point. With a view to have wide range of applications, this principle has been improved, extended, and generalized in many directions (e.g., [1–3]), which contains several novel generalizations. In the present context, an effective generalization given by Jleli and Samet [1] is worth noting wherein the authors introduced the idea of θ-contractions.

In 1986, the idea of an order-theoretic fixed point result was initiated by Turinici [4]. Thereafter, Ran and Reurings [5] established a relatively more natural order-theoretic version, followed by Nieto and Rodríguez-López [6–8]. Thereafter, Samet and Turinici [9] obtained fixed point results under symmetric closure of an amorphous binary relation for nonlinear contractions. Recently, Alam and Imdad [3] obtained a relation-theoretic analog of Banach contraction principle employing an arbitrary binary relation, which unifies several well-known relevant order-theoretic results.

For the sake of completeness, we recollect few basic notions and related results regarding multi-valued mappings.

Let M be a nonempty set. Suppose that (M, ρ) is a metric space and $CB(M)$ the family of all nonempty closed and bounded subsets of M. Let $K(M)$ be the family of all nonempty compact subsets of M. Now, define $\mathcal{H} : CB(M) \times CB(M) \to \mathbb{R}$ by

$$\mathcal{H}(U, V) = max\left\{ \sup_{u \in U} D(u, V), \sup_{v \in V} D(v, U) \right\}, \quad U, V \in CB(M),$$

where $D(u, V) := \inf\{\rho(u, v) : v \in V\}$. Then, $\mathcal{H}$ is a metric on $CB(M)$ known as Pompeiu–Hausdorff metric. Let $\mathcal{P}(M)$ denote the family of all nonempty subsets of M and $S : M \to P(M)$. An element $u \in M$ is said to be a fixed point of S if $u \in Su$ ($Fix(S)$ denotes the set of all such points).

In 1969, Nadler [10] extended Banach contraction principle to multi-valued mappings and begun the study of fixed point theory of multi-valued contractions. Thereafter, vigorous studies were conducted to obtain a variety of generalizations, extensions and applications of Nadler's Theorem (e.g., see, [11–21]). With a similar quest, Hançer et al. [22] extended the concept of θ-contractions to multi-valued mappings and proved two nice fixed point results. Recently, Baghani and Ramezani [23] introduced a new class of multi-valued mappings by utilizing the idea of arbitrary binary relations between two sets and prove some relation-theoretic multi-valued results in a metric space.

Continuing this direction of research, in this paper, we do the following:

- We introduce a relatively new concept of multi-valued $(\theta, \mathcal{R})$-contractions and obtain some relation-theoretic fixed point results for a special class of mappings proposed by Baghani and Ramezani [23], which in turn generalize and extend the results obtained by Hançer et al. [22].
- To exhibit the utility, we provide some illustrative examples.
- We obtain some relation-theoretic existence and uniqueness results for single-valued mappings.
- As consequences of our results, we deduce some corollaries in the setting of ordered-metric spaces.
- We show the applicability of our newly obtained results by investigating the existence and uniqueness of a positive solution for Volterra type integral equation under some suitable conditions.

2. Preliminaries

We begin this section by describing some terminological and notational conventions that are used throughout the paper.

Following [1] and [24], let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(Θ_1) θ is nondecreasing;
(Θ_2) for each sequence $\{\beta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\beta_n) = 1 \iff \lim_{n\to\infty} \beta_n = 0^+$;
(Θ_3) there exist $r \in (0, 1)$ and $\gamma \in (0, \infty]$ such that $\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^r} = \gamma$; and
(Θ_4) θ is continuous.

In addition, we use the following notations:

- $\Theta_{1,2,3,4}$ denotes the set of all functions θ satisfying $(\Theta_1) - (\Theta_4)$.
- $\Theta_{1,2,3}$ denotes the set of all functions θ satisfying $(\Theta_1) - (\Theta_3)$.
- $\Theta_{1,2,4}$ denotes the set of all functions θ satisfying (Θ_1), (Θ_2), and (Θ_4).
- $\Theta_{2,3}$ denotes the set of all functions θ satisfying (Θ_2) and (Θ_3).
- $\Theta_{2,4}$ denotes the set of all functions θ satisfying (Θ_2) and (Θ_4).

For examples of such functions, one may consult the work in [1,24,25]. However, we add the following examples to this effect.

Example 1. *Define* $\theta : (0, \infty) \to (1, \infty)$ *by*

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}}, & \beta \le k, \\ e^{2(k+1)}, & \beta > k, \end{cases}$$

where k *is any fixed real number greater than or equal to 1. Then,* $\theta \in \Theta_{1,2,3}$.

Example 2. *Define* $\theta : (0, \infty) \to (1, \infty)$ *by* $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$*; then,* $\theta \in \Theta_{1,2,4}$.

The notion of θ-contractions was introduced by Jleli and Samet [1] as follows:

Definition 1 ([1])**.** *Let (M,ρ) be a metric space and $\theta \in \Theta_{1,2,3}$. Then, $S : M \to M$ is called a θ-contraction mapping if there exists $\lambda \in (0,1)$ such that*

$$\rho(Su,Sv) > 0 \Rightarrow \theta(\rho(Su,Sv)) \leq [\theta(\rho(u,v))]^{\lambda}, \text{ for all } u,v \in M. \tag{1}$$

Considering this new concept, the authors of [1] proved the following result.

Theorem 1 (Corollary 2.1 of [1])**.** *On a complete metric space, every θ-contraction mapping has a unique fixed point.*

Imdad et al. [25] noticed that Theorem 1 can be proved without the Assumption (Θ_1), from which they introduced the notion of weak θ-contractions. Inspired by this, we also deduce some relation-theoretic results (without Assumption (Θ_1)) for single-valued mappings.

On the other hand, the concept of multi-valued θ-contractions was introduced by Hançer et al. [22] as follows:

Definition 2 ([22])**.** *Let (M,ρ) be a metric space and $S : M \to CB(M)$. Then, S is said to be a multi-valued θ-contraction mapping if there exist $\lambda \in (0,1)$ and $\theta \in \Theta_{1,2,3}$ such that*

$$\mathcal{H}(Su,Sv) > 0 \Rightarrow \theta(\mathcal{H}(Su,Sv)) \leq [\theta(\rho(u,v))]^{\lambda}, \text{ for all } u,v \in M. \tag{2}$$

Utilizing the preceding definition, the authors of [22] proved the following result.

Theorem 2 ([22])**.** *Let (M,ρ) be a complete metric space and $S : M \to K(M)$ a multi-valued θ-contraction for some $\theta \in \Theta_{1,2,3}$. Then, S has a fixed point.*

In addition, Hançer et al. [22] showed that one may replace $K(M)$ by $CB(M)$, by assuming the following additional condition on θ :

(Θ_4') $\theta(\inf B) = \inf \theta(B), \forall B \subset (0,\infty)$ with $\inf B > 0$.

Notice that, if θ satisfies (Θ_1), then it satisfies (Θ_4') if and only if θ is right continuous.

Let $\Theta_{1,2,3,4'}$ be the class of all functions θ satisfying $(\theta_1), (\theta_2), (\theta_3)$ and (θ_4').

Theorem 3 ([22])**.** *Let (M,ρ) be a complete metric space and $S : M \to CB(M)$ be a multi-valued θ-contraction mapping for some $\theta \in \Theta_{1,2,3,4'}$. Then, S has a fixed point.*

To make our paper self contained, we provide some basic relation theoretic notions, definitions, and relevant results described in the following.

A subset $\mathcal{R}$ of $M \times M$ is called a binary relation on M. Trivially, $\emptyset$ and $M \times M$ are binary relations on M known as the empty relation and the universal relation, respectively. A binary relation $\mathcal{R}$ on M is said to be transitive if $(u,v) \in \mathcal{R}$ and $(v,w) \in \mathcal{R}$ implies $(u,w) \in \mathcal{R}$, for all $u,v,w \in M$. Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation. The inverse of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined as $\mathcal{R}^{-1} := \{(u,v) \in M \times M : (v,u) \in \mathcal{R}\}$ and $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$. The elements u and v of M are said to be $\mathcal{R}$-comparable if $(u,v) \in \mathcal{R}$ or $(v,u) \in \mathcal{R}$, which is denoted by $[u,v] \in \mathcal{R}$.

Definition 3 ([3])**.** *Let $\mathcal{R}$ be a binary relation on a nonempty set M. A sequence $\{u_n\} \subseteq M$ is said to be $\mathcal{R}$-preserving if*

$$(u_n, u_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0.$$

Definition 4 ([26])**.** *Let (M, ρ) be a metric space and $\mathcal{R}$ be a binary relation on M. Then, M is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges to some point in M.*

It is worth mentioning that every complete metric space is $\mathcal{R}$-complete, for arbitrary binary relation $\mathcal{R}$. On the other hand, under the universal relation, the notion of $\mathcal{R}$-completeness coincides with the usual completeness.

Definition 5 ([3])**.** *Let (M, ρ) be a metric space and $\mathcal{R}$ a binary relation on M. Then, $\mathcal{R}$ is said to be ρ-self-closed if, whenever $\mathcal{R}$-preserving sequence $\{u_n\}$ converges to u, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$.*

Definition 6 ([26])**.** *Let M be a nonempty set equipped with a binary relation $\mathcal{R}$. Then, M is said to be locally transitive if for any (effective) $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ (with range $A := \{u_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{R}|_A$ is transitive, where $\mathcal{R}|_A = \mathcal{R} \cap (A \times A)$.*

Definition 7 ([3])**.** *Let M be a nonempty set and $S : M \to M$. A binary relation $\mathcal{R}$ on M is called S-closed if for any $u, v \in M$,*

$$(u, v) \in \mathcal{R} \Rightarrow (Su, Sv) \in \mathcal{R}.$$

Definition 8 ([27])**.** *Let (M, ρ) be a metric space, $\mathcal{R}$ a binary relation on M, $S : M \to M$ and $u \in M$. We say that S is $\mathcal{R}$-continuous at u if for any $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ such that $u_n \xrightarrow{\rho} u$, we have $Su_n \xrightarrow{\rho} Su$. Moreover, S is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of M.*

Definition 9 ([27])**.** *A subset $S \subseteq M$ is called $\mathcal{R}$-connected if for each $u, v \in S$, there exists a path in $\mathcal{R}$ from u to v where a path of length n ($n \in \mathbb{N}$) in $\mathcal{R}$ from u to v is a finite sequence $\{u_0, u_1, u_2, ..., u_n\} \subseteq M$ such that $u_0 = u,\ u_n = v$ with $(u_i, u_{i+1}) \in \mathcal{R}$, for each $i \in \{0, 1, ..., n-1\}$.*

Now, we have some definitions which play a crucial role in the forthcoming sections.

Definition 10 ([23])**.** *Let U, V be two nonempty subsets of a nonempty set M and $\mathcal{R}$ a binary relation on M. Define binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ between U and V as follows:*

(i) $(U, V) \in \mathcal{R}_1$ *if* $(u, v) \in \mathcal{R}$, *for all* $u \in U$ *and* $v \in V$.
(ii) $(U, V) \in \mathcal{R}_2$ *if, for each* $u \in U$, *there exists* $v \in V$ *such that* $(u, v) \in \mathcal{R}$.

Remark 1. *Clearly, if $(U, V) \in \mathcal{R}_1$, then $(U, V) \in \mathcal{R}_2$ but the converse is not true in general.*

Definition 11 ([23])**.** *Let (M, ρ) be a metric space equipped with a binary relation $\mathcal{R}$ and $S : M \to CB(M)$. Then, S is called*

(i) *monotone of Type (I) if*

$$u, v \in M,\ (u, v) \in \mathcal{R} \text{ implies that } (Su, Sv) \in \mathcal{R}_1;$$

and

(ii) *monotone of Type (II) if*

$$u, v \in M,\ (u, v) \in \mathcal{R} \text{ implies that } (Su, Sv) \in \mathcal{R}_2.$$

Remark 2. *If S is monotone of Type (I) then by Remark 1 it is monotone of Type (II), but the converse may not be true in general.*

Definition 12. *Let (M,ρ) be a metric space, $\mathcal{R}$ a binary relation on M, $S : M \to CB(M)$ and $u \in M$. We say that S is $\mathcal{R}_{\mathcal{H}}$-continuous at u if for any $\mathcal{R}$-preserving sequence $\{u_n\} \subseteq M$ such that $u_n \xrightarrow{\rho} u$, we have $Su_n \xrightarrow{\mathcal{H}} Su$ (as $n \to \infty$). Moreover, S is called $\mathcal{R}_{\mathcal{H}}$-continuous if it is $\mathcal{R}_{\mathcal{H}}$-continuous at each point of M.*

3. Main Results

We begin this section by introducing the notion of multi-valued $(\theta,\mathcal{R})$-contractions as follows:

Definition 13. *Let (M,ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to CB(M)$. Given $\theta \in \Theta_{1,2,3}$ (or $\theta \in \Theta_{1,2,4}$), we say that S is multi-valued $(\theta,\mathcal{R})$-contraction mapping if there exists $\lambda \in (0,1)$ such that*

$$\theta(\mathcal{H}(Su,Sv)) \leq [\theta(\rho(u,v))]^{\lambda}, \ \ \forall u,v \in M \text{ with } (u,v) \in \mathcal{R}^*, \tag{3}$$

where $(u,v) \in \mathcal{R}^* := \{(u,v) \in \mathcal{R} : \mathcal{H}(Su,Sv) > 0\}$.

Remark 3. *Due to the symmetricity of the metrics ρ and $\mathcal{H}$, it is clear that, if Equation (3) is satisfied for $(u,v) \in \mathcal{R}^*$, then it is also satisfied for $(v,u) \in \mathcal{R}^*$ and so for $[u,v] \in \mathcal{R}^*$.*

Remark 4. *Under the universal relation (in case $\theta \in \Theta_{1,2,3}$), Definition 13 coincides with Definition 2.*

Now, we present our first main result which runs as follows.

Theorem 4. *Let (M,ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to K(M)$. Suppose that the following conditions are fulfilled:*

(a) *S is monotone of Type (I);*
(b) *there exists $u_0 \in M$ such that $(\{u_0\}, Su_0) \in \mathcal{R}_2$;*
(c) *S is multi-valued $(\theta,\mathcal{R})$-contraction with $\theta \in \Theta_{1,2,3}$;*
(d) *M is $\mathcal{R}$-complete; and*
(e) *one of the following holds:*
  - *(e')* *S is $\mathcal{R}_{\mathcal{H}}$-continuous, or*
  - *(e'')* *$\mathcal{R}$ is ρ-self-closed.*

Then, S has a fixed point.

Proof. In view of Assumption (b), there exists $u_0 \in M$ such that $(\{u_0\}, Su_0) \in \mathcal{R}_2$. This implies that there exists $u_1 \in Su_0$ such that $(u_0,u_1) \in \mathcal{R}$. As S is monotone of Type (I), we have $(Su_0, Su_1) \in \mathcal{R}_1$. If $u_1 \in Su_1$, then u_1 is a fixed point of S and we are done. Assume that $u_1 \notin Su_1$, then $Su_0 \neq Su_1$, i.e., $\mathcal{H}(Su_0,Su_1) > 0$. Using Condition (c), we have

$$\theta(\mathcal{H}(Su_0,Su_1)) \leq [\theta(\rho(u_0,u_1))]^{\lambda}. \tag{4}$$

In addition, we have

$$D(u_1,Su_1) \leq \mathcal{H}(Su_0,Su_1). \tag{5}$$

Making use of (Θ_1) and Equations (4) and (5), we have

$$\theta(D(u_1,Su_1)) \leq \theta(\mathcal{H}(Su_0,Su_1)) \leq [\theta(\rho(u_0,u_1))]^{\lambda}. \tag{6}$$

As $u_1 \in Su_0$ and Su_1 is compact, there exists $u_2 \in Su_1$ with $(u_1,u_2) \in \mathcal{R}$ such that

$$D(u_1,Su_1) = \rho(u_1,u_2). \tag{7}$$

Now, from Equations (6) and (7), we have

$$\theta(\rho(u_1, u_2)) \leq [\theta(\rho(u_1, u_0))]^{\lambda}.$$

Recursively, we obtain a sequence $\{u_n\} in M$ such that $u_{n+1} \in Su_n$ with $(u_n, u_{n+1}) \in \mathcal{R}$ (*i.e.*, $\{u_n\}$ is an $\mathcal{R}$-preserving sequence) and if $u_n \notin Su_n$ (for all $n \in \mathbb{N}$), then

$$\theta(\rho(u_n, u_{n+1})) \leq [\theta(\rho(u_n, u_{n-1}))]^{\lambda}, \text{ for all } n \in \mathbb{N}_0. \tag{8}$$

Otherwise, S has a fixed point. Denote $\alpha_n = \rho(u_n, u_{n+1}), \forall n \in \mathbb{N}_0$. Then, $\alpha_n > 0, \forall n \in \mathbb{N}_0$. Now, in view of Equation (8), we have ($\forall n \in \mathbb{N}_0$)

$$\theta(\alpha_n) \leq [\theta(\alpha_{n-1})]^{\lambda} \leq [\theta(\alpha_{n-1})]^{\lambda^2} \leq \cdots \leq [\theta(\alpha_0)]^{\lambda^n},$$

which yields that

$$1 < \theta(\alpha_n) \leq [\theta(\alpha_0)]^{\lambda^n}, \quad \forall n \in \mathbb{N}_0. \tag{9}$$

Taking $n \to \infty$ in Equation (9), we obtain

$$\lim_{n\to\infty} \theta(\alpha_n) = 1,$$

which on using (Θ_2) gives rise to

$$\lim_{n\to\infty} \alpha_n = 0^+. \tag{10}$$

i.e., $\{\alpha_n\}$ is a sequence of positive real numbers converges to 0 (as $n \to \infty$). Using (Θ_3), there exists $r \in (0, 1)$ and $\gamma \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\alpha_n) - 1}{(\alpha_n)^r} = \gamma.$$

There are two cases depending on γ.

Case 1: When $\gamma < \infty$. Take $A = \frac{\gamma}{2} > 0$; then, by the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(\alpha_n) - 1}{(\alpha_n)^r} - \gamma \right| \leq A, \text{ for all } n \geq n_0,$$

which implies that

$$\frac{\theta(\alpha_n) - 1}{(\alpha_n)^r} \geq \gamma - A = A, \text{ for all } n \geq n_0,$$

yielding there by

$$n(\alpha_n)^r \leq nB[\theta(\alpha_n) - 1], (where B = \frac{1}{A}) \text{ for all } n \geq n_0.$$

Case 2: When $\gamma = \infty$. Let $A^* > 0$ be any positive real number. Then, by the definition of limit, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{\theta(\alpha_n) - 1}{(\alpha_n)^r} \geq A^*, \text{ for all } n \geq n_1,$$

which yields

$$n(\alpha_n)^r \leq nB^*[\theta(\alpha_n) - 1], (where B^* = \frac{1}{A^*}) \text{ for all } n \geq n_1.$$

Thus, in both the above cases, there exist $C > 0$ (real number) and a positive integer $n_2 \in \mathbb{N}$ (where $n_2 = max\{n_0, n_1\}$), such that

$$n(\alpha_n)^r \leq nC[\theta(\alpha_n) - 1], \text{ for all } n \geq n_2.$$

Using Equation (9), we have

$$n(\alpha_n)^r \leq nC[[\theta(\alpha_0)]^{\lambda^n} - 1].$$

Taking $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} n(\alpha_n)^r = 0.$$

Therefore, there exists $n_3 \in \mathbb{N}$ such that $n(\alpha_n)^r \leq 1$, for all $n \geq n_3$. Which implies that

$$\alpha_n \leq \frac{1}{n^{\frac{1}{r}}}, \text{ for all } n \geq n_3.$$

Now, our aim is to show that $\{u_n\}$ is a Cauchy sequence, for this let $m, n \in \mathbb{N}$ with $m > n \geq n_2$, then we have

$$\begin{aligned} \rho(u_n, u_m) &\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + ... + \rho(u_{m-1}, u_m) \\ &= \sum_{j=n}^{m-1} \alpha_j \leq \sum_{j=n}^{\infty} \alpha_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r}}}. \end{aligned}$$

As $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{r}}} < \infty$, we get

$$\lim_{n,m\to\infty} \rho(u_n, u_m) = 0.$$

Thus, the sequence $\{u_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in (M, ρ). By Condition (d), M is $\mathcal{R}$-complete, and then there exists $u^* \in M$ such that $\lim_{n\to\infty} u_n = u^*$. Now, in view of Condition (e), we have two alternative cases.

Firstly, if (e') holds, then due to $\mathcal{R}_{\mathcal{H}}$-continuity of S, we must have $\mathcal{H}(Su_n, Su^*) \to 0$ as $n \to \infty$. Now, as $u_{n+1} \in Su_n, \forall n \in \mathbb{N}_0$, we get

$$0 \leq D(u_{n+1}, Su^*) \leq \mathcal{H}(Su_n, Su^*), \forall n \in \mathbb{N}_0,$$

which implies that

$$0 \leq \lim_{n\to\infty} D(u_{n+1}, Su^*) \leq \lim_{n\to\infty} \mathcal{H}(Su_n, Su^*) = 0.$$

That is, $\lim_{n\to\infty} D(u_{n+1}, Su^*) = 0$, from which we obtain $u_{n+1} \in \overline{Su^*}$ (as $n \to \infty$). Since Su^* is closed and $u_{n+1} \to u^*$(as $n \to \infty$), $u^* \in Su^*$. Hence, S has a fixed point.

Secondly, assume that Condition (e'') holds. Then, by Definition 5, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}, \forall k \in \mathbb{N}_0$. In addition, from (θ_1) and Equation (13), we have

$$\mathcal{H}(Su, Sv) < \rho(u, v), \ \ \forall u, v \in M \text{ with } (u, v) \in \mathcal{R}^*.$$

Now, using Condition (c), we obtain

$$D(u_{n_k+1}, Su^*) \leq \mathcal{H}(Su_{n_k}, Su^*) \leq \rho(u_{n_k}, u^*), \ \forall k \in \mathbb{N}_0.$$

Taking limit as $n \to \infty$, we have $D(u^*, Su^*) = 0$, which implies that $u^* \in \overline{Su^*} = Su^*$ (as Su^* is closed). Thus, u^* is a fixed point of S. This finishes the proof. □

Remark 5. *The following question naturally arises: Can we replace $K(M)$ by $CB(M)$ in Theorem 4? The answer to this question is no. The following example substantiates the answer.*

Example 3. *Let $M = [0,2]$ and define a metric ρ on M by (for all $u, v \in M$)*

$$\rho(u,v) = \begin{cases} 0, & u = v, \\ \mu + |u - v|, & u \neq v, \end{cases}$$

where μ is any fixed real number such that $\mu \geq 1$. Define a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(u,v) \in \mathcal{R} \Leftrightarrow \{u,v\} \cap \mathbb{Q} \text{ is singleton, for all } u,v \in M\}.$$

Then, M is $\mathcal{R}$-complete and $\mathcal{R}$ is d-self closed. In addition, (M, ρ) is a bounded metric space. All subsets of M are closed as τ_ρ generates discrete topology. Define a mapping $S : M \to CB(M)$ by

$$Su = \begin{cases} \mathbb{Q}_M, & u \in M \setminus \mathbb{Q}_M, \\ M \setminus \mathbb{Q}_M, & u \in \mathbb{Q}_M, \end{cases}$$

where $\mathbb{Q}_M = \mathbb{Q} \cap M$. Then, S is not compact valued. Now, define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}}, & \beta \leq \mu, \\ e^{2(\mu+1)}, & \beta > \mu. \end{cases}$$

Clearly, $\theta \in \Theta_{1,2,3}$ and does not satisfy (Θ_4). Next, we show that

$$\theta(\mathcal{H}(Su, Sv)) \leq [\theta(\rho(u,v))]^{1/2}, \forall u, v \in M \text{ with } (u,v) \in \mathcal{R}^*.$$

Observe that (for all $(u,v) \in \mathcal{R}^*$)

$$\begin{aligned} &\mathcal{H}(Su, Sv) = \mu \text{ and } \rho(u,v) = \mu + |u - v| > \mu \\ \Rightarrow\ &\theta(\mathcal{H}(Su, Sv)) = e^{\sqrt{\mu}} \text{ and } [\theta(\rho(u,v))]^{1/2} = e^{(\mu+1)} \\ \Rightarrow\ &\theta(\mathcal{H}(Su, Sv)) \leq [\theta(\rho(u,v))]^{1/2}. \end{aligned}$$

Therefore, S is a multi-valued $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{1,2,3}$. Hence, all the conditions of Theorem 4 are satisfied but still S has no fixed point.

Next, we present the following result employing the relatively larger class $CB(M)$ instead of $K(M)$.

Theorem 5. *Let (M, ρ) be a complete metric space endowed with a locally transitive binary relation $\mathcal{R}$ and $S : M \to CB(M)$. Suppose that the following conditions are fulfilled:*

(a) *S is monotone of Type (I);*
(b) *there exists $u_0 \in M$ such that $(\{u_0\}, Su_0) \in \mathcal{R}_2$;*
(c) *S is multi-valued $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{1,2,4}$;*
(d) *M is $\mathcal{R}$-complete; and*
(e) *one of the following holds:*
  - (e') *either S is $\mathcal{R}_{\mathcal{H}}$-continuous; or*
  - (e'') *$\mathcal{R}$ is ρ-self-closed.*

Then, S has a fixed point.

Proof. In view of Assumption (b), there exists $u_0 \in M$ such that $(\{u_0\}, Su_0) \in \mathcal{R}_2$. This implies that there exists $u_1 \in Su_0$ such that $(u_0, u_1) \in \mathcal{R}$. As S is monotone of Type (I), we have $(Su_0, Su_1) \in \mathcal{R}_1$.

Now, if $u_1 \in Su_1$, then u_1 is a fixed point of S and the proof is completed. Assume that $u_1 \notin Su_1$, then $Su_0 \neq Su_1$, *i.e.*, $\mathcal{H}(Su_0, Su_1) > 0$. Now, making use of Condition (c), we have

$$\theta(\mathcal{H}(Su_0, Su_1)) \leq [\theta(\rho(u_0, u_1))]^{\lambda}. \tag{11}$$

In addition, we have

$$D(u_1, Su_1) \leq \mathcal{H}(Su_0, Su_1).$$

Using (Θ_1) and Equation (11), we obtain

$$\theta(D(u_1, Su_1)) \leq \theta(\mathcal{H}(Su_0, Su_1)) \leq [\theta(\rho(u_0, u_1))]^{\lambda}. \tag{12}$$

Due to (Θ_4), we have

$$\theta(D(u_1, Su_1)) = \inf_{v \in Su_1} \theta(\rho(u_1, v)).$$

This together with Equation (12) gives rise to

$$\inf_{v \in Su_1} \theta(\rho(u_1, v)) \leq [\theta(\rho(u_0, u_1))]^{\lambda} < [\theta(\rho(u_0, u_1))]^{\lambda_1}, \tag{13}$$

where $\lambda_1 \in (c, 1)$. From Equation (13), there exists $u_2 \in Su_1$ with $(u_1, u_2) \in \mathcal{R}$ such that

$$\theta(\rho(u_1, u_2)) \leq [\theta(\rho(u_0, u_1))]^{\lambda_1}.$$

Again, if $u_2 \in Su_2$, then we are done. Otherwise, by the same way, we can find $u_3 \in Su_2$ with $(u_2, u_3) \in \mathcal{R}$ such that

$$\theta(\rho(u_2, u_3)) \leq [\theta(\rho(u_1, u_2))]^{\lambda_1}.$$

Continuing this process, we construct a sequence $\{u_n\}$ in M such that $u_{n+1} \in Su_n$ with $(u_n, u_{n+1}) \in \mathcal{R}$ and if $u_n \notin Su_n$, then

$$\theta(\rho(u_n, u_{n+1})) \leq [\theta(\rho(u_{n-1}, u_n))]^{\lambda_1}, \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Otherwise, u_n is a fixed point of S. Denote $\alpha_n = \rho(u_n, u_{n+1})$, for all $n \in \mathbb{N}_0$. Then, $\alpha_n > 0$, for all $n \in \mathbb{N}_0$. Now, in view of Equation (14), we have

$$\theta(\alpha_n) \leq [\theta(\alpha_{n-1})]^{\lambda_1} \leq [\theta(\alpha_{n-1})]^{\lambda_1^2} \leq \ldots \leq [\theta(\alpha_0)]^{\lambda_1^n},$$

which implies that

$$1 < \theta(\alpha_n) \leq [\theta(\alpha_0)]^{\lambda_1^n}, \quad \text{for all } n \in \mathbb{N}_0. \tag{15}$$

Letting $n \to \infty$ in Equation (15), we obtain

$$\lim_{n \to \infty} \theta(\alpha_n) = 1. \tag{16}$$

This together with (Θ_2) gives rise to $\lim_{n \to \infty} \alpha_n = 0^+$, that is

$$\lim_{n \to \infty} \rho(u_n, u_{n+1}) = 0. \tag{17}$$

Now, we show that $\{u_n\}$ is a Cauchy sequence. Let on the contrary $\{u_n\}$ not be Cauchy; then, there exist an $\epsilon > 0$ and two subsequences $\{u_{n(k)}\}$ and $\{u_{m(k)}\}$ of $\{u_n\}$ such that

$$k \leq n(k) < m(k), \quad \rho(u_{m(k)-1}, u_{n(k)}) < \epsilon \leq \rho(u_{m(k)}, u_{n(k)}) \quad \text{for all } k \geq 0, \tag{18}$$

and

$$\lim_{k\to\infty} \rho(u_{m(k)}, u_{n(k)}) = \epsilon. \tag{19}$$

Now, observe that

$$\begin{aligned} \epsilon &\leq \rho(u_{m(k)}, u_{n(k)}) \\ &\leq \rho(u_{m(k)}, u_{(m(k)-1)}) + \rho(u_{m(k)-1}, u_{n(k)-1}) + \rho(u_{n(k)-1}, u_{n(k)}) \\ &\leq \rho(u_{m(k)}, u_{(m(k)-1)}) + \rho(u_{m(k)-1}, u_{n(k)}) + 2\rho(u_{n(k)-1}, u_{n(k)}). \end{aligned}$$

Making use of Equations (17)–(19) and letting $k \to \infty$, we have

$$\lim_{k\to\infty} \rho(u_{m(k)-1}, u_{n(k)-1}) = \epsilon. \tag{20}$$

Which implies that there exists $n_0 \in \mathbb{N}_0$ such that $\rho(u_{m(k)}, u_{n(k)}) > 0$ for all $k \geq n_0$ (due to Equation (19)). Since $\mathcal{R}$ is locally transitive, we have $(u_{n(k)-1}, u_{m(k)-1}) \in \mathcal{R}$ (as $n(k) - 1 < m(k) - 1$). Using Condition (c), we have (for all $k \geq n_0$)

$$\theta(\rho(u_{n(k)}, u_{m(k)})) \leq \theta(\mathcal{H}(Su_{n(k)-1}, Su_{m(k)-1})) \leq [\theta(\rho(u_{n(k)-1}, u_{m(k)-1}))]^{\lambda}. \tag{21}$$

Letting $k \to \infty$ in Equation (21) and making use of (Θ_4) and Equations (19) and (20), we obtain $\theta(\epsilon) \leq \theta(\epsilon)^{\lambda}$, which is a contradiction. Thus, $\{u_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence. The rest of the proof follows same lines as in the proof of Theorem 4. □

Now, we present the following example to exhibit the utility of our results.

Example 4. *Let $M = (0, \infty)$ equipped with the usual metric ρ. Define a sequence $\{\sigma_n\}$ in M by*

$$\sigma_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \textit{for all } n \in \mathbb{N}.$$

Now, consider a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(\sigma_1, \sigma_1), (\sigma_i, \sigma_j) : \textit{for } 1 \leq i < j, \textit{where } i, j \in \mathbb{N}\}.$$

Then, it is obvious that $\mathcal{R}$ is locally transitive and ρ-self-closed. In addition, M is $\mathcal{R}$-complete. Now, define a mapping $S : M \to CB(M)$ by

$$Su = \begin{cases} \{u\}, & \textit{if } 0 < u \leq \sigma_1; \\ \{\sigma_1\}, & \textit{if } \sigma_1 \leq u \leq \sigma_2; \\ \{\sigma_1, \sigma_i + \left(\frac{\sigma_{i+1}-\sigma_i}{\sigma_{i+2}-\sigma_{i+1}}\right)(u - \sigma_{i+1})\}, & \textit{if } \sigma_{i+1} \leq u \leq \sigma_{i+2},\ i = 1, 2, \cdots. \end{cases}$$

Clearly, S is a monotone mapping of Type (I) and $(\{\sigma_1\}, S\sigma_1) \in \mathcal{R}_2$. Now, observe that

$$(\sigma_i, \sigma_j) \in \mathcal{R}, S\sigma_i \neq S\sigma_j \Leftrightarrow (i \geq 1, j > 3).$$

Define a function $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\beta) := e^{\sqrt{\beta e^{\beta}}}, \quad \textit{for all } \beta > 0.$$

Then, $\theta \in \Theta_{1,2,3,4}$. Now, we show that S satisfies Equation (3), that is

$$\mathcal{H}(S\sigma_i, S\sigma_j) \neq 0 \Rightarrow e^{\sqrt{\mathcal{H}(S\sigma_i, S\sigma_j) e^{\mathcal{H}(S\sigma_i, S\sigma_j)}}} \leq e^{\lambda \sqrt{\rho(\sigma_i, \sigma_j) e^{\rho(\sigma_i, \sigma_j)}}}, \quad \textit{for some } \lambda \in (0, 1),$$

or

$$\mathcal{H}(S\sigma_i, S\sigma_j) \neq 0 \Rightarrow \frac{\mathcal{H}(S\sigma_i, S\sigma_j)e^{\mathcal{H}(S\sigma_i,S\sigma_j)-\rho(\sigma_i,\sigma_j)}}{\rho(\sigma_i, \sigma_j)} \leq \lambda^2, \text{ for some } \lambda \in (0,1). \tag{22}$$

Now, consider two cases as follows:

Case 1: *When $i = 1 or 2$ and $j > 3$. In this case, we get*

$$\begin{aligned} \frac{\mathcal{H}(S\sigma_1, S\sigma_j)e^{\mathcal{H}(S\sigma_1,S\sigma_j)-\rho(\sigma_1,\sigma_j)}}{\rho(\sigma_1, \sigma_j)} &= \frac{j^2 - j - 2}{j^2 + j - 2}e^{-j} \\ &\leq e^{-1}. \end{aligned} \tag{23}$$

Case 2: *When $j > i > 2$. We have*

$$\begin{aligned} \frac{\mathcal{H}(S\sigma_i, S\sigma_j)e^{\mathcal{H}(S\sigma_i,S\sigma_j)-\rho(\sigma_i,\sigma_j)}}{\rho(\sigma_i, \sigma_j)} &= \frac{j + i - 1}{j + i + 1}e^{i-j} \\ &\leq e^{-1}. \end{aligned} \tag{24}$$

Therefore, the inequality in Equation (22) is satisfied with $\lambda = e^{-1/2}$. Hence, all the requirements of Theorem 4 (as well as Theorem 5) are fulfilled ($Fix(S) = (0, \sigma_1]$).

Remark 6. *Observe that the results due to Hançer et al. [22] are not usable in the context of Example 4 as S does not satisfy Equation (2) on $(0, \sigma_1]$ and also the underlying space is incomplete.*

By putting $Su = \{Su\}$ (for all $u \in M$), every single valued map can be treated as a multi-valued map. Therefore, using Theorems 4 and 5, we deduce two fixed point results for single valued mappings as follows:

Corollary 1. *Let (M, ρ) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to M$. Suppose the following conditions are fulfilled:*

- *(a)* *$\mathcal{R}$ is S-closed;*
- *(b)* *there exists $u_0 \in M$ such that $(u_0, Su_0) \in \mathcal{R}$;*
- *(c)* *S is $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{2,3}$;*
- *(d)* *M is $\mathcal{R}$-complete; and*
- *(e)* *one of the following holds:*
  - (e') *S is $\mathcal{R}$-continuous; or*
  - (e'') *$\mathcal{R}$ is ρ-self-closed.*

Then, S has a fixed point.

Corollary 2. *Let (M, ρ) be a complete metric space endowed with a locally transitive binary relation $\mathcal{R}$ and $S : M \to M$. Suppose the following conditions are fulfilled:*

- *(a)* *$\mathcal{R}$ is S-closed;*
- *(b)* *there exists $u_0 \in M$ such that $(u_0, Su_0) \in \mathcal{R}$;*
- *(c)* *S is $(\theta, \mathcal{R})$-contraction with $\theta \in \Theta_{2,4}$;*
- *(d)* *M is $\mathcal{R}$-complete; and*
- *(e)* *one of the following holds:*
  - (e') *S is $\mathcal{R}$-continuous; or*
  - (e'') *$\mathcal{R}$ is ρ-self-closed.*

Then, S has a fixed point.

Remark 7. *The monotonicity assumption on θ (namely, (Θ_1)) can be removed in the context of single-valued mappings and hence it is omitted in Corollaries 1 and 2.*

Next, we obtain a corresponding uniqueness result in this sequel as follows.

Theorem 6. *Besides the assumptions of Corollary 1 (or Corollary 2), if $Fix(S)$ is $\mathcal{R}^s$-connected, then the fixed point of S is unique.*

Proof. On the contrary, let us suppose that $u, v \in Fix(S)$ such that $u \neq v$. Then, we construct a path of some finite length n from u to v in $\mathcal{R}^s$, say $\{u = u_0, u_1, u_2, \cdots, u_n = v\} \subseteq Fix(S)$ (where $u_i \neq u_{i+1}$ for each i, $(0 \leq i \leq n-1)$, otherwise $u = v$, a contradiction) with $[u_i, u_{i+1}] \in \mathcal{R}$ for each i $(0 \leq i \leq n-1)$. As $u_i \in Fix(S)$, $Su_i = u_i$, for each $i \in \{0, 1, 2, \cdots, n\}$. By using the fact that S is $(\theta, \mathcal{R})$-contraction, we have (for all i, $(0 \leq i \leq n-1)$)

$$\theta(\rho(u_i, u_{i+1})) = \theta(\rho(Su_i, Su_{i+1})) \leq [\theta(\rho(u_i, u_{i+1}))]^{\lambda}, \text{ where } \lambda \in (0,1),$$

a contradiction. This concludes the proof. □

Remark 8. *If we take $\theta(\beta) = e^{\sqrt{\beta}}$ $(\theta \in \Theta_{2,3})$, then Theorem 6 is a sharpened version of the main result due to Alam and Imdad [3].*

4. Some Consequences in Ordered Metric Spaces

This section is devoted to obtaining some ordered-theoretic corollaries of our newly obtained results. We recall some relevant definitions and notions before presenting our results. Let X be a non-empty set. If (M, ρ) is a metric space and $(M, \preceq)$ is partially ordered, then $(M, \rho, \preceq)$ is called an ordered metric space. Then, $u, v \in M$ are said to be comparable if $u \preceq v$ or $v \preceq u$ holds. Further, a self-mapping S on M is called non-decreasing if $Su \preceq Sv$ whenever $u \preceq v$ for all $u, v \in M$. Moreover, an ordered metric space $(M, \rho, \preceq)$ is regular for every non-decreasing sequence $\{u_n\} \subset M$ convergent to some $u \in M$ if there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $u_{n_k} \preceq u$, $\forall k \in \mathbb{N}_0$.

If we take $\mathcal{R} :=\preceq$ in Definition 10, then we deduce to the following ordered theoretic definition due to Beg and Butt [28].

Definition 14. *Let (M, ρ) be a partially ordered set. Let U and V be any two nonempty subsets of M. The relation between U and V is denoted and defined as follows: $U \prec_1 V$, if for each $u \in U$ there exists $v \in V$ such that $u \preceq v$. In addition, we say that $U \prec_2 V$ whenever for each $u \in U$ and $v \in V$ we have $u \preceq v$.*

In addition, by take $\mathcal{R} :=\preceq$ in Definition 11 we have the following definition.

Definition 15. *Let $(M, \rho, \preceq)$ be an ordered metric space and $S : M \to CB(M)$. Then, S is called monotone of Type (I) if*

$$u, v \in M, \ u \preceq v \text{ implies that } Su \prec_1 Sv;$$

and monotone of Type (II) if

$$u, v \in M, \ u \preceq v \text{ implies that } Su \prec_2 Sv.$$

If we take $\mathcal{R} :=\preceq$ in Theorem 4, we obtain the following corollary.

Corollary 3. *Let $(M, \rho, \preceq)$ be an ordered metric space and $S : M \to K(M)$. Suppose that the following conditions are fulfilled:*

(a) *S is monotone of Type (I);*

(b) *there exists* $u_0 \in M$ *such that* $\{u_0\} \prec_2 Su_0$;
(c) S *is multi-valued* $(\theta, \preceq)$*-contraction with* $\theta \in \Theta_{1,2,3}$;
(d) M *is* $\preceq$*-complete; and*
(e) *one of the following holds:*

(e') S *is* $\preceq_{\mathcal{H}}$*-continuous; or*
(e'') $(M, \rho, \preceq)$ *is regular.*

Then, S *has a fixed point.*

On setting $\mathcal{R} :=\preceq$ in Theorem 5, we obtain the following corollary.

Corollary 4. *Let* $(M, \rho, \preceq)$ *be an ordered metric space and* $S : M \to CB(M)$*. Suppose that the following conditions are fulfilled:*

(a) S *is monotone of Type* (I);
(b) *there exists* $u_0 \in M$ *such that* $\{u_0\} \prec_2 Su_0$;
(c) S *is multi-valued* $(\theta, \preceq)$*-contraction with* $\theta \in \Theta_{1,2,4}$;
(d) M *is* $\preceq$*-complete; and*
(e) *one of the following holds:*

(e') S *is* $\preceq_{\mathcal{H}}$*-continuous; or*
(e'') $(M, \rho, \preceq)$ *is regular.*

Then, S *has a fixed point.*

Similarly, by taking $\mathcal{R} :=\preceq$ in Corollaries 1 and 2 and Theorem 6, we obtain the following results for single valued mapping.

Corollary 5. *Let* $(M, \rho, \preceq)$ *be an ordered metric space and* $S : M \to M$*. Suppose the following conditions are fulfilled:*

(a) S *is non-decreasing;*
(b) *there exists* $u_0 \in M$ *such that* $u_0 \preceq Su_0$;
(c) S *is* $(\theta, \preceq)$*-contraction with* $\theta \in \Theta_{2,3}$;
(d) M *is* $\preceq$*-complete; and*
(e) *one of the following holds:*

(e') S *is* $\preceq$*-continuous; or*
(e'') $(M, \rho, \preceq)$ *is regular.*

Then, S *has a fixed point.*

Corollary 6. *Let* $(M, \rho, \preceq)$ *be an ordered metric space and* $S : M \to M$*. Suppose the following conditions are fulfilled:*

(a) S *is non-decreasing;*
(b) *there exists* $u_0 \in M$ *such that* $(u_0, Su_0) \in \mathcal{R}$;
(c) S *is* $(\theta, \mathcal{R})$*-contraction with* $\theta \in \Theta_{2,4}$;
(d) M *is* $\preceq$*-complete; and*
(e) *one of the following holds:*

(e') S *is* $\preceq$*-continuous; or*

(e'') $(M, \rho, \preceq)$ *is regular.*

Then, S has a fixed point.

Corollary 7. *In addition to the assumptions of Corollary 5 (or Corollary 6), if $u, v \in Fix(T)$ implies that $u \preceq v$ or $v \preceq u$, then the fixed point of S is unique.*

5. Application to Integral Equation

In this section, we show the applicability of some of our newly obtained results by proving existence and uniqueness of a positive solution for the integral equation of Volterra type as follows:

$$u(t) = \int_0^t g(t, r, u(r))dr + \beta(t), \ \ \forall t \in I = [0, 1], \tag{25}$$

where $g : I \times I \times \mathbb{R} \to \mathbb{R}$ is an integrable function and $\beta : I \to [1, \infty)$ is a given function.

Consider $M = \{u \in C(I, \mathbb{R}) : u(t) > 0, \ for\ all\ t \in I\}$, where $C(I, \mathbb{R})$ is the space of all continuous functions $u : I \to \mathbb{R}$ equipped with the Bielecki's norm

$$\|u\| = \sup_{t \in I} e^{-t}|u(t)|.$$

Define a metric ρ on M by $\rho(u, v) = \|u - v\|$, for all $u, v \in M$. Then, (M, ρ) is a metric space which is not complete.

Now, we are equipped to state and prove our result of the section, which runs as follows:

Theorem 7. *Assume that the following conditions are satisfied:*

(a_1) $g(t_1, r_1, u) > 0$, *for all* $u > 0$ *and* $t_1, r_1 \in I$; *and*

(a_2) *g is non-decreasing in the third variable and there exists* $h > 0$ *such that*

$$|g(t, r, u) - g(t, r, v)| \leq \frac{|u(t) - v(t)|}{h\|u - v\| + 1},$$

$\forall\ t, r \in I$ *and* $u, v > 0$ *with* $uv \geq (u \vee v)$, *where* $u \vee v = u$ *or* v.

Then, the integral in Equation (25) has a positive solution.

Proof. Let us define a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(u, v) \in \mathcal{R} \Leftrightarrow u(t)v(t) \geq (u(t) \vee v(t)), \ \text{for all } t \in I\}.$$

Since $C(I, \mathcal{R})$ is a Banach space with Bielecki's norm, then, for any $\mathcal{R}$-preserving Cauchy sequence $\{u_n\}$ in M, it converges to some point $u \in C(I, \mathbb{R})$. Now, fix $t \in I$, then, by the definition of $\mathcal{R}$, we have

$$u_n(t)u_{n+1}(t) \geq (u_n(t) \vee u_{n+1}(t)), \ \text{for all } n \in \mathbb{N}.$$

As $u_n(t) > 0$, $\forall\ n \in \mathbb{N}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}(t) \geq 1$, $\forall k \in \mathbb{N}$. This subsequence $\{u_{n_k}(t)\}$ of real numbers converges to $u(t)$, which gives rise to $u(t) \geq 1$. As $t \in I$ is arbitrary, we have $u \geq 1$ and consequently $u \in M$. Therefore, (M, ρ) is $\mathcal{R}$-complete. In a similar fashion, one may prove that $\mathcal{R}$ is ρ-self-closed.

Now, consider a mapping $S : M \to M$ defined by

$$S(u(t)) = \int_0^t g(t, r, u(r))dr + \beta(t), \quad u \in C(I, \mathbb{R}).$$

Clearly, the solutions of Equation (25) are nothing but fixed points of S.

Now, for all $u, v \in M$ with $(u, v) \in \mathcal{R}$ and $t \in I$, we have

$$\begin{aligned} &S(u(t)) = \int_0^t g(t, r, u(r))dr + \beta(t) \geq 1 \\ &\Rightarrow S(u(t))S(v(t)) \geq S(u(t)) \end{aligned}$$

so that by the definition of $\mathcal{R}$, we have $(Su, Sv) \in \mathcal{R}$, *i.e.*, $\mathcal{R}$ is S-closed. By the definition of $\mathcal{R}$, it is clear that $\mathcal{R}$ is also locally transitive. Furthermore, for any $u \in M$, $(u, Su) \in \mathcal{R}$.
Next, for all $u, v \in M$ with $(u, v) \in \mathcal{R}$ and $t \in I$, consider

$$\begin{aligned} |S(u(t)) - S(v(t))| &= \left| \int_0^t (g(t, r, u(r)) - g(t, r, v(r)))dr \right| \\ &\leq \int_0^t |(g(t, r, u(r)) - g(t, r, v(r)))| \, dr \\ &\leq \int_0^t \frac{1}{h\|u - v\| + 1} (|u - v|e^{-t})e^t dr \\ &\leq \frac{1}{h\|u - v\| + 1} \int_0^t \|u - v\|e^t dr \\ &\leq \frac{\|u - v\|}{h\|u - v\| + 1} e^t. \end{aligned}$$

Thus, we obtain

$$|S(u(t)) - S(v(t))|e^{-t} \leq \frac{\|u - v\|}{h\|u - v\| + 1}, \ \forall t \in I.$$

Taking supremum over both the sides, we have

$$\|Su - Sv\| \leq \frac{\|u - v\|}{h\|u - v\| + 1},$$

$$\frac{-1}{\|Su - Sv\|} \leq \frac{-1}{\|u - v\|} - h,$$

or

$$\frac{-1}{\rho(Su, Sv)} \leq \frac{-1}{\rho(u, v)} - h.$$

Now, define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, then $\theta \in \Theta_{1,2,4}$. In addition, S satisfies Equation (13) with this θ (and $\lambda = e^{-h}, h > 0$). Therefore, all the requirements of Corollary 2 are fulfilled. Consequently, S has a fixed point. □

Next, we obtain a corresponding uniqueness result of Theorem 7 as follows.

Theorem 8. *Besides the assumptions of Theorem 7, if* $Fix(S) \subseteq \{u \in M : u(t) \geq 1, \ \forall t \in I\}$*, then the solution of the integral in Equation (25) is unique.*

Proof. Due to Theorem 7, the set $Fix(S)$ is nonempty. Now, if $Fix(S) \subseteq \{u \in M : u(t) \geq 1, \ \forall t \in I\}$, then, by the definition of $\mathcal{R}$, we have $Fix(S)$ is $\mathcal{R}^s$-connected. Hence, Theorem 6 ensures that $Fix(S)$ is a singleton set. Thus, the solution of the integral in Equation (25) is unique. This establishes our result. □

6. Conclusions

In this paper, we present two fixed point results for a special class of multi-valued mappings proposed by Baghani and Ramezani [23] via $(\theta, \mathcal{R})$-contractions employing an amorphous binary relation on metric spaces without completeness, which in turn generalize and extend the results obtained by Hançer et al. [22] in respect of underlying space, involved binary relation, and contractive condition. Some illustrative examples are also furnished to exhibit the utility of our obtained results besides deducing some relation-theoretic existence and uniqueness results for single-valued mappings. In addition, we show the applicability of our results by investigating the existence and uniqueness of a positive solution for Volterra type integral equation under some suitable conditions. For future research, we propose to study these problems in relatively larger classes of metric spaces particularly in semi-metric spaces and partial metric spaces.

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