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Existence and Multiplicity Results for Nonlocal Boundary Value Problems with Strong Singularity

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Abstract: In this paper, we study singular φ-Laplacian nonlocal boundary value problems with a nonlinearity which does not satisfy the L^1-Carathéodory condition. The existence, nonexistence and/or multiplicity results of positive solutions are established under two different asymptotic behaviors of the nonlinearity at ∞.

Keywords: multiplicity of positive solutions; sup-multiplicative-like function; singular weight; nonlocal boundary conditions

1. Introduction

Consider the following singular φ-Laplacian boundary value problem (BVP)

$$(w(t)\varphi(u'(t)))' + \lambda h(t) f(t, u(t)) = \{0, \quad t \in (0,1), \tag{1}$$

$$u(0) = \int_0^1 u(r) d\alpha_1(r), u(1) = \int_0^1 u(r) d\alpha_2(r), \tag{2}$$

where $w \in C([0,1], (0,\infty))$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $\lambda \in [0,\infty)$ is a parameter, $h \in C((0,1),(0,\infty))$ and $f \in C([0,1] \times (0,\infty), \mathbb{R})$.

Throughout this paper, the following hypotheses are assumed, unless otherwise stated.

$(A1)$ there exist increasing homeomorphisms $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that

$$\varphi(x)\psi_1(y) \le \varphi(yx) \le \varphi(x)\psi_2(y) \text{ for all } x, y \in [0,\infty). \tag{3}$$

$(A2)$ For $i = 1,2$, α_i is monotone increasing on $[0,1]$ satisfying

$$\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0,1).$$

All integrals in (2) are meant in the sense of Riemann-Stieljes. By a solution u to BVP (1) and (2), we mean $u \in C^1(0,1) \cap C[0,1]$ with $w\varphi(u') \in C^1(0,1)$ satisfies the Equation (1) and the boundary conditions (2).

The condition $(A1)$ on the odd increasing homeomorphism φ was first introduced by Wang in [1] where the existence, nonexistence and/or multiplicity of positive solutions to quasilinear elliptic equations were studied. Later on, the condition $(A1)$ was weakened by some researchers. For example, Karakostas ([2,3]) introduced a sup-multiplicative-like function as an odd increasing homeomorphism φ satisfies the following condition.

(F_1) There exists an increasing homeomorphism $\psi_1 : [0,\infty) \to [0,\infty)$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \text{ for all } x, y \in [0,\infty). \tag{4}$$

The author investigated several sufficient conditions for the existence of positive solutions to the one dimensional φ-Laplacian equation with deviated arguments. Any function of the form

$$\varphi(s) = \sum_{k=1}^{n} c_k |s|^{p_k - 2} s$$

is sup-multiplicative-like, where $c_k \geq 0$ and $p_k \in (1, \infty)$ for $1 \leq k \leq n$ and $c_1 c_n > 0$ for some $n \in \mathbb{N}$ (see, e.g., [2,4]). Lee and Xu ([5,6]) generalized the condition $(A1)$ to the one with ψ_2 is a function not requiring that $\psi_2(0) = 0$ and studied the existence of positive solutions to singularly weighted nonlinear systems. In [7], it was pointed out that the condition $(A1)$ is equivalent to the one (F_1). Consequently, the condition $(A1)$ is equivalent to those in [2,3,5,6].

Due to a wide range of applications in mathematics and physics (see, e.g., [8–14]), p-Laplacian or more generalized Laplacian problems have been extensively studied. For example, when $\varphi(s) = |s|^{p-2}s$ for some $p \in (1, \infty)$, $w \equiv 1$ and $h \in \mathcal{H}_\varphi$, Agarwal, Lü and O'Regan [15] investigated the existence and multiplicity of positive solutions to BVP (1) and (2) with $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$ under various assumptions on the nonlinearity $f = f(t, u)$ at $u = 0$ and ∞. When $\varphi(s) = s$, $w \equiv 1$ and $\lambda = 1$, Webb and Infante [16] considered problem (1) with various nonlocal boundary conditions involving a Stieltjes integral with a signed measure and gave several sufficient conditions on the nonlinearity $f = f(t, u)$ for the existence and multiplicity of positive solutions via fixed point index theory. When $\varphi(s) = |s|^{p-2}s$ for some $p \in (1, \infty)$, $w \equiv 1$, $\lambda = 1$ and $h \in \mathcal{H}_\varphi$, Kim [17] investigated sufficient conditions on the nonlinearity $f = f(t, u)$ for the existence and multiplicity of positive solutions to problem (1) with multi-point boundary conditions.

Xu, Qin and Li [18] studied the following three-point boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda g(u(t)) + k(u(t)) = 0, \;\; t \in (0,1), \\ u(0) = 0, \; u(1) = u(\eta), \end{cases} \tag{5}$$

where $p > 1$, $\varphi_p(s) = |s|^{p-2}s$, $\eta \in (0,1)$ and $g, h \in C([0, \infty), [0, \infty))$ are strictly increasing. Under the suitable assumptions on g and k such that g is p-sublinear at 0 and k is p-superlinear at ∞, the exact number of pseudo-symmetric positive solutions to problem (5) was studied.

Recently, Son and Wang [19] considered the following p-Laplacian system with nonlinear boundary conditions

$$\begin{cases} (\varphi_p(u_i'))' + \lambda h_i(t) f_i(u_j) = 0, \;\; t \in (0,1), \\ u_i(0) = 0 = a_i u_i'(1) + c_i(\lambda, u_j(1), u_i(1)) u_i(1), \end{cases} \tag{6}$$

where $i, j \in \{1, 2\}$, $i \neq j$, $\varphi_p(s) = |s|^{p-2}s$ for some $p \in (1, \infty)$, $c_i = c_i(\lambda, r, s)s$ is nondecreasing for $s \in (0, \infty)$ and $f_i \in C((0, \infty), \mathbb{R})$. Under several assumptions on h_i and f_i, the existence and multiplicity of positive solutions to problem (6) were shown.

Bachouche, Djebali and Moussaoui [20] considered the following φ-Laplacian problem with nonlocal boundary conditions involving bounded linear operators L_0, L_1

$$\begin{cases} (\varphi(u'))' + \lambda f(t, u, u') = 0, \;\; t \in (0,1), \\ u(0) = L_0(u), \; u(1) = L_1(u). \end{cases} \tag{7}$$

Here φ satisfies the following inequality

$$\varphi(sx) \leq \varphi(s)\varphi(x) \text{ for all } s, x \in [0, \infty)$$

and the nonlinearity $f = f(t, u, v)$ satisfies L^1-Carathéodory condition. The authors showed the existence of a positive solution or a nonnegative solution to problem (7).

For more general φ which does not satisfy $(A1)$, Kaufmann and Milne [21] considered BVP (1) and (2) with $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$, $0 \le h \in L^1(0,1)$ with $h \not\equiv 0$ and $f = f(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and showed the existence of a positive solution for all $\lambda > 0$ under the assumptions on the nonlinearity f which induce the sublinear nonlinearity provided $\varphi(s) = |s|^{p-1}s$ with $p > 1$. Recently, for an odd increasing homeomorphism φ satisfying $(A1)$, Kim and Jeong [4] studied various existence results for positive solutions to BVP (1) and (2) with $\lambda = 1$. For other interesting results, we refer the reader to [22–47] and the references therein.

Let $\xi : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism. Then we denote by $\mathcal{H}_\xi$ the set

$$\left\{ g \in C((0,1),(0,\infty)) : \int_0^1 \xi^{-1}\left(\left| \int_s^{\frac{1}{2}} g(\tau) d\tau \right| \right) ds < \infty \right\}.$$

It is well known that

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{8}$$

and

$$L^1(0,1) \cap C(0,1) \subseteq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$$

(see, e.g., ([7], Remark 1)).

Recall that we say that $g : (0,1) \times [0,\infty) \to \mathbb{R}$ satisfies L^1-Carathéodory condition if

(i) $g(\cdot, u)$ is measurable for all $u \in [0,\infty)$;
(ii) $g(t, \cdot)$ is continuous for almost all $u \in [0,\infty)$;
(iii) for every $r > 0$, there exists $h_r \in L^1(0,1)$ such that

$$|g(t,u)| \le h_r(t) \text{ for a.e. } t \in (0,1) \text{ and all } u \in [0,r].$$

Throughout this paper, we assume $h \in \mathcal{H}_\varphi$. Since there may be a function $h \in \mathcal{H}_\varphi \setminus L^1(0,1)$ (see, e.g., Remark 2 below), the nonlinearity $h(t)f(t,u)$ in the equation (1) may not satisfy the L^1-Carathéodory condition. Consequently, the solution space should be taken as $C[0,1]$, since the solutions to BVP (1) and (2) may not be in $C^1[0,1]$ unlike References [20–22] where the nonlinearity satisfies the L^1-Carathéodory condition. The lack of solution regularity and the boundary conditions (2) make it difficult to get the desired result.

The rest of this article is organized as follows. In Section 2, we give some preliminaries which are crucial for proving the main results in this paper. In Section 3, the main results (Theorems 2–4) are proved and some examples which illustrate the main results are given. Finally, the summary of this paper is given in Section 4.

2. Preliminaries

Throughout this section, we assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ and $h \in \mathcal{H}_\varphi$ hold. The usual maximum norm in a Banach space $C[0,1]$ of continuous functions on $[0,1]$ is denoted by

$$\|u\|_\infty := \max_{t \in [0,1]} |u(t)| \text{ for } u \in C[0,1],$$

and let

$$\mathcal{K} := \{ u \in \mathcal{P} : u(t) \ge \rho_w \|u\|_\infty \text{ for } t \in [\alpha, \beta] \text{ and } u \text{ satisfies (2) } \}$$

be a cone in $C[0,1]$. Here, $\mathcal{P} := C([0,1],[0,\infty))$, α and β are any fixed constants satisfying $0 < \alpha < \beta < 1$, $w_0 := \min_{t \in [0,1]} w(t) > 0$ and

$$\rho_w := \min\{\alpha, 1-\beta\}\psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{w_0}\right)\right]^{-1} \in (0,1].$$

For $r > 0$, let

$$\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty < r\}, \partial\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty = r\}$$

and

$$\overline{\mathcal{K}}_r := \mathcal{K}_r \cup \partial\mathcal{K}_r.$$

Now, we introduce a solution operator related to BVP (1) and (2). Let $(\lambda, u) \in (0,\infty) \times \mathcal{K}$ be given. Define functions $\nu^1_{\lambda,u}, \nu^2_{\lambda,u} : (0,1) \to (-\infty, \infty)$ by, for $x \in (0,1)$,

$$\nu^1_{\lambda,u}(x) = A_1 \int_0^1 \int_0^r I_{\lambda,u}(s,x)dsd\alpha_1(r) + \int_0^x I_{\lambda,u}(s,x)ds$$

and

$$\nu^2_{\lambda,u}(x) = -A_2 \int_0^1 \int_r^1 I_{\lambda,u}(s,x)dsd\alpha_2(r) - \int_x^1 I_{\lambda,u}(s,x)ds.$$

Here

$$A_i := (1-\hat{\alpha}_i)^{-1} \in [1,\infty) \text{ for } i = 1,2$$

and

$$I_{\lambda,u}(s,x) = \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_s^x h(\tau)f(\tau,u(\tau))d\tau\right).$$

Remark 1. *We give the properties of $I_{\lambda,u}$ for any given $(\lambda,u) \in (0,\infty) \times \mathcal{K}$ as follows.*

(1) $I_{\lambda,u}(x,y) > 0$ *and* $I_{\lambda,u}(y,x) < 0$ *for any x, y satisfying* $0 < x < y < 1$.
(2) $I_{\lambda,u}(s,x_1) < I_{\lambda,u}(s,x_2)$ *for any $s \in (0,1)$ and* $0 < x_1 < x_2 < 1$.
(3) *Let $x \in (0,1)$ be given. Then $I_{\lambda,u}(\cdot,x) \in L^1(0,1)$. Moreover, for any $\epsilon \in [0, \min\{x, 1-x\})$, there exists $C^* = C^*(x,\epsilon,\lambda,u) > 0$ satisfying*

$$\int_0^1 |I_{\lambda,u}(s,x)|\, ds \le C^*. \tag{9}$$

Indeed, by (8),

$$\begin{aligned}
&\int_0^1 |I_{\lambda,u}(s,x)|\, ds \\
=\ & \int_0^x |I_{\lambda,u}(s,x)|\, ds + \int_x^1 |I_{\lambda,u}(s,x)|\, ds \\
=\ & \int_0^x \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_s^x h(\tau)f(\tau,u(\tau))d\tau\right) ds + \int_x^1 \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_x^s h(\tau)f(\tau,u(\tau))d\tau\right) ds \\
\le\ & \int_0^x \varphi^{-1}\left(\frac{\lambda M_u}{w_0}\int_s^x h(\tau)d\tau\right) ds + \int_x^1 \varphi^{-1}\left(\frac{\lambda M_u}{w_0}\int_x^s h(\tau)d\tau\right) ds \\
\le\ & \psi_1^{-1}\left(\frac{\lambda M_u}{w_0}\right)\left[\int_0^{x+\epsilon} \varphi^{-1}\left(\int_s^{x+\epsilon} h(\tau)d\tau\right) ds + \int_{x-\epsilon}^1 \varphi^{-1}\left(\int_{x-\epsilon}^s h(\tau)d\tau\right) ds\right] =: C^*.
\end{aligned}$$

Here

$$M_u := \max\{f(x,u(x)) : x \in [0,1]\} > 0.$$

The following lemmas (Lemmas 1–3) can be proved by the similar arguments in [4] (Section 2) and [39] (Section 2). For the sake of completeness, we give the proofs of them.

Lemma 1. *Assume that* $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ *and* $h \in \mathcal{H}_\varphi$ *hold, and let* $(\lambda, u) \in (0,\infty) \times \mathcal{K}$ *be given. Then there exists a unique point* $\sigma = \sigma(\lambda, u) \in (0,1)$ *satisfying*

$$v^1_{\lambda,u}(\sigma) = v^2_{\lambda,u}(\sigma).$$

Proof. From Remark 1, it follows that $v^1_{\lambda,u}$ is a strictly increasing continuous function on $(0,1)$ and $v^2_{\lambda,u}$ is a strictly decreasing continuous function on $(0,1)$.

Next, we prove

$$\lim_{x \to 0^+} v^1_{\lambda,u}(x) \in [-\infty, 0].$$

In order to show it, we rewrite $v^1_{\lambda,u}(x)$ by, for $x \in (0,1)$,

$$\begin{aligned} v^1_{\lambda,u}(x) &= A_1 \left(\int_0^1 \int_0^r I_{\lambda,u}(s,x) ds d\alpha_1(r) + \left(1 - \int_0^1 d\alpha_1(r)\right) \int_0^x I_{\lambda,u}(s,x) ds \right) \\ &= A_1 \left(\int_0^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) + \int_0^x I_{\lambda,u}(s,x) ds \right). \end{aligned}$$

For any $x \in (0,1)$, by Remark 1 (1),

$$\begin{aligned} &\int_0^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) \\ = &- \int_0^x \int_r^x I_{\lambda,u}(s,x) ds d\alpha_1(r) + \int_x^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) \le 0, \end{aligned}$$

which implies

$$v^1_{\lambda,u}(x) \le A_1 \int_0^x I_{\lambda,u}(s,x) ds \text{ for any } x \in (0,1). \tag{10}$$

By (8), for any $x \in (0, 1/2)$,

$$\begin{aligned} 0 \le \int_0^x I_{\lambda,u}(s,x) ds &= \int_0^x \varphi^{-1} \left(\frac{\lambda}{w(s)} \int_s^x h(\tau) f(\tau, u(\tau)) d\tau \right) \\ &\le \psi_1^{-1} \left(\frac{\lambda M_u}{w_0} \right) \int_0^x \varphi^{-1} \left(\int_s^{\frac{1}{2}} h(\tau) d\tau \right) ds, \end{aligned}$$

where

$$M_u = \max\{ f(x, u(x)) : x \in [0,1] \} > 0.$$

From $h \in \mathcal{H}_\varphi$, it follows that

$$\lim_{x \to 0^+} \int_0^x I_{\lambda,u}(s,x) ds = 0.$$

Combining this and (10) yields

$$\lim_{x \to 0^+} v^1_{\lambda,u}(x) \in [-\infty, 0].$$

Next we will show

$$\lim_{x \to 1^-} v^1_{\lambda,u}(x) \in (0, \infty].$$

For any $x \in (0,1)$,

$$\begin{aligned} \nu^1_{\lambda,u}(x) = A_1 \quad \Big[\quad & \int_0^x \int_0^r I_{\lambda,u}(s,x) ds d\alpha_1(r) \\ & + \int_x^1 \int_0^x I_{\lambda,u}(s,x) ds d\alpha_1(r) + \int_x^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) \Big] \\ & + \int_0^x I_{\lambda,u}(s,x) ds. \end{aligned}$$

From

$$\lambda h(\tau) f(\tau, u(\tau)) > 0 \text{ for any } \tau \in (0,1),$$

it follows that

$$\lim_{x \to 1^-} \int_0^x I_{\lambda,u}(s,x) ds = \lim_{x \to 1^-} \int_0^x \varphi^{-1} \left(\frac{\lambda}{w(s)} \int_s^x h(\tau) f(\tau, u(\tau)) d\tau \right) \in (0, \infty].$$

For any $x \in (0,1)$,

$$\int_0^x \int_0^r I_{\lambda,u}(s,x) ds d\alpha_1(r) + \int_x^1 \int_0^x I_{\lambda,u}(s,x) ds d\alpha_1(r) > 0.$$

For any $x > 1/2$, by (8),

$$\begin{aligned} \left| \int_x^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) \right| &= \int_x^1 \int_x^r \varphi^{-1} \left(\frac{\lambda}{w(s)} \int_x^s h(\tau) f(\tau, u(\tau)) d\tau \right) ds d\alpha_1(r) \\ &\leq \psi_1^{-1} \left(\frac{\lambda M_u}{w_0} \right) \int_x^1 \int_x^1 \varphi^{-1} \left(\int_{\frac{1}{2}}^s h(\tau) d\tau \right) ds d\alpha_1(r) \\ &\leq \psi_1^{-1} \left(\frac{\lambda M_u}{w_0} \right) \int_x^1 d\alpha_1(r) \int_x^1 \varphi^{-1} \left(\int_{\frac{1}{2}}^s h(\tau) d\tau \right) ds. \end{aligned}$$

Combining this and the fact $h \in \mathcal{H}_\varphi$ yields

$$\lim_{x \to 1^-} \int_x^1 \int_x^r I_{\lambda,u}(s,x) ds d\alpha_1(r) = 0.$$

Consequently

$$\lim_{x \to 1^-} \nu^1_{\lambda,u}(x) \in (0, \infty].$$

Similarly, it can be shown that

$$\lim_{x \to 0^+} \nu^2_{\lambda,u}(x) \in (0, \infty] \text{ and } \lim_{x \to 1^-} \nu^2_{\lambda,u}(x) \in [-\infty, 0].$$

Thus, by continuity and strict monotonicity of $\nu^1_{\lambda,u}$ and $\nu^2_{\lambda,u}$, there exists a unique point $\sigma \in (0,1)$ satisfying

$$\nu^1_{\lambda,u}(\sigma) = \nu^2_{\lambda,u}(\sigma).$$

□

Define an operator $T : [0,\infty) \times \mathcal{K} \to C[0,1]$ by

$$T(0,u) = 0 \text{ for } u \in \mathcal{K},$$

and for $(\lambda, u) \in (0,\infty) \times \mathcal{K}$,

$$T(\lambda,u)(t) = \begin{cases} A_1 \int_0^1 \int_0^r I_{\lambda,u}(s,\sigma) ds d\alpha_1(r) + \int_0^t I_{\lambda,u}(s,\sigma) ds, & \text{if } 0 \leq t \leq \sigma, \\ -A_2 \int_0^1 \int_r^1 I_{\lambda,u}(s,\sigma) ds d\alpha_2(r) - \int_t^1 I_{\lambda,u}(s,\sigma) ds, & \text{if } \sigma \leq t \leq 1, \end{cases} \tag{11}$$

where $\sigma = \sigma(\lambda, u)$ is the unique point satisfying $v^1_{\lambda,u}(\sigma) = v^2_{\lambda,u}(\sigma)$ in Lemma 1. By the definition of $\sigma = \sigma(\lambda, u)$, T is well defined and

$$T(\lambda, u)(\sigma) = v^1_{\lambda,u}(\sigma).$$

Moreover, $T(\lambda, u)$ is strictly increasing on $[0, \sigma)$ and is strictly decreasing on $(\sigma, 1]$.

Lemma 2. *Assume that* $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ *and* $h \in \mathcal{H}_\varphi$ *hold. Then*

$$T(\lambda, u) \in \mathcal{K} \text{ for any } (\lambda, u) \in [0,\infty) \times \mathcal{K}$$

and

$$T(\lambda, u)(\sigma) = \|T(\lambda, u)\|_\infty > 0 \text{ for any } (\lambda, u) \in (0,\infty) \times \mathcal{K}.$$

Moreover, u is a positive solution to BVP (1) *and* (2) *if and only if* $T(\lambda, u) = u$ *for some* $(\lambda, u) \in (0,\infty) \times \mathcal{K}$.

Proof. First, we show that

$$T(\lambda, u) \in \mathcal{K} \text{ for any } (\lambda, u) \in [0,\infty) \times \mathcal{K}.$$

Clearly,

$$T(0, u) = 0 \in \mathcal{K} \text{ for any } u \in \mathcal{K}.$$

Let $(\lambda, u) \in (0,\infty) \times \mathcal{K}$ be given. Then, by (11),

$$(T(\lambda, u))'(s) = I_{\lambda,u}(s, \sigma) \text{ for } s \in (0,1),$$

which implies, for $r \in [0,1]$,

$$T(\lambda, u)(r) = T(\lambda, u)(0) + \int_0^r I_{\lambda,u}(s, \sigma)ds. \tag{12}$$

Since

$$\begin{aligned} T(\lambda, u)(0) &= A_1 \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r) \\ &= \frac{1}{1-\hat{\alpha}_1} \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r), \end{aligned}$$

integrating (12) from 0 to 1,

$$\begin{aligned} \int_0^1 T(\lambda, u)(r)d\alpha_1(r) &= \int_0^1 T(\lambda, u)(0)d\alpha_1(r) + \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r) \\ &= \hat{\alpha}_1 T(\lambda, u)(0) + (1-\hat{\alpha}_1)T(\lambda, u)(0) \\ &= T(\lambda, u)(0). \end{aligned}$$

Similarly, it can be shown that

$$T(\lambda, u)(1) = \int_0^1 T(\lambda, u)(r)d\alpha_2(r).$$

Thus $T(\lambda, u)$ satisfies the boundary conditions (2). Since $T(\lambda, u)$ is strictly increasing on $[0, \sigma)$ and is strictly decreasing on $(\sigma, 1]$,

$$T(\lambda, u)(t) \geq \min\{T(\lambda, u)(0), T(\lambda, u)(1)\} \text{ for } t \in [0,1].$$

We only consider the case

$$\min\{T(\lambda, u)(t) : 0 \leq t \leq 1\} = T(\lambda, u)(0),$$

since the case

$$\min\{T(\lambda,u)(t) : 0 \le t \le 1\} = T(\lambda,u)(1)$$

is similar. Then

$$T(\lambda,u)(0) = \int_0^1 T(\lambda,u)(r)d\alpha_1(r) \ge \hat{\alpha}_1 T(\lambda,u)(0),$$

which implies

$$T(\lambda,u)(0) \ge 0,$$

since

$$\hat{\alpha}_1 = \int_0^1 d\alpha_1(r) \in [0,1).$$

Consequently,

$$T(\lambda,u)(t) \ge 0 \text{ for all } t \in [0,1], \text{ i.e., } T(\lambda,u) \in \mathcal{P}.$$

Clearly

$$T(\lambda,u)(\sigma) = \|T(\lambda,u)\|_\infty > 0,$$

since $T(\lambda,u)$ is strictly increasing on $[0,\sigma)$ and is strictly decreasing on $(\sigma,1]$.

For $t \in [0,\sigma]$, by (8),

$$\begin{aligned} T(\lambda,u)(t) &= T(\lambda,u)(0) + \int_0^t \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_s^\sigma h(\tau)f(\tau,u(\tau))d\tau\right)ds \\ &\ge T(\lambda,u)(0) + \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)q_1(t). \end{aligned} \quad (13)$$

Here

$$q_1(t) := \int_0^t \varphi^{-1}\left(\lambda\int_s^\sigma h(\tau)f(\tau,u(\tau))d\tau\right)ds \text{ for } t \in [0,\sigma].$$

Similarly,

$$\|T(\lambda,u)\|_\infty = T(\lambda,u)(\sigma) \le T(\lambda,u)(0) + \psi_1^{-1}\left(\frac{1}{w_0}\right)q_1(\sigma). \quad (14)$$

Since

$$q_1'(t) = \varphi^{-1}\left(\lambda\int_t^\sigma h(\tau)f(\tau,u(\tau))d\tau\right) > 0 \text{ for } t \in (0,\sigma),$$

q_1' is a strictly decreasing function on $(0,\sigma]$. Consequently, q_1 is a strictly increasing concave function on $[0,\sigma]$ with $q_1(0) = 0$, so that

$$q_1(t) \ge tq_1(\sigma) \text{ for } t \in [0,\sigma].$$

Consequently, by (13) and (14),

$$\begin{aligned} T(\lambda,u)(t) - T(\lambda,u)(0) &\ge \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)q_1(t) \\ &\ge t\psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)q_1(\sigma) \\ &\ge t\rho_1(\|T(\lambda,u)\|_\infty - T(\lambda,u)(0)), \end{aligned}$$

where

$$\rho_1 := \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{w_0}\right)\right]^{-1} \in (0,1].$$

Consequently, for $t \in [0, \sigma]$,

$$\begin{aligned} T(\lambda, u)(t) &\geq \rho_1 t \|T(\lambda, u)\|_\infty + (1 - \rho_1 t) T(\lambda, u)(0) \\ &\geq \rho_1 t \|T(\lambda, u)\|_\infty. \end{aligned}$$

Similarly, it can be shown that

$$T(\lambda, u)(t) \geq \rho_1 (1 - t) \|T(\lambda, u)\|_\infty \text{ for } t \in [\sigma, 1].$$

Then

$$T(\lambda, u)(t) \geq \rho_1 \min\{t, 1 - t\} \|T(\lambda, u)\|_\infty \text{ for } t \in [0, 1], \tag{15}$$

and consequently

$$T(\lambda, u)(t) \geq \rho_w \|T(\lambda, u)\|_\infty \text{ for } t \in [\alpha, \beta],$$

i.e.,

$$T(\lambda, u) \in \mathcal{K}.$$

Assume that

$$T(\lambda, u) = u \text{ for some } (\lambda, u) \in (0, \infty) \times \mathcal{K}.$$

From direct differentiation and the definition of $\mathcal{K}$, it follows that u is a nonnegative solution to BVP (1) and (2). Since $\lambda > 0$, $T(\lambda, u) \neq 0$, and by (15),

$$u(t) = T(\lambda, u)(t) > 0 \text{ for } t \in (0, 1).$$

Consequently, u is a positive solution to BVP (1) and (2) with $\lambda > 0$.

Let u_λ be a positive solution to BVP (1) and (2). Then

$$0 \leq u_\lambda(0) < \|u_\lambda\|_\infty.$$

Indeed, assume on the contrary that $u_\lambda(0) = \|u_\lambda\|_\infty > 0$. Since

$$0 \leq u_\lambda(0) = \int_0^1 u_\lambda(r) d\alpha_1(r) \leq \hat{\alpha}_1 \|u_\lambda\|_\infty = \hat{\alpha}_1 u_\lambda(0).$$

Then $\|u_\lambda\|_\infty = u_\lambda(0) = 0$, which contradicts the fact that u_λ is a positive solution to BVP (1) and (2). Similarly, it can be shown that

$$0 \leq u_\lambda(1) < \|u_\lambda\|_\infty.$$

Consequently, there exists a point $\sigma_\lambda \in (0, 1)$ satisfying

$$\|u_\lambda\|_\infty = u_\lambda(\sigma_\lambda).$$

Integrating the Equation (1) with $u = u_\lambda$ yields

$$u_\lambda(r) = u_\lambda(0) + \int_0^r I_{\lambda, u_\lambda}(s, \sigma_\lambda) ds = u_\lambda(1) - \int_r^1 I_{\lambda, u_\lambda}(s, \sigma) ds \text{ for } r \in [0, 1].$$

By boundary conditions (2) with $u = u_\lambda$,

$$u_\lambda(0) = A_1 \int_0^1 \int_0^r I^1_{\lambda, u_\lambda}(s, \sigma_\lambda) ds d\alpha_1(r)$$

and

$$u_\lambda(1) = -A_2 \int_0^1 \int_r^1 I^1_{\lambda, u_\lambda}(s, \sigma_\lambda) ds d\alpha_2(r).$$

Consequently

$$u_\lambda \equiv T(\lambda, u_\lambda) \in \mathcal{K}.$$

Clearly $\lambda > 0$, since

$$T(0, u) = 0 \text{ for all } u \in \mathcal{K}.$$

Thus, the proof is complete. □

Lemma 3. *Assume that* $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ *and* $h \in \mathcal{H}_\varphi$ *hold. Let* $L > 0$ *be given and let* (λ_n, u_n) *be a bounded sequence in* $(0,\infty) \times \mathcal{K}$ *with*

$$|\lambda_n| + \|u_n\|_\infty \leq L.$$

If $\lim_{n\to\infty} \sigma_n \in \{0,1\}$, *then*

$$T(\lambda_n, u_n)(\sigma_n) = \|T(\lambda_n, u_n)\|_\infty \to 0$$

and

$$\lambda_n \to 0 \text{ as } n \to \infty.$$

Here $\sigma_n = \sigma(\lambda_n, u_n)$ *is the unique point satisfying*

$$\nu^1_{\lambda_n,u_n}(\sigma_n) = \nu^2_{\lambda_n,u_n}(\sigma_n) \text{ for each } n \in \mathbb{N}.$$

Proof. We only prove the case

$$\lim_{n\to\infty} \sigma_n = 0,$$

since the case $\lim_{n\to\infty} \sigma_n = 1$ can be dealt similarly. Since there exist positive constants N_1, N_2 satisfying

$$\lambda_n N_1 \leq \lambda_n f(t,u) \leq N_2 \text{ for all } (t,u) \in [0,1] \times [0,L] \text{ and all } n,$$

by (8) and (10),

$$\begin{aligned}
\|T(\lambda_n, u_n)\|_\infty &= A_1 \int_0^1 \int_0^r I_{\lambda_n,u_n}(s,\sigma_n) ds d\alpha_1(r) + \int_0^{\sigma_n} I_{\lambda_n,u_n}(s,\sigma_n) ds \\
&\leq A_1 \int_0^{\sigma_n} I_{\lambda_n,u_n}(s,\sigma_n) ds \\
&\leq A_1 \psi_1^{-1}\left(\frac{N_2}{w_0}\right) \int_0^{\sigma_n} \varphi^{-1}\left(\int_s^{\sigma_n} h(\tau) d\tau\right) ds.
\end{aligned}$$

Then, from $h \in \mathcal{H}_\varphi$, it follows that

$$\|T(\lambda_n, u_n)\|_\infty \to 0 \text{ as } n \to \infty. \tag{16}$$

Since $T(\lambda_n, u_n)(1) \geq 0$ for all n, by (8),

$$\begin{aligned}
\|T(\lambda_n, u_n)\|_\infty &= T(\lambda_n, u_n)(\sigma_N) \\
&= T(\lambda_n, u_n)(1) - \int_{\sigma_n}^1 I_{\lambda_n,u_n}(s,\sigma_n) ds \\
&\geq -\int_{\sigma_n}^1 I_{\lambda_n,u_n}(s,\sigma_n) ds \\
&= \int_{\sigma_n}^1 \varphi^{-1}\left(\frac{\lambda_n}{w(s)} \int_{\sigma_n}^s h(\tau) f(\tau, u_n(\tau)) d\tau\right) ds \\
&\geq \psi_2^{-1}\left(\frac{\lambda_n N_1}{\|w\|_\infty}\right) \int_{\sigma_n}^1 \varphi^{-1}\left(\int_{\sigma_n}^s h(\tau) d\tau\right) ds \geq 0.
\end{aligned}$$

Since $h(t) > 0$ for all $t \in (0,1)$, by (16),

$$\lambda_n \to 0 \text{ as } n \to \infty.$$

□

Using Lemma 3 and (8), by the similar arguments in the proof of [17] (Lemma 2.4) and [48] (Lemma 3.3) , one can prove the complete continuity of the operator $T = T(\lambda, u)$. We only state the result as follows.

Lemma 4. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ and $h \in \mathcal{H}_\varphi$ hold. Then the operator $T : [0,\infty) \times \mathcal{K} \to \mathcal{K}$ is completely continuous, i.e., compact and continuous.*

We recall a well-known theorem for the existence of a global continuum of solutions by Leray and Schauder [49]:

Theorem 1. (see, e.g., [50] (Corollary 14.12)) *Let X be a Banach space with $X \neq \{0\}$ and let $\mathcal{K}$ be a cone in X. Consider*

$$x = T(\lambda, x), \tag{17}$$

where $\lambda \in [0,\infty)$ and $u \in \mathcal{K}$. If $T : [0,\infty) \times \mathcal{K} \to \mathcal{K}$ is completely continuous and $T(0,u) = 0$ for all $u \in \mathcal{K}$, there exists an unbounded solution component $\mathcal{C}$ of (17) in $[0,\infty) \times \mathcal{K}$ emanating from $(0,0)$.

Since

$$T(0,u) = 0 \text{ for all } u \in \mathcal{K},$$

by Lemmas 2–4 and Theorem 1, one has the following proposition.

Proposition 1. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$ and $h \in \mathcal{H}_\varphi$ hold. Then there exists an unbounded solution component $\mathcal{C}$ emanating from $(0,0)$ in $[0,\infty) \times \mathcal{K}$ satisfying (i) $\mathcal{C} \cap (\{0\} \times \mathcal{K}) = \{(0,0)\}$ and (ii) for any $(\lambda, u) \in \mathcal{C} \setminus \{(0,0)\}$, u is a positive solution to BVP (1) and (2) with $\lambda > 0$.*

3. Main Results

First, we give a list of hypotheses on $f = f(t,s)$ which are used in this section:

(F_0) $\displaystyle\lim_{s\to\infty} \min_{t\in[0,1]} \frac{f(t,s)}{\psi_1(s)} = 0.$

(F_0') $\displaystyle\lim_{s\to\infty} \max_{t\in[0,1]} \frac{f(t,s)}{\varphi(s)} = 0.$

(F_∞) There exists a nondegenerate interval $[\alpha,\beta] \subseteq (0,1)$ satisfying

$$\lim_{s\to\infty} \min_{t\in[\alpha,\beta]} \frac{f(t,s)}{\varphi(s)} = \infty.$$

For convenience, let

$$\gamma := \frac{\alpha+\beta}{2}.$$

Since α and β are any fixed constants in the cone $\mathcal{K}$ satisfying $0 < \alpha < \beta < 1$,

$$0 < \alpha < \gamma < \beta < 1.$$

When we need the assumption (F_∞), let α and β in the cone $\mathcal{K}$ be the same constants in the assumption (F_∞).

Lemma 5. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$, (F_∞) and $h \in \mathcal{H}_\varphi$ hold. Then there exists $\bar{\lambda} > 0$ such that BVP (1) and (2) has no positive solutions for any $\lambda > \bar{\lambda}$.*

Proof. Let u be a positive solution to BVP (1) and (2) with $\lambda > 0$ and let $\sigma \in (0,1)$ be the unique point satisfying $u(\sigma) = \|u\|_\infty$. Since $f \in C([0,1] \times [0,\infty), (0,\infty))$, by (F_∞), there exists $\hat{C} > 0$ satisfying

$$f(t,s) > \hat{C}\varphi(s) \text{ for } (t,s) \in [\alpha,\beta] \times [0,\infty).$$

We only give the proof for the case $\sigma \geq \gamma$, since the case $\sigma < \gamma$ can be dealt similarly. Then

$$u(t) \geq u(\alpha) \text{ for } t \in [\alpha,\gamma],$$

which implies

$$f(t,u(t)) > \hat{C}\varphi(u(t)) \geq \hat{C}\varphi(u(\alpha)) \text{ for } t \in [\alpha,\gamma].$$

By Lemma 2 and (8),

$$\begin{aligned}
u(\alpha) &= u(0) + \int_0^\alpha I_{\lambda,u}(s,\sigma)ds \\
&\geq \int_0^\alpha \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_s^\sigma h(\tau)f(\tau,u(\tau))d\tau\right) ds \\
&\geq \int_0^\alpha \varphi^{-1}\left(\frac{\lambda}{w(s)}\int_\alpha^\gamma h(\tau)f(\tau,u(\tau))d\tau\right) ds \\
&\geq \int_0^\alpha \varphi^{-1}\left(\int_\alpha^\gamma h(\tau)d\tau \frac{\lambda\hat{C}\varphi(u(\alpha))}{\|w\|_\infty}\right) ds \\
&\geq \int_\alpha^\gamma \varphi^{-1}\left(\int_\alpha^\gamma h(\tau)d\tau\varphi(u(\alpha))\right) ds\psi_2^{-1}(\frac{\lambda\hat{C}}{\|w\|_\infty}) \\
&\geq \int_\alpha^\gamma \varphi^{-1}\left(\int_\alpha^\gamma h(\tau)d\tau\right) ds\psi_2^{-1}\left(\frac{\lambda\hat{C}}{\|w\|_\infty}\right) u(\alpha) \\
&\geq C_h\psi_2^{-1}\left(\frac{\lambda\hat{C}}{\|w\|_\infty}\right) u(\alpha).
\end{aligned}$$

Here

$$C_h := \min\left\{\int_\alpha^\gamma \varphi^{-1}\left(\int_\alpha^\gamma h(\tau)d\tau\right) ds, \int_\gamma^\beta \varphi^{-1}\left(\int_\gamma^\beta h(\tau)d\tau\right) ds\right\} > 0.$$

Thus

$$\lambda \leq \psi_2\left(\frac{1}{C_h}\right)\frac{\|w\|_\infty}{\hat{C}} =: \bar{\lambda}.$$

□

Lemma 6. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$, (F_∞) and $h \in \mathcal{H}_\varphi$ hold. Let $I > 0$ be given. Then there exists $M_I > 0$ such that $\|u\|_\infty \leq M_I$ for any positive solutions u to BVP (1) and (2) with $\lambda \in [I,\infty)$.*

Proof. Suppose to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ satisfying u_n is a positive solutions to BVP (1) and (2) with $\lambda = \lambda_n \in [I,\infty)$ and $||u_n||_\infty \to \infty$ as $n \to \infty$.

Take

$$C^* = \frac{\|w\|_\infty\psi_2(\alpha^{-1})}{(\gamma-\alpha)Ih_0} + 1,$$

where

$$h_0 := \min\{h(t) : t \in [\alpha,\beta]\} > 0.$$

By (F_∞), there exists $K > 0$ such that

$$f(t,s) > C^*\varphi(s) \text{ for } (t,s) \in [\alpha,\beta] \times (K,\infty).$$

By Lemma 2,

$$u_n(t) \geq \rho_w \|u_n\|_\infty \text{ for } t \in [\alpha,\beta].$$

Then, for sufficiently large $N > 0$,

$$u_N(t) \geq K \text{ for } t \in [\alpha,\beta],$$

which implies

$$\lambda_N h(t) f(t,u_N(t)) \geq Ih_0 C^* \varphi(u_N(t)) \text{ for all } t \in [\alpha,\beta].$$

Let $\sigma_N \in (0,1)$ be a unique point satisfying

$$u_N(\sigma_N) = \|u_N\|_\infty.$$

We only consider the case $\sigma_N \geq \gamma$, since the case $\sigma_N < \gamma$ can be dealt in a similar manner. By (8) and the fact that

$$u_N(t) \geq u_N(\alpha) \text{ for } t \in [\alpha,\sigma_N],$$

one has

$$\begin{aligned} u_N(\alpha) &= u_N(0) + \int_0^\alpha \varphi^{-1}\left(\frac{\lambda_N}{w(s)}\int_s^{\sigma_N} h(\tau)f(\tau,u_N(\tau))d\tau\right) ds \\ &\geq \int_0^\alpha \varphi^{-1}\left(\frac{\lambda_N}{w(s)}\int_\alpha^{\gamma} h(\tau)f(\tau,u_N(\tau))d\tau\right) ds \\ &\geq \alpha\varphi^{-1}(\|w\|_\infty^{-1}(\gamma-\alpha)Ih_0C^*\varphi(u_N(\alpha))) \\ &\geq \alpha\psi_2^{-1}(\|w\|_\infty^{-1}(\gamma-\alpha)Ih_0C^*)u_N(\alpha), \end{aligned}$$

which implies

$$C^* \leq \frac{\|w\|_\infty \psi_2(\alpha^{-1})}{(\gamma-\alpha)Ih_0}.$$

However, this contradicts the choice of C^*. Thus the proof is complete. □

Theorem 2. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$, (F_∞) and $h \in \mathcal{H}_\varphi$ hold. Then there exists $\lambda_* > 0$ such that BVP (1) and (2) has at least two positive solutions u_λ^1 and u_λ^2 for $\lambda \in (0,\lambda_*)$, at least one positive solution for $\lambda = \lambda_*$ and no positive solutions for $\lambda > \lambda_*$. Moreover, for $\lambda \in (0,\lambda_*)$, two positive solutions u_λ^1 and u_λ^2 satisfy*

$$\|u_\lambda^1\|_\infty \to 0 \text{ and } \|u_\lambda^2\|_\infty \to \infty \text{ as } \lambda \to 0^+.$$

Proof. Set

$$\lambda_* := \sup\{\hat{\lambda} > 0 : \text{BVP (1) and (2) has at least two positive solution for all } \lambda \in (0,\hat{\lambda})\}.$$

Then, by Proposition 1, Lemmas 5 and 6, $\lambda_* \in (0,\infty)$ is well-defined. Indeed, let $\{(\lambda_n,u_n)\}$ be a sequence in the unbounded solution component $\mathcal{C}$ defined in Proposition 1 satisfying

$$\lambda_n + \|u_n\|_\infty \to \infty \text{ as } n \to \infty.$$

By Lemma 5,

$$\lambda_n \leq \bar{\lambda},$$

which implies

$$\|u_n\|_\infty \to \infty \text{ as } n \to \infty.$$

From Lemma 6, it follows that $\lambda_n \to 0^+$ as $n \to \infty$. Consequently, the shape of the continuum of $\mathcal{C}$ is determined, so that BVP (1) and (2) has two positive solutions u_λ^1, u_λ^2 for all small $\lambda > 0$ such that

$$\|u_\lambda^1\|_\infty \to 0 \text{ and } \|u_\lambda^2\|_\infty \to \infty \text{ as } \lambda \to 0^+.$$

By Lemma 5, there are no positive solutions to BVP (1) and (2) for all $\lambda > \bar{\lambda}$. Thus, $\lambda_* \in (0, \infty)$ is well-defined.

By the definition of λ_*, BVP (1) and (2) has at least two positive solutions for $\lambda \in (0, \lambda_*)$. Let $\{(\lambda_n, u_n)\}$ be a sequence such that

$$u_n = T(\lambda_n, u_n) \text{ for each } n \text{ and } \lambda_n \to \lambda_* \text{ as } n \to \infty.$$

By the compactness of T and Lemma 5, there exists a subsequence, say it again $\{(\lambda_n, u_n)\}$, satisfying

$$u_n = T(\lambda_n, u_n) \to u_* \text{ in } C[0,1] \text{ as } n \to \infty.$$

Since

$$(\lambda_n, u_n) \to (\lambda_*, u_*) \text{ in } [0, \infty) \times \mathcal{K},$$

from the continuity of T, it follows that

$$u_* = T(\lambda_*, u_*).$$

Thus BVP (1) and (2) has at least one positive solution for $\lambda = \lambda_*$.

To complete the proof of Theorem 2, it suffices to show that there are no positive solutions to BVP (1) and (2) for $\lambda > \lambda_*$. Assume on the contrary that there exists $\lambda_1 \in (\lambda_*, \infty)$ such that BVP (1) and (2) has a positive solution u_1 for $\lambda = \lambda_1$. We will show that there are two positive solutions to BVP (1) and (2) for all $\lambda \in (0, \lambda_1)$, which contradicts the definition of λ_*.

Let $\lambda \in (0, \lambda_1)$ be fixed and set

$$\epsilon = \frac{1}{2}\left(\frac{\lambda_1}{\lambda} - 1\right) \min_{t \in [0,1]} f(t, u_1(t)) > 0.$$

By the continuity of $f = f(t,s)$, there exists $\delta = \delta(\lambda) > 0$ such that if $x, y \in [0, \|u_1\| + 1]$ and $|x - y| < 2\delta$, then

$$|f(t,x) - f(t,y)| < \epsilon, \ t \in [0,1].$$

We claim that $\beta(t) = u_1(t) + \delta$ satisfies

$$(w(t)\varphi(\beta'(t)))' + \lambda h(t) f(t, \beta(t)) < 0, \ \ t \in (0,1). \tag{18}$$

Indeed, assume on the contrary that β does not satisfy (18), i.e., there exists $t_0 \in (0,1)$ such that

$$(w(t_0)\varphi(\beta'(t_0)))' + \lambda h(t_0) f(t_0, \beta(t_0)) \geq 0.$$

Since $\beta'(t) = u_1'(t)$ for all $t \in (0,1)$,

$$\begin{aligned} \lambda h(t_0) f(t_0, \beta(t_0)) &\geq -(w(t_0)\varphi(\beta'(t_0)))' \\ &= -(w(t_0)\varphi(u_1'(t_0)))' \\ &= \lambda_1 h(t_0) f(t_0, u_1(t_0)), \end{aligned}$$

which implies

$$f(t_0, \beta(t_0)) \geq \frac{\lambda_1}{\lambda} f(t_0, u_1(t_0)). \tag{19}$$

From

$$|\beta(t_0) - u_1(t_0)| = \delta < 2\delta,$$

it follows that

$$\epsilon + f(t_0, u_1(t_0)) > f(t_0, \beta(t_0)).$$

Consequently, by (19),

$$\epsilon \geq \left(\frac{\lambda_1}{\lambda} - 1\right) f(t_0, u_1(t_0)),$$

which contradicts the choice of ϵ. Thus, $\beta(t) = u_1(t) + \delta$ satisfies (18).

Consider the following modified problem

$$\begin{cases} (w(t)\varphi(u'(t)))' + \lambda h(t) f(t, \gamma(t, u(t))) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(r) d\alpha_1(r), u(1) = \int_0^1 u(r) d\alpha_2(r), \end{cases} \tag{20}$$

where $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by, for $t \in [0,1]$,

$$\gamma(t,s) = \begin{cases} \beta(t), & \text{if } \ s \geq \beta(t), \\ s, & \text{if } \ 0 < s < \beta(t), \\ 0, & \text{if } \ s \leq 0. \end{cases}$$

Let u be a positive solution to problem (20). We show that $u(t) \leq \beta(t)$ for $t \in [0,1]$. If not, there exists $t_0 \in [0,1]$ satisfying

$$x(t_0) = \max\{x(t) : t \in [0,1]\} > 0,$$

where

$$x(t) = u(t) - \beta(t) \text{ for } t \in [0,1].$$

If $\hat{\alpha}_1 = 0$, then $u(0) = 0 < \delta = \beta(0)$ and $x(0) < 0 < x(t_0)$. If $\hat{\alpha}_1 \in (0,1)$, then

$$\begin{aligned} x(0) &= u(0) - \beta(0) = u(0) - (u_1(0) + \delta) \\ &= \int_0^1 u(r) d\alpha_1(r) - \left(\int_0^1 u_1(r) d\alpha_1(r) + \delta\right) \\ &< \int_0^1 x(r) d\alpha_1(r) \leq \hat{\alpha}_1 x(t_0) < x(t_0). \end{aligned}$$

Similarly, $x(1) < x(t_0)$. Consequently, $t_0 \in (0,1)$ and $x'(t_0) = 0$, i.e.,

$$u'(t_0) = \beta'(t_0). \tag{21}$$

For some $t^* \in (0, t_0)$,

$$x(t^*) < x(t_0) \tag{22}$$

and $x(t) > 0$ for $t \in [t^*, t_0]$, i.e.,

$$u(t) > \beta(t), \ t \in [t^*, t_0]. \tag{23}$$

By (18) and (23), for $t \in [t^*, t_0]$,

$$\begin{aligned} -(w(t)\varphi(u'(t)))' &= \lambda h(t) f(t, \gamma(t, u(t))) \\ &= \lambda h(t) f(t, \beta(t)) \\ &< -(w(t)\varphi(\beta'(t)))'. \end{aligned}$$

Integrating this from t to t_0, by (21),

$$u'(t) \leq \beta'(t) \text{ for } t \in [t^*, t_0).$$

Integrating it again from t^* to t_0,

$$u(t_0) - u(t^*) \leq \beta(t_0) - \beta(t^*),$$

which contradicts (22). Thus

$$u(t) \leq \beta(t) \text{ for } t \in [0,1],$$

which implies

$$\gamma(t, u(t)) = u(t) \text{ for all } t \in [0,1].$$

Consequently u is a positive solution to BVP (1) and (2).

Since $\hat{\alpha}_1 \in [0,1)$, it is easy to see that $u(0) < \beta(0)$ and $u(1) < \beta(1)$. Indeed,

$$\begin{aligned} u(0) &= \int_0^1 u(r) d\alpha_1(r) \\ &\leq \int_0^1 \beta(r) d\alpha_1(r) = \int_0^1 (u_1(r) + \delta) d\alpha_1(r) = \int_0^1 u_1(r) d\alpha_1(r) + \delta\hat{\alpha}_1 \\ &< u_1(0) + \delta = \beta(0). \end{aligned}$$

Similarly, it can be shown that $u(1) < \beta(1)$.

Set

$$\Omega = \{u \in C[0,1] : \ -1 < u(t) < \beta(t), \ t \in [0,1]\}.$$

Then Ω is a bounded open subset in $C[0,1]$. We claim that $u \in \Omega \cap \mathcal{K}$. Assume on the contrary that there exist t_1, t_2 and $\delta_1 > 0$ such that

$$0 < t_1 - \delta_1 < t_1 \leq t_2 < t_2 + \delta_1 < 1,$$

$$u(t) = \beta(t) \text{ for } t \in [t_1, t_2]$$

and

$$u(t) < \beta(t) \text{ for } t \in [t_1 - \delta_1, t_1) \cup (t_2, t_2 + \delta_1].$$

Since β satisfies (18),

$$\max\{(w(t)\varphi(\beta'(t)))' + \lambda h(t) f(t, \beta(t)) : t \in [t_1 - \delta_1, t_2 + \delta_1]\} =: -\epsilon_1 < 0. \tag{24}$$

Set

$$\epsilon_2 = \frac{\epsilon_1}{\lambda h^*} > 0, \tag{25}$$

where

$$h^* := \max\{h(t) : t \in [t_1 - \delta_1, t_2 + \delta_1]\}.$$

Then there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$ and $x, y \in [0, ||\beta||_\infty + 1]$, then

$$|f(t,x) - f(t,y)| < \epsilon_2,$$

and there exists an interval $[a,b] \subset (t_1 - \delta_1, t_2 + \delta_1)$ such that

$$(u - \beta)'(a) > 0, \ (u - \beta)'(b) < 0$$

and

$$-\delta_2 < \gamma(t, u(t)) - \beta(t) = u(t) - \beta(t) \leq 0, \ t \in [a,b].$$

Consequently

$$w(a)[\varphi(u'(a)) - \varphi(\beta'(a))] > 0, w(b)[\varphi(u'(b)) - \varphi(\beta'(b))] < 0$$

and

$$f(t, \gamma(t, u(t))) = f(t, u(t)) < f(t, \beta(t)) + \epsilon_2, \ t \in [a, b].$$

Then, by (24) and (25),

$$\begin{aligned}
0 &> w(b)[\varphi(u'(b)) - \varphi(\beta'(b))] - w(a)[\varphi(u'(a)) - \varphi(\beta'(a))], \\
&= \left[w(b)\varphi(u'(b)) - w(a)\varphi(u'(a))\right] - \left[w(b)\varphi(\beta'(b)) - \varphi(\beta'(a))\right] \\
&= \int_a^b ((w(t)\varphi(u'(t)))' - (w(t)\varphi(\beta'(t)))')dt \\
&= \int_a^b (-\lambda h(t) f(t, \gamma(t, u(t))) - (w(t)\varphi(\beta'(t)))')dt \\
&> \int_a^b (-\lambda h(t)[f(t, \beta(t)) + \epsilon_2] - (w(t)\varphi(\beta'(t)))')dt \\
&= \int_a^b \left(-\lambda h(t)\epsilon_2 - \left[(w(t)\varphi(\beta'(t)))' + \lambda h(t) f(t, \beta(t))\right]\right) dt \\
&\geq \int_a^b (-\lambda \epsilon_2 h(t) + \epsilon_1)dt \geq \int_a^b (-\lambda \epsilon_2 h^* + \epsilon_1)dt = 0.
\end{aligned}$$

This is a contradiction. Thus $u \in \Omega \cap \mathcal{K}$.

Since BVP (1) and (2) is equivalent to problem (20) on $\Omega \cap \mathcal{K}$, by Lemmas 5 and 6 and the same argument in the proof of [51] (Theorem 1.1), one can conclude that BVP (1) and (2) has at least two positive solutions for $\lambda_* < \lambda < \lambda_1$. Thus the proof is complete. □

Lemma 7. *Assume that $(A1), (A2)$, $f \in C([0,1] \times [0, \infty), (0, \infty))$, either (F_0) and $h \in \mathcal{H}_\varphi$ or (F_0') and $h \in \mathcal{H}_{\psi_1}$ hold. Let $L > 0$ be given. Then there exists $M_L > 0$ such that $\|u\|_\infty \leq M_L$ for any positive solutions u to BVP (1) and (2) with $\lambda \in [0, L]$.*

Proof. We give the proof for the case that (F_0) and $h \in \mathcal{H}_\varphi$, since the case (F_0') and $h \in \mathcal{H}_{\psi_1}$ can be proved in a similar manner.

Set

$$M := (4L)^{-1} w_0 \psi_1(h_*^{-1}) > 0,$$

where

$$h_* = \max\left\{A_1 \int_0^\gamma \varphi^{-1}\left(\int_s^\gamma h(\tau)d\tau\right) ds, A_2 \int_\gamma^1 \varphi^{-1}\left(\int_\gamma^s h(\tau)d\tau\right) ds\right\}.$$

By (F_0), there exists $s_M > 0$ such that

$$f(t, s) \leq M\psi_1(s) \text{ for } (t, s) \in [0, 1] \times [s_M, \infty). \tag{26}$$

Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to BVP (1) and (2) with $\lambda = \lambda_n \in (0, L]$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Set

$$C_M = \max\{f(t, s) : (t, s) \in [0, 1] \times [0, s_M]\} > 0.$$

Then there exists $N > 0$ satisfying

$$\|u_N\|_\infty \geq \psi_1^{-1}\left(\frac{C_M}{M}\right),$$

which implies

$$C_M \leq M\psi_1(\|u_N\|_\infty).$$

Consequently, by the definition of C_M and (26),

$$f(t,s) \le C_M + M\psi_1(s) \le 2M\psi_1(\|u_N\|_\infty) \text{ for } (t,s) \in [0,1] \times [0, \|u_N\|_\infty]. \tag{27}$$

Let σ_N be a unique point satisfying $\|u_N\|_\infty = u_N(\sigma_N)$. Assume that $\sigma_N \le \gamma$, since the case $\sigma_N > \gamma$ can be dealt in a similar manner. Then, by (8), (10) and (27),

$$\begin{aligned}
\|u_N\|_\infty &= u_N(\sigma_N) = T(\lambda_N, u_N)(\sigma_N) = v^1_{\lambda_N,u_N}(\sigma_N) \\
&\le A_1 \int_0^{\sigma_N} I_{\lambda_N,u_N}(s,\sigma_N)ds \\
&= A_1 \int_0^{\sigma_N} \varphi^{-1}\left(\frac{\lambda_n}{w(s)} \int_s^{\sigma_N} h(\tau)f(\tau,u_N(\tau))d\tau\right) ds \\
&\le A_1 \int_0^{\gamma} \varphi^{-1}\left(\frac{2LM}{w_0} \int_s^{\gamma} h(\tau)d\tau\psi_1(\|u_N\|_\infty))\right) ds \\
&\le A_1 \int_0^{\gamma} \varphi^{-1}\left(\frac{2LM}{w_0} \int_s^{\gamma} h(\tau)d\tau\right) ds\|u_N\|_\infty \\
&\le A_1 \int_0^{\gamma} \varphi^{-1}\left(\int_s^{\gamma} h(\tau)d\tau\right) ds\psi_1^{-1}\left(\frac{2LM}{w_0}\right)\|u_N\|_\infty \\
&\le h_*\psi_1^{-1}\left(\frac{2LM}{w_0}\right)\|u_N\|_\infty < \|u_N\|_\infty.
\end{aligned}$$

Here the choice of M is used in the last inequality. This contradiction completes the proof. □

Remark 2. *The assumptions* (F_0) *and* $h \in \mathcal{H}_\varphi$ *are different from the ones* (F_0') *and* $h \in \mathcal{H}_{\psi_1}$ *in Theorem 3. Indeed, let*

$$\varphi(s) = s + s^2 \text{ and } \psi_1(s) = \min\{s, s^2\} \text{ for } s \in [0,\infty).$$

Then the first inequality in (3) *is satisfied. Clearly,* (F_0) *implies* (F_0')*, since*

$$\varphi(1)\psi_1(s) \le \varphi(s) \text{ for all } s \in [0,\infty).$$

Let $f(t,s) = 1 + s^{\frac{3}{2}}$ *for* $(t,s) \in [0,1] \times [0,\infty)$*. Then*

$$\lim_{s\to\infty} \frac{1+s^{\frac{3}{2}}}{\varphi(s)} = 0, \text{ but } \lim_{s\to\infty} \frac{1+s^{\frac{3}{2}}}{\psi_1(s)} = \infty.$$

Consequently, (F_0') *does not imply* (F_0)*. Since* $\mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi$*, we give an example of* h *satisfying* $h \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1}$*. Let*

$$h(t) = t^{-2} \text{ for } t \in (0,1].$$

From

$$\varphi^{-1}(s) = \frac{-1+\sqrt{1+4s}}{2} \text{ and } \psi_1^{-1}(s) = \max\{\sqrt{s}, s\} \text{ for } s \in [0,\infty).$$

it follows that

$$\varphi^{-1}\left(\int_s^{\frac{1}{2}} \tau^{-2}d\tau\right) = \varphi^{-1}\left(s^{-1}-2\right) = \frac{-1+\sqrt{1+4(s^{-1}-2)}}{2} \in L^1\left(0,\frac{1}{2}\right)$$

and

$$\psi_1^{-1}\left(\int_s^{\frac{1}{2}} \tau^{-2}d\tau\right) = \psi_1^{-1}\left(s^{-1}-2\right) = s^{-1}-2 \notin L^1\left(0,\frac{1}{3}\right).$$

Consequently

$$h \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1},$$

since $h \in C(0,1]$.

Theorem 3. *Assume that* $(A1), (A2)$, $f \in C([0,1] \times [0,\infty), (0,\infty))$, *either* (F_0) *and* $h \in \mathcal{H}_\varphi$ *or* (F_0') *and* $h \in \mathcal{H}_{\psi_1}$ *hold. Then for any* $\lambda \in (0,\infty)$, *there exists a positive solution* u_λ *to BVP* (1) *and* (2) *such that*

$$\|u_\lambda\|_\infty \to 0 \text{ as } \lambda \to 0^+ \text{ and } \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty.$$

Proof. Set

$$\lambda^* = \sup\{\lambda \in [0,\infty) : (\lambda, u_\lambda) \in \mathcal{C}\}.$$

Here $\mathcal{C}$ is the unbounded solution component in Proposition 1. Then, by Lemma 7, $\lambda^* = \infty$. Indeed, assume on the contrary that $\lambda^* < \infty$. Then, by Lemma 7, all solutions u_λ to problem (1) satisfying $(\lambda, u_\lambda) \in \mathcal{C}$ are bounded in $C[0,1]$. This contradicts the fact that the solution component $\mathcal{C}$ is unbounded in $[0,\infty) \times \mathcal{K}$. Thus, $\lambda^* = \infty$, and for any $\lambda \in (0,\infty)$, there exists a positive solution u_λ to BVP (1) and (2) satisfying

$$(\lambda, u_\lambda) \in \mathcal{C} \text{ and } \|u_\lambda\|_\infty \to 0 \text{ as } \lambda \to 0^+.$$

Next we show that

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty.$$

Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ in $\mathcal{C}$ such that

$$\lambda_n \to \infty \text{ as } n \to \infty,$$

but there exists $m > 0$ satisfying

$$\|u_n\|_\infty \le m \text{ for all } n.$$

Since $f \in C([0,1] \times [0,\infty), (0,\infty)$, there exists $\delta_m > 0$ satisfying

$$f(t, u_n(t)) \ge \delta_m \text{ for all } t \in [0,1] \text{ and all } n.$$

For each n, let σ_n be the unique point satisfying $u_n(\sigma_n) = \|u_n\|_\infty$. Suppose that $\sigma_n \ge \gamma$ (the case $\sigma_n < \gamma$ is similar). Then, by (8),

$$\begin{aligned}
\|u_n\|_\infty &\ge u_n(\gamma) \\
&= u_n(0) + \int_0^\gamma \varphi^{-1}\left(\frac{1}{w(s)} \int_s^{\sigma_n} \lambda_n h(\tau) f(\tau, u_n(\tau)) d\tau\right) ds \\
&\ge \int_0^\gamma \varphi^{-1}\left(\frac{1}{w(s)} \int_0^\gamma h(\tau) d\tau \lambda_n \delta_m\right) ds \\
&\ge \int_0^\gamma \varphi^{-1}\left(\frac{1}{w(s)} \int_0^\gamma h(\tau) d\tau\right) ds \psi_2^{-1}(\lambda_n \delta_n) \to \infty \text{ as } n \to \infty,
\end{aligned}$$

which contradicts the fact that $\|u_n\|_\infty \le m$ for all n. Thus, the proof is complete. □

Remark 3. *Assume that* $f \in C([0,1] \times \mathbb{R}_+, (0,\infty))$ *and* $\hat{\alpha}_i \in (0,1)$ *for* $i = 1,2$. *Then, for any positive solutions* u *to BVP* (1) *and* (2),

$$u(t) \ge \hat{\rho}\|u\|_\infty \text{ for all } t \in [0,1]. \tag{28}$$

Here

$$\hat{\rho} := \rho_1 \min\left\{\int_0^1 \min\{r, 1-r\} d\alpha_i(r) : i = 1,2\right\} \in (0,1)$$

and

$$\rho_1 = \psi_2^{-1}\left(\frac{1}{\|w\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{w_0}\right)\right]^{-1} \in (0,1].$$

In fact, by (2) *and* (15), *for* $t \in [0,1]$,

$$\begin{aligned} u(t) &\geq \min\{u(0), u(1)\} \\ &= \min\left\{\int_0^1 u(r)d\alpha_i(r) : i = 1,2\right\} \\ &= \min\left\{\int_0^1 T(\lambda,u)(r)d\alpha_i(r) : i = 1,2\right\} \\ &\geq \min\left\{\int_0^1 \rho_1 \min\{r, 1-r\}\|T(\lambda,u)\|_\infty d\alpha_i(r) : i = 1,2\right\} \\ &= \rho_1 \min\left\{\int_0^1 \min\{r,1-r\}d\alpha_i(r) : i = 1,2\right\}\|u\|_\infty = \hat{\rho}\|u\|_\infty. \end{aligned}$$

Theorem 4. *Assume that* $(A1), (A2)$, $\hat{\alpha}_i \in (0,1)$ *for* $i = 1,2$ *and* $f \in C([0,1] \times (0,\infty), \mathbb{R})$ *satisfies* $f(t,s) > 0$ *for all* $(t,s) \in [0,1] \times [M,\infty)$ *and for some* $M > 0$.

(1) *Assume that* (F_∞) *and* $h \in \mathcal{H}_\varphi$ *hold. Then there exists* $\lambda_\infty > 0$ *such that BVP* (1) *and* (2) *has at least one positive solution* u_λ *for any* $\lambda \in (0,\lambda_\infty)$ *satisfying*

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to 0^+.$$

(2) *Assume that either* (F_0) *and* $h \in \mathcal{H}_\varphi$ *or* (F_0') *and* $h \in \mathcal{H}_{\psi_1}$ *hold. Then there exists* $\lambda_0 > 0$ *such that BVP* (1) *and* (2) *has at least one positive solution* u_λ *for any* $\lambda \in (\lambda_0,\infty)$ *satisfying*

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty.$$

Proof. We only give the proof of (2) with the case (F_0) and $h \in \mathcal{H}_\varphi$, since other cases can be proved in a similar manner.

Consider the following modified problem

$$\begin{cases} (w(t)\varphi(u'(t)))' + \lambda h(t) f_1(t,u(t)) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r), \end{cases} \tag{29}$$

where

$$f_1(t,s) = \begin{cases} f(t,M), & \text{for } (t,s) \in [0,1] \times [0,M), \\ f(t,s), & \text{for } (t,s) \in [0,1] \times [M,\infty). \end{cases}$$

Then, by (F_0), $f_1 \in C([0,1] \times [0,\infty), (0,\infty))$ satisfies

$$\lim_{s\to\infty} \min_{t\in[0,1]} \frac{f_1(t,s)}{\psi_1(s)} = 0.$$

By Theorem 3, problem (29) has at least one positive solution u_λ for any $\lambda \in (\lambda_0,\infty)$ satisfying $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Since

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty,$$

there exists $\lambda_0 > 0$ such that positive solutions u_λ satisfy

$$\|u_\lambda\|_\infty \geq \hat{\rho}^{-1}M \text{ for any } \lambda \in (\lambda_0,\infty).$$

By Remark 3, for $\lambda \in (\lambda_0,\infty)$,

$$u_\lambda(t) \geq M \text{ for } t \in [0,1].$$

Consequently

$$f_1(t,u_\lambda(t)) = f(t,u_\lambda(t)) \text{ for } t \in [0,1]$$

and u_λ becomes the positive solution to BVP (1) and (2) for $\lambda \in (\lambda_0, \infty)$. Thus the proof is complete. □

Finally, we give some examples to illustrate the main results (Theorem 2, Theorems 3 and 4) obtained in this section.

Example 1. *Consider the following problem*

$$\begin{cases} ((t+1)^{-1}\varphi(u'(t)))' + \lambda h(t) f(t, u(t)) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r), \end{cases} \tag{30}$$

where φ is defined by

$$\varphi(s) = s + |s|s \text{ for } s \in (-\infty, \infty)$$

and

$$\alpha_1(r) = \frac{1}{2}r^2 \text{ and } \alpha_2(r) = \frac{1}{3}r^3 \text{ for } r \in [0,1].$$

Then it is easy to see that $(A1)$ is satisfied with

$$\psi_1(y) = \min\{y, y^2\} \text{ and } \psi_2(y) = \max\{y, y^2\} \text{ for } y \in [0, \infty)$$

and $(A2)$ holds with

$$\hat{\alpha}_1 = \frac{1}{2} \text{ and } \hat{\alpha}_2 = \frac{1}{3}.$$

Note that

$$\psi_1^{-1}(s) = \max\{\sqrt{s}, s\} \text{ and } \psi_2^{-1}(s) = \min\{\sqrt{s}, s\} \text{ for } s \in [0, \infty).$$

From

$$w_0 = \frac{1}{2} \text{ and } \|w\|_\infty = 1,$$

it follows that

$$\psi_2^{-1}(\|w\|_\infty) = \psi_2^{-1}(1) = 1 \text{ and } \psi_1^{-1}(w_0) = \psi_1^{-1}(2) = 2.$$

Consequently

$$\rho_w = \frac{1}{2}\min\{\alpha, 1-\beta\} \in (0,1) \text{ for any } \alpha, \beta \in (0,1).$$

(1) Let

$$h(t) = t^{-2} \text{ for } t \in (0,1].$$

Then $h \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1}$(see Remark 2).

(i) Let f be any positive continuous function satisfying

$$f(t,s) = \left(\frac{7}{8} + \sin s + t\right)(1+s^3) \text{ for } (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [0, \infty).$$

Then (F_∞) is satisfied with

$$\alpha = \frac{1}{4} \text{ and } \beta = \frac{3}{4}.$$

By Theorem 2, there exists $\lambda_* > 0$ such that problem (30) has at least two positive solutions u_λ^1, u_λ^2 for any $\lambda \in (0, \lambda_*)$, at least one positive solution for $\lambda = \lambda_*$ and no positive solutions for $\lambda > \lambda_*$. Moreover, two positive solutions u_λ^1 and u_λ^2 for $\lambda \in (0, \lambda_*)$ satisfy

$$\|u_\lambda^1\|_\infty \to 0 \text{ and } \|u_\lambda^2\|_\infty \to \infty \text{ as } \lambda \to 0^+.$$

(ii) Let $f : (0,\infty) \to \mathbb{R}$ *be defined by*

$$f(t,s) = \begin{cases} \frac{3s-2+t}{2s}, & \text{for } (t,s) \in [0,1] \times (0,2], \\ \frac{4+t}{4} + (s-2)^3, & \text{for } (t,s) \in [0,1] \times (2,\infty). \end{cases}$$

Then (F_∞) *is satisfied for any* α, β *satisfying*

$$0 < \alpha < \beta < 1 \text{ and } f(t,s) > 0 \text{ for all } (t,s) \in [0,1] \times [2,\infty).$$

By Theorem 4 (1), there exists $\lambda_\infty > 0$ *such that problem (30) has at least one positive solution* u_λ *for all* $\lambda \in (\lambda_\infty, \infty)$ *satisfying*

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to 0^+.$$

(2) Let

$$h(t) = t^{-\frac{3}{2}} \text{ for } t \in (0,1].$$

Then $h \in \mathcal{H}_{\psi_1}$*, since* $\psi_1^{-1}(s) = s$ *for* $s \geq 1$.

(i) Let f *be defined by*

$$f(t,s) = \left(\cos t + s^{\frac{3}{2}}\right) \text{ for } (t,s) \in [0,1] \times (0,\infty).$$

Then (F_0') *are satisfied. By Theorem 3, problem (30) has at least one positive solution* u_λ *for any* $\lambda \in (0,\infty)$ *satisfying*

$$\|u_\lambda\|_\infty \to 0 \text{ as } \lambda \to 0^+ \text{ and } \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty.$$

(ii) Let f *be defined by*

$$f(t,s) = \frac{3s-2+t}{2s} \text{ for } (t,s) \in [0,1] \times (0,\infty).$$

Then (F_0') *is satisfied and* $f(t,s) > 0$ *for all* $(t,s) \in [0,1] \times [1,\infty)$*. By Theorem 4 (2), there exists* $\lambda_0 > 0$ *such that problem (30) has at least one positive solution* u_λ *for any* $\lambda \in (\lambda_0, \infty)$ *satisfying*

$$\|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty.$$

4. Conclusions

In this work, the existence, nonexistence and/or multiplicity of positive solutions to BVP (1) and (2) were studied. If the nonlinearity $f = f(t,u) \in C([0,1] \times [0,\infty), (0,\infty))$ is superlinear at $u = \infty$, it is not hard to show the result that, for some $\lambda_*^1, \lambda_*^2 > 0$, BVP (1) and (2) has at least two positive solutions u_λ^1 and u_λ^2 for $\lambda \in (0, \lambda_*^1)$, at least one positive solution for $\lambda \in [\lambda_*^1, \lambda_*^2]$ and no positive solutions for $\lambda > \lambda_*^2$. This result is partial since there is no information on the multiplicity of positive solutions for $\lambda \in [\lambda_*^1, \lambda_*^2)$. By the lack of solution regularity and the boundary conditions (2), it is not obvious to show $\lambda_*^1 = \lambda_*^2$. In Theorem 2, when the nonlinearity $f = f(t,u) \in C([0,1] \times [0,\infty), (0,\infty))$ is superlinear at $u = \infty$, the global result for positive solutions to BVP (1) and (2) with respect to the parameter λ (i.e., $\lambda_*^1 = \lambda_*^2$) was shown. In Theorem 3, when the nonlinearity $f = f(t,u) \in C([0,1] \times [0,\infty), (0,\infty))$ is sublinear at $u = \infty$, the existence of one positive solution for all $\lambda > 0$ was shown. Theorems 2 and 3 extend the results in [7] for problem (1) with Dirichlet boundary conditions ($\hat{\alpha}_1 = \hat{\alpha}_2 = 0$) to the ones for problem (1) with Riemann-Stieltjes integral boundary conditions in some ways. In Theorem 4, when $\hat{\alpha}_1\hat{\alpha}_2 \neq 0$ and the sign-changing nonlinearity $f = f(t,u) \in C([0,1] \times (0,\infty), \mathbb{R})$ may be singular at $u = 0$, the existence of one positive solution was shown for all small $\lambda > 0$ when $f = f(t,u)$ is superlinear at $u = \infty$, and the existence of one positive solution was shown for all large $\lambda > 0$ when $f = f(t, u)$ is sublinear at $u = \infty$.

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