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Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p -Laplacian Like Operator

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Abstract: This paper aims to study the oscillatory properties of fourth-order advanced differential equations with p -Laplacian like operator. By using the technique of Riccati transformation and the theory of comparison with first-order delay equations, we will establish some new oscillation criteria for this equation. Some examples are considered to illustrate the main results.

Keywords: oscillation; advanced differential equations; p -Laplacian equations; comparison theorem

1. Introduction

In the last decades, many researchers from all fields of science, technology and engineering have devoted their attention to introducing more sophisticated analytical and numerical techniques to solve and analyze mathematical models arising in their fields.

Fourth-order advanced differential equations naturally appear in models concerning physical, biological, chemical phenomena applications in dynamical systems, mathematics of networks, and optimization. They also appear in the mathematical modeling of engineering problems to study electrical power systems, materials and energy, elasticity, deformation of structures, and soil settlement [1]. The p -Laplace equations have some applications in continuum mechanics, see for example [2–4].

An active and essential research area in the above investigations is to study the sufficient criterion for oscillation of delay differential equations. In fact, during this decade, Several works have been accomplished in the development of the oscillation theory of higher order delay and advanced equations by using the Riccati transformation and the theory of comparison between first and second-order delay equations, (see [5–12]). Further, the oscillation theory of fourth and second order delay equations has been studied and developed by using integral averaging technique and the Riccati transformation, (see [13–27]). The study of oscillation has been carried to fractional equations in the setting of fractional operators with singular and nonsingular kernels, as well (see [28,29] and the references therein).

We provide oscillation properties of the fourth order advanced differential equation with a p -Laplacian like operator

$$\left(b(v) |y'''(v)|^{p-2} y'''(v) \right)' + \sum_{i=1}^j q_i(v) g(y(\eta_i(v))) = 0, \quad (1)$$

where $v \geq v_0$ and $j \geq 1$. Throughout this paper, we assume that:

$$\begin{aligned} (D_1) \quad & p > 1 \text{ is a real number,} \\ (D_2) \quad & q_i, \eta_i \in C([v_0, \infty), \mathbb{R}), q_i(v) \geq 0, \\ (D_3) \quad & \eta_i(v) \geq v, \lim_{v \rightarrow \infty} \eta_i(v) = \infty, i = 1, 2, \dots, j, \\ (D_4) \quad & g \in C(\mathbb{R}, \mathbb{R}) \text{ such that} \end{aligned} \quad g(x) / |x|^{p-2} x \geq k > 0, \text{ for } x \neq 0. \quad (2)$$

$$(D_5) \quad b \in C^1([v_0, \infty), \mathbb{R}), b(v) > 0, b'(v) \geq 0 \text{ and under the condition}$$

$$\int_{v_0}^{\infty} \frac{1}{b^{1/(p-1)}(s)} ds = \infty. \quad (3)$$

In fact, our aim in this paper is complete and improves the results in [5–7]. For the sake of completeness, we first recall and discuss these results. Li et al. [3] examined the oscillation of equation

$$\left(a(v) |z'''(v)|^{p-2} z'''(v) \right)' + \sum_{i=1}^j q_i(v) |w(\delta_i(v))|^{p-2} w(\delta_i(v)) = 0,$$

where $p > 1$ is a real number. The authors used the Riccati transformation and integral averaging technique. Park et al. [8] used Riccati technique to obtain necessary and sufficient conditions for oscillation of

$$\left(a(v) |w^{(\kappa-1)}(v)|^{p-2} w^{(\kappa-1)}(v) \right)' + q(v) g(w(\delta(v))) = 0,$$

where κ is even and under the condition

$$\int_{v_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty.$$

Agarwal and Grace [5] considered the equation

$$\left(\left(y^{(\kappa-1)}(v) \right)^\gamma \right)' + q(v) y^\gamma(\eta(v)) = 0, \quad (4)$$

where κ is even and they proved it oscillatory if

$$\liminf_{v \rightarrow \infty} \int_v^{\eta(v)} (\eta(s) - s)^{\kappa-2} \left(\int_s^\infty q(v) dv \right)^{1/\gamma} ds > \frac{(\kappa-2)!}{e}. \quad (5)$$

Agarwal et al. in [6] studied Equation (4) and obtained the criterion of oscillation

$$\limsup_{v \rightarrow \infty} v^{\gamma(\kappa-1)} \int_v^\infty q(s) ds > ((\kappa-1)!)^\gamma. \quad (6)$$

Authors in [7] studied oscillatory behavior of (4) where $\gamma = 1$, κ is even and if there exists a function $\delta \in C^1([v_0, \infty), (0, \infty))$, also, they proved it oscillatory by using the Riccati transformation if

$$\int_{v_0}^\infty \left(\delta(s) q(s) - \frac{(\kappa-2)! (\delta'(s))^2}{2^{3-2\kappa} s^{\kappa-2} \delta(s)} \right) ds = \infty. \quad (7)$$

To compare the conditions, we apply the previous results to the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(3v) = 0, v \geq 1, \quad (8)$$

1. By applying condition (5) on Equation (8), we get

$$q_0 > 13.6.$$

2. By applying condition (6) on Equation (8), we get

$$q_0 > 18.$$

3. By applying condition (7) on Equation (8), we get

$$q_0 > 576.$$

From the above we find the results in [6] improves results [7]. Moreover, the results in [5] improves results [6,7], we see this clearly in the Section 3. Thus, the motivation in studying this paper is complement and improve results [5–7].

We will need the following lemmas.

Lemma 1 ([18]). *If the function y satisfies $y^{(i)}(v) > 0$, $i = 0, 1, \dots, n$, and $y^{(n+1)}(v) < 0$, then*

$$\frac{y(v)}{v^n/n!} \geq \frac{y'(v)}{v^{n-1}/(n-1)!}.$$

Lemma 2 ([10]). *Suppose that $y \in C^n([v_0, \infty), (0, \infty))$, $y^{(n)}$ is of a fixed sign on $[v_0, \infty)$, $y^{(n)}$ not identically zero and there exists a $v_1 \geq v_0$ such that*

$$y^{(n-1)}(v)y^{(n)}(v) \leq 0,$$

for all $v \geq v_1$. If we have $\lim_{v \rightarrow \infty} y(v) \neq 0$, then there exists $v_\lambda \geq v_1$ such that

$$y(v) \geq \frac{\lambda}{(n-1)!} v^{n-1} |y^{(n-1)}(v)|,$$

for every $\lambda \in (0, 1)$ and $v \geq v_\lambda$.

Lemma 3 ([21]). *Let γ be a ratio of two odd numbers, $V > 0$ and U are constants. Then*

$$Ux - Vx^{(\gamma+1)/\gamma} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{U^{\gamma+1}}{V^\gamma}, \quad V > 0.$$

Lemma 4 ([15]). *Assume that y is an eventually positive solution of (1). Then, there exist two possible cases:*

$$\begin{aligned} (\mathbf{S}_1) \quad & y(v) > 0, y'(v) > 0, y''(v) > 0, y'''(v) > 0, y^{(4)}(v) \leq 0, \\ (\mathbf{S}_2) \quad & y(v) > 0, y'(v) > 0, y''(v) < 0, y'''(v) > 0, y^{(4)}(v) \leq 0, \end{aligned}$$

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

2. Oscillation Criteria

In this section, we shall establish some oscillation criteria for Equation (1).

Lemma 5. *Assume that y be an eventually positive solution of (1) and $(\mathbf{S}_1)$ holds. If*

$$\pi(v) := \delta(v) \left(\frac{b(v)(y'''(v))^{p-1}}{y^{p-1}(v)} \right), \quad (9)$$

where $\delta \in C^1([v_0, \infty), (0, \infty))$, then

$$\pi'(v) \leq \frac{\delta'(v)}{\delta(v)} \pi(v) - k\delta(v) \sum_{i=1}^j q_i(v) - \frac{(p-1)\varepsilon v^2}{2(\delta(v)b(v))^{\frac{1}{p-1}}} \pi(v)^{\frac{p}{p-1}}, \quad (10)$$

for all $v > v_1$, where v_1 large enough.

Proof. Let y is an eventually positive solution of (1) and (S_1) holds. Thus, from Lemma 2, we get

$$y'(v) \geq \frac{\varepsilon}{2} v^2 y'''(v), \quad (11)$$

for every $\varepsilon \in (0, 1)$ and for all large v . From (9), we see that $\pi(v) > 0$ for $v \geq v_1$, and

$$\begin{aligned} \pi'(v) &= \delta'(v) \frac{b(v)(y'''(v))^{p-1}}{y^{p-1}(v)} + \delta(v) \frac{(b(y''')^{p-1})'(v)}{y^{p-1}(v)} \\ &\quad - (p-1)\delta(v) \frac{y^{p-2}(v)y'(v)b(v)(y'''(v))^{p-1}}{y^{2(p-1)}(v)}. \end{aligned}$$

Using (11) and (9), we obtain

$$\begin{aligned} \pi'(v) &\leq \frac{\delta'(v)}{\delta(v)} \pi(v) + \delta(v) \frac{(b(v)(y'''(v))^{p-1})'(v)}{y^{p-1}(v)} \\ &\quad - (p-1)\delta(v) \frac{\varepsilon}{2} v^2 \frac{b(v)(y'''(v))^p}{y^p(v)} \\ &\leq \frac{\delta'(v)}{\delta(v)} \pi(v) + \delta(v) \frac{(b(v)(y'''(v))^{p-1})'(v)}{y^{p-1}(v)} \\ &\quad - \frac{(p-1)\varepsilon v^2}{2(\delta(v)b(v))^{\frac{1}{p-1}}} \pi(v)^{\frac{p}{p-1}}. \end{aligned} \quad (12)$$

From (1) and (12), we get

$$\pi'(v) \leq \frac{\delta'(v)}{\delta(v)} \pi(v) - k\delta(v) \frac{\sum_{i=1}^j q_i(v) y^{p-1}(\eta_i(v))}{y^{p-1}(v)} - \frac{(p-1)\varepsilon v^2}{2(\delta(v)b(v))^{\frac{1}{p-1}}} \pi(v)^{\frac{p}{p-1}}.$$

Note that $y'(v) > 0$ and $\eta_i(v) \geq v$, thus, we find

$$\pi'(v) \leq \frac{\delta'(v)}{\delta(v)} \pi(v) - k\delta(v) \sum_{i=1}^j q_i(v) - \frac{(p-1)\varepsilon v^2}{2(\delta(v)b(v))^{\frac{1}{p-1}}} \pi(v)^{\frac{p}{p-1}}.$$

The proof is complete. $\square$

Lemma 6. Assume that y be an eventually positive solution of (1) and (S_2) holds. If

$$\xi(v) := \sigma(v) \frac{y'(v)}{y(v)}. \quad (13)$$

where $\sigma \in C^1([v_0, \infty), (0, \infty))$, then

$$\xi'(v) \leq \frac{\sigma'(v)}{\sigma(v)} \xi(v) - \sigma(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{\sigma(v)} \xi(v)^2, \quad (14)$$

for all $v > v_1$, where v_1 large enough.

Proof. Let y is an eventually positive solution of (1) and (S_2) holds. Integrating (1) from v to m and using $y'(v) > 0$, we obtain

$$b(m) (y'''(m))^{p-1} - b(v) (y'''(v))^{p-1} = - \int_v^m \sum_{i=1}^j q_i(s) g(y(\eta_i(s))) ds.$$

By virtue of $y'(v) > 0$ and $\eta_i(v) \geq v$, we get

$$b(m) (y'''(m))^{p-1} - b(v) (y'''(v))^{p-1} \leq -ky^{p-1}(v) \int_v^u \sum_{i=1}^j q_i(s) ds.$$

Letting $m \rightarrow \infty$, we see that

$$b(v) (y'''(v))^{p-1} \geq ky^{p-1}(v) \int_v^\infty \sum_{i=1}^j q_i(s) ds$$

and so

$$y'''(v) \geq y(v) \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)}.$$

Integrating again from v to ∞ , we get

$$y''(v) + y(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv \leq 0. \quad (15)$$

From the definition of $\xi(v)$, we see that $\xi(v) > 0$ for $v \geq v_1$. By differentiating, we find

$$\xi'(v) = \frac{\sigma'(v)}{\sigma(v)} \xi(v) + \sigma(v) \frac{y''(v)}{y(v)} - \frac{1}{\sigma(v)} \xi(v)^2. \quad (16)$$

From (15) and (16), we obtain

$$\xi'(v) \leq \frac{\sigma'(v)}{\sigma(v)} \xi(v) - \sigma(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{\sigma(v)} \xi(v)^2.$$

The proof is complete. $\square$

Theorem 1. Assume that there exist positive functions $\delta, \sigma \in C^1([v_0, \infty), (0, \infty))$ such that

$$\int_{v_0}^\infty \left(k\delta(s) \sum_{i=1}^j q_i(s) - \frac{2^{p-1}b(s)(\delta'(s))^p}{p^p(s^2\varepsilon\delta(s))^{p-1}} \right) ds = \infty, \quad (17)$$

for some $\varepsilon \in (0, 1)$, and either

$$\int_{v_0}^\infty \sum_{i=1}^j q_i(s) ds = \infty \quad (18)$$

or

$$\int_{v_0}^{\infty} \left(\sigma(s) \int_v^{\infty} \left(\frac{k}{b(v)} \int_v^{\infty} \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{4\sigma(s)} (\sigma'(s))^2 \right) ds = \infty. \quad (19)$$

Then every solution of (1) is oscillatory.

Proof. Assume that y is eventually positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases (S_1) and (S_2).

Assume that case (S_1) holds. From Lemma 5, we get that (10) holds. Using Lemma 3 with

$$U = \delta'(v) / \delta(v), \quad V = (p-1) \varepsilon v^2 / \left(2(\delta(v) b(v))^{\frac{1}{(p-1)}} \right) \text{ and } x = \pi(v),$$

we get

$$\frac{\delta'(v)}{\delta(v)} \pi(v) - \frac{(p-1) \varepsilon v^2}{2(\delta(v) b(v))^{\frac{1}{(p-1)}}} \pi(v)^{\frac{p}{(p-1)}} \leq -\frac{2^{p-1} b(v) (\delta'(v))^p}{p^p (v^2 \varepsilon \delta(v))^{p-1}}. \quad (20)$$

From (10) and (20), we obtain

$$\pi'(v) \leq -k \delta(v) \sum_{i=1}^j q_i(v) + \frac{2^{p-1} b(v) (\delta'(v))^p}{p^p (v^2 \varepsilon \delta(v))^{p-1}}.$$

Integrating from v_1 to v , we get

$$\int_{v_1}^v \left(k \delta(s) \sum_{i=1}^j q_i(s) - \frac{2^{p-1} b(s) (\delta'(s))^p}{p^p (s^2 \varepsilon \delta(s))^{p-1}} \right) ds \leq \pi(v_1),$$

for every $\varepsilon \in (0, 1)$, which contradicts (17).

Let case (S_2) holds. Integrating (1) from m to v , we conclude that

$$-b(m) (y'''(m))^{p-1} = - \int_m^v \sum_{i=1}^j q_i(s) g(y(\eta_i(s))) ds.$$

By virtue of $y'(v) > 0$ and $\eta_i(v) \geq v$, we get

$$\int_m^v \sum_{i=1}^j q_i(s) ds \leq \frac{b(m) (y'''(m))^{p-1}}{k y^{p-1}(m)},$$

which contradicts (18).

From Lemma 6, we get that (14) holds. Using Lemma 3 with

$$U = \sigma'(v) / \sigma(v), \quad V = 1/\sigma(v) \text{ and } x = \xi(v),$$

we get

$$\frac{\sigma'(v)}{\sigma(v)} \xi(v) - \frac{1}{\sigma(v)} \xi^2(v) \leq -\frac{1}{4\sigma(v)} (\sigma'(v))^2. \quad (21)$$

From (14) and (21), we obtain

$$\xi'(v) \leq -\sigma(v) \int_v^{\infty} \left(\frac{k}{b(v)} \int_v^{\infty} \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv + \frac{1}{4\sigma(v)} (\sigma'(v))^2. \quad (22)$$

Integrating from v_1 to v , we get

$$\int_{v_1}^v \left(\sigma(s) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\gamma} dv - \frac{1}{4\sigma(s)} (\sigma'(s))^2 \right) ds \leq \xi(v_1),$$

which contradicts (19). The proof is complete. $\square$

When putting $\delta(v) = v^3$ and $\sigma(v) = v$ into Theorem 1, we get the following oscillation criteria:

Corollary 1. Let (3) hold. Assume that

$$\int_{v_0}^\infty \left(s^3 \sum_{i=1}^j q_i(s) - \frac{2^{p-1} 3^p s^{-3(p-1)+2} b(s)}{p^p \varepsilon^{p-1}} \right) ds = \infty, \quad (23)$$

or some $\varepsilon \in (0, 1)$. If (18) holds and

$$\int_{v_0}^\infty \left(s \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{4s} \right) ds = \infty, \quad (24)$$

then every solution of (1) is oscillatory.

In the next theorem, we compare the oscillatory behavior of (1) with the first-order differential equations:

Theorem 2. Assume that (3) holds. If the differential equations

$$\theta'(v) + k \sum_{i=1}^j q_i(v) \left(\frac{\varepsilon v^2}{2b^{1/\gamma}(v)} \right)^{p-1} \theta(\eta(v)) = 0 \quad (25)$$

and

$$\phi'(v) + v\phi(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv = 0 \quad (26)$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases (S_1) and (S_2).

In the case where (S_1) holds, from Lemma 2, we get

$$y(v) \geq \frac{\varepsilon v^2}{2b^{1/(p-1)}(v)} \left(b^{1/(p-1)}(v) y'''(v) \right),$$

for every $\varepsilon \in (0, 1)$ and for all large v . Thus, if we set

$$\theta(v) = b(v) (y'''(v))^{p-1} > 0,$$

then we see that ξ is a positive solution of the inequality

$$\theta'(v) + k \sum_{i=1}^j q_i(v) \left(\frac{\varepsilon v^2}{2b^{1/(p-1)}(v)} \right)^{p-1} \theta(\eta(v)) \leq 0. \quad (27)$$

From ([27], Theorem 1), we conclude that the corresponding Equation (25) also has a positive solution, which is a contradiction.

In the case where (S_2) holds, from Lemma 1, we get

$$y(v) \geq vy'(v), \quad (28)$$

From (28) and (15), we get

$$y''(v) + vy'(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv \leq 0.$$

Thus, if we set

$$\phi(v) = y'(v),$$

then we see that ζ is a positive solution of the inequality

$$\phi'(v) + v\phi(v) \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv \leq 0. \quad (29)$$

It is well known (see ([27], Theorem 1)) that the corresponding Equation (26) also has a positive solution, which is a contradiction. The proof is complete. $\square$

Corollary 2. Assume that (3) holds. If

$$\liminf_{v \rightarrow \infty} \int_{\eta(v)}^v \sum_{i=1}^j q_i(s) \left(\frac{\varepsilon s^2}{2b^{1/(p-1)}(s)} \right)^{p-1} ds > \frac{((n-1)!)^{p-1}}{e} \quad (30)$$

and

$$\liminf_{v \rightarrow \infty} \int_{\eta(v)}^v v \int_v^\infty \left(\frac{k}{b(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv ds > \frac{1}{e}, \quad (31)$$

then every solution of (1) is oscillatory.

3. Examples

For an application of Corollary 1, we give the following example:

Example 1. Consider a differential equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(2v) = 0, \quad v \geq 1, \quad (32)$$

where $q_0 > 0$ is a constant. Note that $p = 2$, $b(v) = 1$, $q(v) = q_0/v^4$ and $\eta(v) = 2v$. If we set $k = 1$, then condition (23) becomes

$$\begin{aligned} \int_{v_0}^\infty \left(s^3 \sum_{i=1}^j q_i(s) - \frac{2^{p-1} 3^p s^{-3(p-1)+2} b(s)}{p^p \varepsilon^{p-1}} \right) ds &= \int_{v_0}^\infty \left(\frac{q_0}{s} - \frac{9}{2\varepsilon s} \right) ds \\ &= \left(q_0 - \frac{9}{2\varepsilon} \right) \int_{v_0}^\infty \frac{1}{s} ds \\ &= \infty \quad \text{if } q_0 > 4.5 \end{aligned}$$

and condition (24) becomes

$$\begin{aligned} \int_{v_0}^{\infty} \left(s \int_v^{\infty} \left(\frac{k}{b(v)} \int_v^{\infty} \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{4s} \right) ds &= \int_{v_0}^{\infty} \left(\frac{q_0}{6s} - \frac{1}{4s} \right) ds \\ &= \infty, \text{ if } q_0 > \frac{3}{2}. \end{aligned}$$

Therefore, from Corollary 1, all solution Equation (32) is oscillatory if $q_0 > 4.5$.

Remark 1. We compare our result with the known related criteria for oscillation of this equation are as follows:

The condition	(5)	(6)	(7)	our condition
The criterion	$q_0 > 25.5$	$q_0 > 18$	$q_0 > 1728$	$q_0 > 4.5$

Therefore, it is clear that we see our result improves results [5–7].

For an application of Theorem 1, we give the following example.

Example 2. Consider a differential equation

$$(v(y'''(v)))' + \frac{a}{v^3}y(cv) = 0, \quad v \geq 1, \quad (33)$$

where $c > 0$ and $a > 1$ is a constant. Note that $p = 2$, $b(v) = v$, $q(v) = a/v^3$.

If we set $k = 1$, $\delta(s) = \sigma(s) = s^2$, then conditions (17) and (19) become

$$\begin{aligned} \int_{v_0}^{\infty} \left(k\delta(s) \sum_{i=1}^j q_i(s) - \frac{2^{p-1}b(s)(\delta'(s))^p}{p^p(s^2\varepsilon\delta(s))^{p-1}} \right) ds &= \int_{v_0}^{\infty} \left(\frac{a}{s} - \frac{2}{s\varepsilon} \right) ds \\ &= \left(a - \frac{2}{\varepsilon} \right) \int_{v_0}^{\infty} \frac{1}{s} ds \\ &= \infty \text{ if } a > \frac{2}{\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \int_{v_0}^{\infty} \left(\sigma(s) \int_v^{\infty} \left(\frac{k}{b(v)} \int_v^{\infty} \sum_{i=1}^j q_i(s) ds \right)^{1/(p-1)} dv - \frac{1}{4\sigma(s)} (\sigma'(s))^2 \right) ds &= \int_{v_0}^{\infty} \left(\frac{a}{4} - \frac{1}{4} \right) ds \\ &= \infty, \text{ if } q_0 > 1. \end{aligned}$$

for some constant $\varepsilon \in (0, 1)$. Hence, by Theorem 1, every solution of Equation (33) is oscillatory if

$$a > \frac{2}{\varepsilon}.$$

Remark 2. By applying condition (23) in Equation (8), we find

$$q_0 > 4.5,$$

while the conditions that we obtained in the introduction as follows:

The condition	(5)	(6)	(7)	our condition
The criterion	$q_0 > 13.6$	$q_0 > 18$	$q_0 > 576$	$q_0 > 4.5$

Therefore, our result improves results [5–7].

4. Conclusions

This paper is concerned with the oscillatory properties of the fourth-order differential equations with p -Laplacian like operators. New oscillation criteria are established, and they essentially improves the related contributions to the subject. In this paper the following methods were used:

1. Riccati transformations technique.
2. Method of comparison with first-order differential equations.

Further, in the future work we get some oscillation criteria of (1) under the condition $\int_{v_0}^{\infty} \frac{1}{b^{1/(p-1)}(s)} ds < \infty$.

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