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Derivation of Logarithmic and Logarithmic Hyperbolic Tangent Integrals Expressed in Terms of Special Functions

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Abstract: The derivation of integrals in the table of Gradshteyn and Ryzhik in terms of closed form solutions is always of interest. We evaluate several of these definite integrals of the form $\int_0^\infty \log(1 \pm e^{-\alpha y}) R(k, a, y) dy$ in terms of a special function, where $R(k, a, y)$ is a general function and k, a and α are arbitrary complex numbers, where $\operatorname{Re}(\alpha) > 0$.

Keywords: Aton Winckler; hyperbolic tangent; logarithmic function; definite integral; hankel contour; Cauchy integral; Gradshteyn and Ryzhik; Bierens de Haan

1. Introduction

We will derive integrals as indicated in the abstract in terms of special functions, which are summarized in Table 1. Some special cases of these integrals have been reported in Gradshteyn and Ryzhik [1]. In 1861, Winckler [2] derived the integrals $\int_0^\infty \frac{\log(1 \pm e^{-ay})}{a^2 + y^2} dy$ when $a = 1$ and we will consider these as well. In our case the constants in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [3]. Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \quad (1)$$

This method involves using a form of Equation (1) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Integrals Involving the Product of Logarithmic Functions

2.1. Definite Integral of the Contour Integral

We use the method in [3]. In Cauchy's integral formula we replace y by $iy + \log(a)$ and $-iy + \log(a)$ then add these two equations, followed by multiplying both sides by $\frac{1}{2} \log(1 + e^{-\alpha y})$ to get

$$\frac{1}{2k!} \left((-iy + \log(a))^k + (iy + \log(a))^k \right) \log(1 + e^{-\alpha y}) = \frac{1}{2\pi i} \int_C a^w w^{-k-1} \cos(wy) \log(1 + e^{-\alpha y}) dw \quad (2)$$

the logarithmic function is defined in Section (4.1) in [4]. We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{2k!} \int_0^\infty \frac{\log(1 + e^{-\alpha y})}{((-iy + \log(a))^k + (iy + \log(a))^k)^{-1}} dy &= \frac{1}{2\pi i} \int_0^\infty \int_C a^w w^{-k-1} \cos(wy) \log(1 + e^{-\alpha y}) dw dy \\
&= \frac{1}{2\pi i} \int_C \left(\int_0^\infty \cos(wy) \log(1 + e^{-\alpha y}) dy \right) \frac{a^w dw}{w^{k+1}} \\
&= \frac{1}{2\pi i} \int_C \left(\frac{1}{2} a^w w^{-k-3} \alpha - \frac{1}{2} a^w \pi w^{-k-2} \operatorname{csch}\left(\frac{\pi w}{\alpha}\right) \right) dw
\end{aligned} \quad (3)$$

from Equation (1.4.4.43) in [5] and the integral is valid for α , k and a complex, $-1 < \operatorname{Re}(w) < 0$, $\operatorname{Re}(\alpha) > 0$. In a similar manner we can derive the second integral formula the same as above and multiplying by $\frac{1}{2} \log(1 - e^{-\alpha y})$ we get

$$\frac{1}{2k!} \left((-iy + \log(a))^k + (iy + \log(a))^k \right) \log(1 - e^{-\alpha y}) = \frac{1}{2\pi i} \int_C a^w w^{-k-1} \cos(wy) \log(1 - e^{-\alpha y}) dw \quad (4)$$

We then take the definite integral over $y \in [0, \infty)$ of both sides to get

$$\begin{aligned}
\frac{1}{2k!} \int_0^\infty \frac{\log(1 - e^{-\alpha y})}{((-iy + \log(a))^k + (iy + \log(a))^k)^{-1}} dy &= \frac{1}{2\pi i} \int_0^\infty \int_C a^w w^{-k-1} \cos(wy) \log(1 - e^{-\alpha y}) dw dy \\
&= \frac{1}{2\pi i} \int_C \left(\int_0^\infty \cos(wy) \log(1 - e^{-\alpha y}) dy \right) \frac{a^w dw}{w^{k+1}} \\
&= \frac{1}{2\pi i} \int_C \left(\frac{1}{2} a^w w^{-k-3} \alpha - \frac{1}{2} a^w \pi w^{-k-2} \operatorname{ctnh}\left(\frac{\pi w}{\alpha}\right) \right) dw
\end{aligned} \quad (5)$$

from Equation (1.4.4.44) in [5].

2.2. Infinite Sum of the Contour Integral

Again, using the method in [3], replacing y with $\pi(2p+1)/\alpha + \log(a)$ and replacing k with $k+1$ to yield

$$\frac{(\pi(2p+1)/\alpha + \log(a))^{k+1}}{(k+1)!} = \frac{1}{2\pi i} \int_C \frac{e^{w(\pi(2p+1)/\alpha)}}{w^{k+2}} a^w dw \quad (6)$$

followed by taking the infinite sum of both sides of Equation (6) with respect to p over $[0, \infty)$ to get

$$\begin{aligned}
\frac{2^{k+1} \pi^{k+2} \alpha^{-k-1}}{(k+1)!} \zeta\left(-1-k, \frac{1}{2} + \frac{\alpha \log(a)}{2\pi}\right) &= \frac{1}{2\pi i} \sum_{p=0}^\infty \int_C \frac{e^{w(\pi(2p+1)/\alpha)}}{w^{k+2}} a^w dw \\
&= \frac{1}{2\pi i} \int_C \sum_{p=0}^\infty \frac{e^{w(\pi(2p+1)/\alpha)}}{w^{k+2}} a^w dw \\
&= -\frac{1}{4i} \int_C a^w \operatorname{csch}\left(\frac{\pi w}{\alpha}\right) w^{-k-2} dw
\end{aligned} \quad (7)$$

from (1.232.3) in [1] and $\operatorname{Im}(w) > 0$ for the convergence of the sum and if the $\operatorname{Re}(k) < 0$ then the argument of the sum over p cannot be zero for some value of p . We use (9.521.1) in [1] for the Hurwitz zeta function $\zeta(s, u)$.

Similarly, using the method in [3], replacing y with $2\pi(p+1)/\alpha + \log(a)$ and replacing k with $k+1$ to yield

$$\frac{(2\pi(p+1)/\alpha + \log(a))^{k+1}}{(k+1)!} = \frac{1}{2\pi i} \int_C \frac{e^{w(2\pi(p+1)/\alpha)}}{w^{k+2}} a^w dw \quad (8)$$

followed by taking the infinite sum of both sides of Equation (8) with respect to p over $[0, \infty)$ to get

$$\begin{aligned}
\frac{2^{k+1} \pi^{k+2} \alpha^{-k-1}}{(k+1)!} \zeta\left(-1-k, 1 + \frac{\alpha \log(a)}{2\pi}\right) &= \frac{1}{2\pi i} \sum_{p=0}^\infty \int_C \frac{e^{w(2\pi(p+1)/\alpha)}}{w^{k+2}} a^w dw \\
&= \frac{1}{2\pi i} \int_C \sum_{p=0}^\infty \frac{e^{w(2\pi(p+1)/\alpha)}}{w^{k+2}} a^w dw \\
&= -\frac{1}{4i} \int_C a^w w^{-k-2} dw - \frac{1}{4i} \int_C a^w \operatorname{ctnh}\left(\frac{\pi w}{\alpha}\right) w^{-k-2} dw
\end{aligned} \quad (9)$$

similar to the formulas in (1.232.1) in [1] where

$$\operatorname{ctnh}(x) = 1 + 2 \sum_{k=1}^{\infty} e^{-2kx} \quad (10)$$

and $\operatorname{Re}(x) > 0$. To obtain the first contour integral in the last line of Equation (3) we use the Cauchy formula by replacing y by $\log(a)$, k by $k+2$, and multiplying both sides by $\alpha/2$ and simplifying we get

$$\frac{\alpha \log^{k+2}(a)}{2(k+2)!} = \frac{1}{4\pi i} \int_C \alpha a^w w^{-3-k} dw \quad (11)$$

To obtain the first contour integral in the last line in Equation (9) we use the Cauchy formula by replacing y by $\log(a)$, k by $k+1$, and multiplying both sides by $-\pi/2$ and simplifying we get

$$-\frac{\pi \log^{k+1}(a)}{2(k+1)!} = -\frac{1}{4i} \int_C a^w w^{-2-k} dw \quad (12)$$

Since the right hand-side of Equation (3) is equal to the addition of (7) and (11), we can equate the left hand-sides and simplify using a general parameter b where $a = e^b$ to get

$$\int_0^{\infty} \frac{\log(1 + e^{-\alpha y})}{((-iy+b)^k + (iy+b)^k)^{-1}} dy = \frac{(2\pi)^{k+2} \alpha^{-k-1}}{(k+1)} \zeta\left(-1-k, \frac{1}{2} + \frac{\alpha b}{2\pi}\right) + \frac{\alpha b^{k+2}}{(k+1)(k+2)} \quad (13)$$

We can write down an equivalent formula for the corresponding Hurwitz Zeta function for the second integral using Equations (5), (9), (11) and (12),

$$\begin{aligned} \int_0^{\infty} \frac{\log(1 - e^{-\alpha y})}{((-iy+b)^k + (iy+b)^k)^{-1}} dy &= \frac{(2\pi)^{k+2} \alpha^{-k-1}}{(k+1)} \zeta\left(-1-k, 1 + \frac{\alpha b}{2\pi}\right) \\ &+ \frac{\pi b^{k+1}}{k+1} + \frac{\alpha b^{k+2}}{(k+1)(k+2)} \end{aligned} \quad (14)$$

3. Special Cases of the Definite Integrals

3.1. When α Is Replaced by $2\pi\alpha$

We take the second, mixed partial derivative of (13) with respect to b and k then set $k=0$ and $b=1$ to get

$$\begin{aligned} \int_0^{\infty} \frac{\log(1 + e^{-2\pi\alpha y})}{1+y^2} dy &= -\pi \left(\alpha + \zeta\left(0, \frac{1}{2} + \alpha\right) (\log(\alpha) + 1) + \frac{\partial^2}{\partial k \partial b} \zeta\left(-1, \frac{1}{2} + \alpha\right) \right) \\ &= \pi \left(\alpha \log(\alpha) - \frac{\partial^2}{\partial k \partial b} \zeta\left(-1, \frac{1}{2} + \alpha\right) \right) \\ &= -\pi \left(\alpha + \log \left[\frac{\Gamma\left(\frac{1}{2} + \alpha\right)}{\alpha^\alpha \sqrt{2\pi}} \right] \right) \end{aligned} \quad (15)$$

from (3.10) in [6] where $\operatorname{Re}(\alpha) > 0$. This integral is listed in Equation (4.319.2) in [1] but the result given is in error. We also note that $\frac{\partial^2}{\partial r \partial s} \zeta(r, s)$ is the second partial derivative with respect to k and a of $\zeta(r, s)$ and $\frac{\partial}{\partial s} \zeta(r, s) = -r\zeta(r+1, s)$ from (9.521.1) in [1].

3.2. When α Is Replaced by $2\pi\alpha$

We take the second, mixed partial derivative of (14) with respect to b and k then set $k = 0$ and $b = 1$ to get

$$\begin{aligned}\int_0^\infty \frac{\log(1 - e^{-2\pi\alpha y})}{1 + y^2} dy &= -\pi \left(\alpha + \zeta(0, 1 + \alpha) (\log(\alpha) + 1) + \frac{\partial^2}{\partial k \partial b} \zeta(-1, 1 + \alpha) \right) \\ &= -\frac{\pi}{2} \left(2\alpha + \log \left[\frac{\Gamma^2(1 + \alpha)}{2\pi\alpha^{2\alpha+1}} \right] \right)\end{aligned}\quad (16)$$

from (3.10) in [6] where $\operatorname{Re}(\alpha) > 0$. This integral is listed in Equation (4.319.1) in [1] but the result given is in error.

3.3. When $\alpha = \pi$ and $b = 1$

We subtract Equation (13) from (14) then take the first derivative of this difference with respect to k then set $k = 0$ to get

$$\begin{aligned}\int_0^\infty \log(1 + y^2) \log \left(\tanh \left(\frac{\pi y}{2} \right) \right) dy &= \frac{\pi}{2} + \frac{3}{2} \pi \log(\pi) - \frac{3}{2} \pi \log(2\pi) + 4\pi \zeta'(-1, 1) - 4\pi \zeta' \left(-1, \frac{3}{2} \right) \\ &= \pi \left(1 + \frac{2 \log(2)}{3} - 6 \log(A) \right)\end{aligned}\quad (17)$$

from (3) in [7,8], where $A = e^{(1/12 - \zeta'(-1))}$ and A is the Glaisher-Kinkelin constant. The value of $\zeta(-1)$ is given in [9].

3.4. When $\alpha = \pi/2$ and $b = 1$

We subtract Equation (13) from (14) then take the first derivative of this difference with respect to k then set $k = 0$ to get

$$\begin{aligned}\int_0^\infty \log(1 + y^2) \log \left(\tanh \left(\frac{\pi y}{4} \right) \right) dy &= \pi \left(-1 - 2\zeta \left(-1, \frac{3}{4} \right) (8 \log(2) - 4) + 2\zeta \left(-1, \frac{5}{4} \right) (8 \log(2) - 4) \right) \\ &\quad + \pi \left(8\zeta' \left(-1, \frac{3}{4} \right) - 8\zeta' \left(-1, \frac{5}{4} \right) \right) \\ &= \pi - 4C\end{aligned}\quad (18)$$

from (1.16) in [6], from (3) in [7,8], where C is Catalan's constant.

3.5. An Integral Involving the Logarithmic Hyperbolic Tangent and Quadratic Denominator

We subtract Equation (13) from (14) then we take the second, mixed partial derivative with respect to b and k then set $k = 0$ then simplify to get

$$\begin{aligned}\int_0^\infty \frac{\log \left(\tanh \left(\frac{\alpha y}{2} \right) \right)}{b^2 + y^2} dy &= \frac{\pi}{2b} \left(\log(b) - 2\zeta \left(0, \frac{\pi + b\alpha}{2\pi} \right) \left(\log \left(\frac{2\pi}{\alpha} \right) - 1 \right) \right) \\ &\quad + \frac{\pi}{2b} \left(2\zeta \left(0, 1 + \frac{b\alpha}{2\pi} \right) \left(\log \left(\frac{2\pi}{\alpha} \right) - 1 \right) + \right) \\ &\quad + \frac{\pi}{2a} \left(2 \frac{\partial^2}{\partial k \partial b} \zeta \left(-1, \frac{\pi + a\alpha}{2\pi} \right) - 2 \frac{\partial^2}{\partial k \partial b} \zeta \left(-1, 1 + \frac{b\alpha}{2\pi} \right) \right) \\ &= \frac{\pi}{2b} \left(1 + \log(b) - \log \left(\frac{2\pi}{\alpha} \right) + 2 \frac{\partial^2}{\partial k \partial b} \zeta \left(-1, \frac{\pi + a\alpha}{2\pi} \right) - 2 \frac{\partial^2}{\partial k \partial b} \zeta \left(-1, 1 + \frac{a\alpha}{2\pi} \right) \right) \\ &= \frac{\pi}{2b} \left(\log \left[\frac{b\alpha}{2\pi} \left(\frac{\Gamma(\frac{\pi + b\alpha}{2\pi})}{\Gamma(1 + \frac{b\alpha}{2\pi})} \right)^2 \right] \right)\end{aligned}\quad (19)$$

from (3.10) in [6], where $\operatorname{Re}(b) > 0$, $\operatorname{Re}(\alpha) > 0$ and this equation is a general case for Equation (2.6.39.21) in [10].

3.6. When $k = -3$, $b = 1$ and $\alpha = \frac{\pi}{2}$

We subtract Equation (13) from (14) then simplify to get

$$\begin{aligned} \int_0^\infty \frac{1-3y^2}{(1+y^2)^3} \log\left(\tanh\left(\frac{\pi y}{4}\right)\right) dy &= \frac{1}{2} \left(-\frac{\pi}{2} + \frac{\pi}{16}(\pi^2 - 8C) - \frac{\pi}{16} \zeta\left(2, \frac{5}{4}\right) \right) \\ &= \frac{\pi}{4}(1 - 2C) \end{aligned} \quad (20)$$

from (12.11.17) in [11] and C is Catalan's constant.

4. Generalizations and Table of Integrals

In this section we summarized the integrals evaluated in this work in the form of a table (see Table 1).

Table 1. Table of Integrals.

$Q(y)$	$\int_0^\infty Q(y) dy$
$\frac{\log(1+e^{-2\pi\alpha y})}{1+y^2}$	$-\pi \left(\alpha + \log \left[\frac{\Gamma(\frac{1}{2}+\alpha)}{\alpha^{\frac{1}{2}} \sqrt{2\pi}} \right] \right)$
$\frac{\log(1-e^{-2\pi\alpha y})}{1+y^2}$	$-\frac{\pi}{2} \left(2\alpha + \log \left[\frac{\Gamma^2(1+\alpha)}{2\pi\alpha^{2\alpha+1}} \right] \right)$
$\log(1+y^2) \log\left(\tanh\left(\frac{\pi y}{2}\right)\right)$	$\pi \left(1 + \frac{2\log(2)}{3} - 6\log(A) \right)$
$\log(1+y^2) \log\left(\tanh\left(\frac{\pi y}{4}\right)\right)$	$\pi - 4C$
$\frac{\log\left(\tanh\left(\frac{ay}{2}\right)\right)}{b^2+y^2}$	$\frac{\pi}{2b} \left(\log \left[\frac{b\alpha}{2\pi} \left(\frac{\Gamma(\frac{\pi+b\alpha}{2\pi})}{\Gamma(1+\frac{b\alpha}{2\pi})} \right)^2 \right] \right)$
$\frac{1-3y^2}{(1+y^2)^3} \log\left(\tanh\left(\frac{\pi y}{4}\right)\right)$	$\frac{\pi}{4}(1 - 2C)$

5. Summary

In this article we derived some interesting definite integrals derived by famous mathematicians, Aton Winckler, David Bierens de Haan and Gradshteyn and Ryzhik in terms of special functions. We were able to produce a closed form solution for an integrals tabled in Bierens de Haan [12] and Gradshteyn and Ryzhik [1] previously derived using analytical methods. We will be looking at other integrals using this contour integral method for future work.

In this article we looked at definite integrals and in some cases expressed them in terms of the logarithm of the gamma function $\log(\Gamma(z))$. The logarithm of the gamma function is used in discrete mathematics, number theory and other fields of science.

The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram. We noted in some cases the closed form solutions in [1,2] are in error.

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