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P-Tensor Product for Group C^*-Algebras

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Abstract: In this paper, we introduce new tensor products $\overset{p}{\otimes}(1 \leq p \leq +\infty)$ on $C^*_{\ell_p}(\Gamma) \otimes C^*_{\ell_p}(\Gamma)$ and $\overset{c_0}{\otimes}$ on $C^*_{c_0}(\Gamma) \otimes C^*_{c_0}(\Gamma)$ for any discrete group Γ. We obtain that for $1 \leq p < +\infty$ $C^*_{\ell_p}(\Gamma) \overset{max}{\otimes} C^*_{\ell_p}(\Gamma) = C^*_{\ell_p}(\Gamma) \overset{p}{\otimes} C^*_{\ell_p}(\Gamma)$ if and only if Γ is amenable; $C^*_{c_0}(\Gamma) \overset{max}{\otimes} C^*_{c_0}(\Gamma) = C^*_{c_0}(\Gamma) \overset{c_0}{\otimes} C^*_{c_0}(\Gamma)$ if and only if Γ has Haagerup property. In particular, for the free group with two generators $\mathbb{F}_2$ we show that $C^*_{\ell_p}(\mathbb{F}_2) \overset{p}{\otimes} C^*_{\ell_p}(\mathbb{F}_2) \ncong C^*_{\ell_q}(\mathbb{F}_2) \overset{q}{\otimes} C^*_{\ell_q}(\mathbb{F}_2)$ for $2 \leq q < p \leq +\infty$.

Keywords: p-tensor product; amenability; Haagerup property

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1. Introduction

When $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, it can happen that numerous different norms make $\mathcal{A} \odot \mathcal{B}$ into a pre-C*-algebra. In other words, $\mathcal{A} \odot \mathcal{B}$ may carry more than one C*-norms. For example, the spatial (or minimal) tensor product norm $\| \cdot \|_{min}$ and the maximal tensor product $\| \cdot \|_{max}$ are always C*-norms on $\mathcal{A} \odot \mathcal{B}$. As the names suggest, the spatial (minimal) tensor norm is the smallest C*-norm one can place on $\mathcal{A} \odot \mathcal{B}$ and the maximal is the largest. In general these norms do not agree. In 1995, Junge and Pisier [1] proved that

$$\mathbb{B}(\ell_2) \overset{max}{\otimes} \mathbb{B}(\ell_2) \neq \mathbb{B}(\ell_2) \overset{min}{\otimes} \mathbb{B}(\ell_2).$$

In 2014, Ozawa and Pisier [2] demonstrated that $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$ admits $2^{\aleph_0}$ distinct C*-norms. Ozawa and Pisier also showed that $C^*_\lambda(\mathbb{F}_n) \otimes C^*_\lambda(\mathbb{F}_n)$ admits $2^{\aleph_0}$ distinct C*-norms where $\mathbb{F}_n$ is the noncommutative free group on $n \geq 2$ generators. Recently, Wiersma generalized Ozawa and Pisier's result. In [3], Wiersma proved that $C^*_\lambda(\Gamma_1) \otimes C^*_\lambda(\Gamma_2)$ and $C^*(\Gamma_1) \otimes C^*_\lambda(\Gamma_2)$ admit $2^{\aleph_0}$ distinct C*-norms where Γ_1 and Γ_2 are discrete groups containing copies of noncommutative free groups. In the other respect, Kirchberg [4] proved the following striking theorem:

$$C^*(\mathbb{F}) \overset{max}{\otimes} \mathbb{B}(\mathcal{H}) = C^*(\mathbb{F}) \overset{min}{\otimes} \mathbb{B}(\mathcal{H})$$

for any free group $\mathbb{F}$. Kirchberg's famous QWEP conjecture is one of the most important open problems in the theory of operator algebras. Kirchberg showed that QWEP conjecture is equivalent to

$$C^*(\mathbb{F}_2) \overset{max}{\otimes} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \overset{min}{\otimes} C^*(\mathbb{F}_2).$$

Brown and Guentner introduced a new C^*-completion of the group ring of a countable discrete group Γ in [5]. In the following, we first recall some results in [5].

Let Γ be a countable discrete group and π be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, we denote the matrix coefficient of π by

$$\pi_{\xi,\eta}(s) = \langle \pi(s)\xi | \eta \rangle .$$

It is clear that $\pi_{\xi,\eta} \in \ell_\infty(\Gamma)$.

Let D be an algebraic two-side ideal of $\ell_\infty(\Gamma)$. If there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\pi_{\xi,\eta} \in D$ for all $\xi, \eta \in \mathcal{H}_0$, then π is called $D-$representation. If D is invariant under the left and right translation of Γ on $\ell_\infty(\Gamma)$, then it is said to be translation invariant. In this paper, we always assume that D is a non-zero translation invariant ideal of $\ell_\infty(\Gamma)$. For each $p \in [1, +\infty)$, we denote the norm on $\ell_p(\Gamma)$ by

$$|f|_p = (\sum_{s\in\Gamma} |f^p(s)|)^{\frac{1}{p}} \qquad \text{for} \qquad f \in \ell_p(\Gamma).$$

We denote by $c_0(\Gamma)$ the functions of $\ell_\infty(\Gamma)$ with vanishing at infinity. It is clear that $\ell_p(\Gamma)$ and $c_0(\Gamma)$ are non-trivial translation invariant ideals of $\ell_\infty(\Gamma)$.

The C^*-algebra $C_D^*(\Gamma)$ is the C^*-completion of the group ring $\mathbb{C}\Gamma$ by$\|\cdot\|_D$, where for $\forall f \in \mathbb{C}\Gamma$,

$$\|f\|_D = \sup\{\|\pi(f)\| : \pi \ \ \textit{is a } D-\textit{representation}\ \}.$$

We denote by $C^*(\Gamma)$ the full group C^*-algebra and by $C_\lambda{}^*(\Gamma)$ the reduced group C^*-algebra, where $C^*(\Gamma)$ is the completion of $C(\Gamma)$ with respect to the norm

$$\|x\|_u = \sup\{\|\pi(x)\| : \pi \ \ \textit{is a cyclic representation}\ \}.$$

and $C_\lambda{}^*(\Gamma)$ is the completion of $C(\Gamma)$ with the norm

$$\|x\|_r = \sup\{\|\lambda(x)\| : \pi \ \ \textit{is a left regular representation}\ \}.$$

In [5], the following results are obtained:

(1) $C^*(\Gamma) = C_{l_\infty}{}^*(\Gamma)$ and $C_\lambda{}^*(\Gamma) = C_{c_c}{}^*(\Gamma)$; Where $C_c(\Gamma)$ is the function of finitely supported functions on Γ.

(2) $C_{l_p}{}^*(\Gamma) = C_\lambda{}^*(\Gamma)$ for every $p \in [1, 2]$;

(3) $C^*(\Gamma) = C_D{}^*(\Gamma)$ if and only if there exists a sequence $\{h_n\}$ of positive definite functions in D such that $h_n \to 1$;

(4) Γ is amenable if and only if $C^*(\Gamma) = C_{c_c}{}^*(\Gamma)$;

(5) Γ has the Haagerup property if and only if $C^*(\Gamma) = C_{c_0}{}^*(\Gamma)$.

In this paper, we introduce new tensor products $\overset{p}{\otimes}(1 \le p \le +\infty)$ on $C^*_{\ell_p}(\Gamma) \otimes C^*_{\ell_p}(\Gamma)$ and $\overset{c_0}{\otimes}$ on $C^*_{c_0}(\Gamma) \otimes C^*_{c_0}(\Gamma)$ for any discrete group Γ. We obtain that for $1 \le p < +\infty$, $C^*_{\ell_p}(\Gamma) \overset{max}{\otimes} C^*_{\ell_p}(\Gamma) = C^*_{\ell_p}(\Gamma) \overset{p}{\otimes} C^*_{\ell_p}(\Gamma)$ if and only if Γ is amenable; $C^*_{c_0}(\Gamma) \overset{max}{\otimes} C^*_{c_0}(\Gamma) = C^*_{c_0}(\Gamma) \overset{c_0}{\otimes} C^*_{c_0}(\Gamma)$ if and only if Γ has Haagerup property. In last section, for the free group with two generators $\mathbb{F}_2$ we show that $C^*_{\ell_p}(\mathbb{F}_2) \overset{p}{\otimes} C^*_{\ell_p}(\mathbb{F}_2) \ncong C^*_{\ell_q}(\mathbb{F}_2) \overset{q}{\otimes} C^*_{\ell_q}(\mathbb{F}_2)$ for $2 \le q < p \le +\infty$.

2. Amenability and Haagerup Property

Definition 1. *For a discrete group Γ and $1 \le p \le +\infty$, we define*

$$C^*_{\ell_p}(\Gamma) \overset{p}{\otimes} C^*_{\ell_p}(\Gamma) \triangleq C^*_{\ell_p}(\Gamma \times \Gamma).$$

We need to check that $\overset{p}{\otimes}$ is a C^*-tensor product of $C^*_{\ell_p}(\Gamma)$ and $C^*_{\ell_p}(\Gamma)$. First we will show that the map $x \to x \otimes e$ from $C^*_{\ell_p}(\Gamma)$ into $C^*_{\ell_p}(\Gamma \times \Gamma)$ is isometric, where e is the unit of Γ. For $x = \sum\limits_{s \in \Gamma} a_s s \in \mathbb{C}\Gamma$ and the unit e of Γ, $x \otimes e \in \mathbb{C}(\Gamma) \otimes \mathbb{C}(\Gamma) \subseteq \mathbb{C}(\Gamma \times \Gamma)$. We compute

$$
\begin{aligned}
\| x \otimes e \|_{\ell_p} &= \sup\{\| \pi(x \otimes e) \| \,|\pi : \Gamma \times \Gamma \to \mathcal{B}(\mathcal{H}) \text{ is } \ell_p(\Gamma \times \Gamma) - \text{representation}\} \\
&= \sup\{\| \pi(\sum_{s \in \Gamma} a_s s \otimes e) \| \,|\pi : \Gamma \times \Gamma \to \mathcal{B}(\mathcal{H}) \text{ is } \ell_p(\Gamma \times \Gamma) - \text{representation}\} \\
&= \sup\{\| \sum_{s \in \Gamma} a_s \pi(s \otimes e) \| \,|\pi : \Gamma \times \Gamma \to \mathcal{B}(\mathcal{H}) \text{ is } \ell_p(\Gamma \times \Gamma) - \text{representation}\} \\
&\leq \sup\{\| \sum_{s \in \Gamma} a_s \sigma(s) \| \,|\sigma \quad \text{is} \quad \ell_p(\Gamma) - \text{representation}\} \\
&\leq \sup\{\| \sigma(x) \| \,|\sigma \text{ is } \ell_p(\Gamma) - \text{representation}\} \\
&= \| x \|_{\ell_p},
\end{aligned}
$$

since it is easy to check that $s \to \pi(s \otimes e)$ is an $\ell_p(\Gamma) -$ representation.

Conversely, we have

$$
\begin{aligned}
\| x \otimes e \|_{\ell_p} &= \sup\{\| \pi(x \otimes e) \| \,|\pi : \Gamma \times \Gamma \to \mathcal{B}(\mathcal{H}) \text{ is } \ell_p(\Gamma \times \Gamma) - \text{representation}\} \\
&= \sup\{\| \sum_{s \in \Gamma} a_s \pi(s \otimes e) \| \,|\pi : \Gamma \times \Gamma \to \mathcal{B}(\mathcal{H}) \text{ is } \ell_p(\Gamma \times \Gamma) - \text{representation}\} \\
&\geq \sup\{\| \sum_{s \in \Gamma} a_s \sigma_s \otimes \sigma_e \| \,|\sigma \text{ is } \ell_p(\Gamma) - \text{representation}\} \\
&= \sup\{\| \sum_{s \in \Gamma} a_s \sigma_s \| \,|\sigma \text{ is } \ell_p(\Gamma) - \text{representation}\} \\
&= \| x \|_{\ell_p},
\end{aligned}
$$

since it is routine to show that $(s,t) \in \Gamma \times \Gamma \to \sigma_s \otimes \sigma_t \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is an $\ell_p(\Gamma \times \Gamma)$-representation. Under this identification, we have

$$\mathbb{C}(\Gamma \times \Gamma) \subseteq C^*_{\ell_p}(\Gamma) \odot C^*_{\ell_p}(\Gamma) \subseteq C^*_{\ell_p}(\Gamma \times \Gamma).$$

This implies that Definition 1 is well defined.

If $1 \leq p \leq 2$, it follows from Proposition 2.11 in [5] that

$$
\begin{aligned}
C^*_\lambda(\Gamma) \overset{p}{\otimes} C^*_\lambda(\Gamma) &= C^*_{\ell_p}(\Gamma) \overset{p}{\otimes} C^*_{\ell_p}(\Gamma) \\
&= C^*_{\ell_p}(\Gamma \times \Gamma) \\
&= C^*_\lambda(\Gamma \times \Gamma) \\
&= C^*_\lambda(\Gamma) \overset{min}{\otimes} C^*_\lambda(\Gamma).
\end{aligned}
$$

This shows that $\overset{p}{\otimes} = \overset{min}{\otimes}$ for $1 \leq p \leq 2$. If $p = \infty$, we have

$$
\begin{aligned}
C^*(\Gamma) \overset{\infty}{\otimes} C^*(\Gamma) &= C^*_{\ell_\infty}(\Gamma) \overset{\infty}{\otimes} C^*_{\ell_\infty}(\Gamma) \\
&= C^*_{\ell_\infty}(\Gamma \times \Gamma) \\
&= C^*(\Gamma \times \Gamma) \\
&= C^*(\Gamma) \overset{max}{\otimes} C^*(\Gamma).
\end{aligned}
$$

This shows that

$$C^*_\lambda(\Gamma) \overset{p}{\otimes} C^*_\lambda(\Gamma) = C^*_\lambda(\Gamma) \overset{min}{\otimes} C^*_\lambda(\Gamma).$$

Theorem 1. *For $1 \leq p < +\infty$, $C^*_{\ell_p}(\Gamma) \overset{max}{\otimes} C^*_{\ell_p}(\Gamma) = C^*_{\ell_p}(\Gamma) \overset{p}{\otimes} C^*_{\ell_p}(\Gamma)$ if and only if Γ is amenable.*

Proof. Suppose that Γ is amenable, $\|\cdot\|_{min}=\|\cdot\|_{max}$ on $\mathbb{C}(\Gamma)$. Since

$$\|\cdot\|_{min}\leq\|\cdot\|_{\ell_p}\leq\|\cdot\|_{max}$$

on $\mathbb{C}(\Gamma)$, we have $\|\cdot\|_{min}=\|\cdot\|_{p}=\|\cdot\|_{max}$ on $\mathbb{C}(\Gamma)$. This implies that $C^*_\lambda(\Gamma)=C^*_{\ell_p}(\Gamma)=C^*(\Gamma)$. Thus

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*(\Gamma)\overset{max}{\otimes}C^*(\Gamma)=C^*(\Gamma\times\Gamma).$$

Since $\Gamma\times\Gamma$ is also amenable, it follows from the Definition 1 that

$$C^*_{\ell_p}(\Gamma)\overset{p}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma\times\Gamma)=C^*(\Gamma\times\Gamma).$$

Therefore

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{p}{\otimes}C^*_{\ell_p}(\Gamma).$$

Conversely, we suppose that

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{p}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma\times\Gamma).$$

Then $C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)$ has a faithful $\ell_p(\Gamma\times\Gamma)$-representation $\pi : C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)\to\mathcal{B}(\mathcal{H})$ and by taking an infinite direct sum if necessary, we can assume $\pi(C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma))$ contains no compact operators. By Glimm's Lemma [6], for any state φ of $\pi(C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma))$, there exist orthonormal vectors $v_n\in\mathcal{H}$ such that

$$\langle\pi(x)v_n|v_n\rangle\to\varphi(\pi(x)),\qquad\forall x\in C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma\times\Gamma).$$

Choose φ the trivial state, we have

$$\langle\pi(x)v_n|v_n\rangle\to 1,\qquad\forall x\in C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma\times\Gamma).$$

In particular,

$$\langle\pi_{s,t}v_n|v_n\rangle\to 1,\qquad\forall s,t\in\Gamma.$$

Since π is a $\ell_p(\Gamma\times\Gamma)$-representation, we can approximate the v_n's with vectors having associated matrix coefficients in $\ell_p(\Gamma\times\Gamma)$. Thus we may assume that $\pi_{v_n,v_n}\in\ell_p(\Gamma\times\Gamma)$ for each n, where $\pi_{v_n,v_n}(s,t)=\langle\pi_{s,t}v_n|v_n\rangle$. Since π_{v_n,v_n} are positive definite functions in $\ell_p(\Gamma\times\Gamma)$ tending pointwise to one, it follows from the Remark 2.13 in [5] that $\Gamma\times\Gamma$ is amenable and so is Γ. □

Theorem 2. *For $1\leq p<+\infty$, $C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{min}{\otimes}C^*_{\ell_p}(\Gamma)$ if and only if Γ is amenable.*

Proof. Suppose that Γ is amenable, we have

$$C^*(\Gamma)=C^*_{\ell_p}(\Gamma)=C^*_\lambda(\Gamma)$$

and

$$C^*(\Gamma\times\Gamma)=C^*_{\ell_p}(\Gamma\times\Gamma)=C^*_\lambda(\Gamma\times\Gamma).$$

Thus

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*(\Gamma)\overset{max}{\otimes}C^*(\Gamma)=C^*(\Gamma\times\Gamma)$$

and

$$C^*_{\ell_p}(\Gamma)\overset{min}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\lambda}(\Gamma)\overset{min}{\otimes}C^*_{\lambda}(\Gamma)=C^*_{\lambda}(\Gamma\times\Gamma).$$

Therefore

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{min}{\otimes}C^*_{\ell_p}(\Gamma).$$

Conversely, suppose that $C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{min}{\otimes}C^*_{\ell_p}(\Gamma)$. Since

$$\|\cdot\|_{min}\leq\|\cdot\|_{\ell_p}\leq\|\cdot\|_{max}$$

on the algebraic tensor product $C^*_{\ell_p}(\Gamma)\odot C^*_{\ell_p}(\Gamma)$,

$$C^*_{\ell_p}(\Gamma)\overset{max}{\otimes}C^*_{\ell_p}(\Gamma)=C^*_{\ell_p}(\Gamma)\overset{p}{\otimes}C^*_{\ell_p}(\Gamma).$$

It follows from Theorem 1 that Γ is amenable. □

Corollary 1. *For free group $\mathbb{F}_n(2\leq n\leq+\infty)$, we have*

$$C^*_{\ell_p}(\mathbb{F}_n)\overset{max}{\otimes}C^*_{\ell_p}(\mathbb{F}_n)\neq C^*_{\ell_p}(\mathbb{F}_n)\overset{min}{\otimes}C^*_{\ell_p}(\mathbb{F}_n)\qquad\forall 1\leq p<+\infty.$$

It is well known that the famous QWEP conjecture is equivalent to

$$C^*(\mathbb{F}_2)\overset{max}{\otimes}C^*(\mathbb{F}_2)=C^*(\mathbb{F}_2)\overset{min}{\otimes}C^*(\mathbb{F}_2).$$

From Proposition 2.10 in [5], $C^*(\Gamma)=C^*_{\ell_\infty}(\Gamma)$. Compare with Corollary 1, maybe we can get some ideas about QWEP.

Definition 2. *For a discrete group Γ, we define*

$$C^*_{c_0}(\Gamma)\overset{c_0}{\otimes}C^*_{c_0}(\Gamma)\triangleq C^*_{c_0}(\Gamma\times\Gamma).$$

By a similar argument after Definition 1, we can show that Definition 2 is well defined also.

Theorem 3. $C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma)\overset{c_0}{\otimes}C^*_{c_0}(\Gamma)$ *if and only if Γ has Haagerup property.*

Proof. The proof is similar to the argument in Theorem 1. Suppose that Γ has Haagerup property. It is well known that $\Gamma\times\Gamma$ also has Haagerup property. Thus it follows from Corollary 3.4 in [5] that we have

$$C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*(\Gamma)\overset{max}{\otimes}C^*(\Gamma)=C^*(\Gamma\times\Gamma)$$

and

$$C^*_{c_0}(\Gamma)\overset{c_0}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma\times\Gamma)=C^*(\Gamma\times\Gamma).$$

So $C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma)\overset{c_0}{\otimes}C^*_{c_0}(\Gamma)$.

Conversely, suppose that

$$C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma)\overset{c_0}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma\times\Gamma).$$

Then $C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)$ has a faithful $C_0(\Gamma\times\Gamma)$-representation

$$\pi : C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)\to\mathcal{B}(\mathcal{H})$$

and by taking an infinite direct sum if necessary, we can assume $\pi(C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma))$ contains no compact operators. By Glimm's Lemma [6], for any state φ of $\pi(C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma))$, there exist orthonormal vectors $v_n\in\mathcal{H}$ such that

$$\langle\pi(x)v_n|v_n\rangle\to\varphi(\pi(x)),\qquad\forall x\in C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma\times\Gamma).$$

Choose φ the trivial state, we have

$$\langle\pi(x)v_n|v_n\rangle\to 1,\qquad\forall x\in C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma\times\Gamma).$$

In particular,

$$\langle\pi_{s,t}v_n|v_n\rangle\to 1,\qquad\forall s,t\in\Gamma.$$

Approximating the v_n's with vectors having associated matrix coefficients in $c_0(\Gamma\times\Gamma)$, we may assume that $\pi_{v_n,v_n}\in c_0(\Gamma\times\Gamma)$ for each n. Therefore $\{\pi_{v_n,v_n}\}$ is a sequence of positive definite functions in $c_0(\Gamma\times\Gamma)$ tending pointwise to one, this implies that $\Gamma\times\Gamma$ has Haagerup property and so does Γ. □

Corollary 2. *If $C^*_{c_0}(\Gamma)\overset{max}{\otimes}C^*_{c_0}(\Gamma)=C^*_{c_0}(\Gamma)\overset{min}{\otimes}C^*_{c_0}(\Gamma)$, then Γ has Haagerup property.*

3. P-Tensor Product on $\mathbb{F}_2$

In this section, we mainly consider the p-tensor product $\overset{p}{\otimes}$ on the free group with two generators $\mathbb{F}_2$.

We recall that a function φ: $\Gamma\to C$ is said to be positive definite if the matrix

$$[\varphi(s^{-1}t)]_{s,t\in\mathbb{F}}\in M_{\mathbb{F}}(C)$$

is positive for every finite set $\mathbb{F}\subset\Gamma$.

Proposition 1. *Let $\mathbb{F}_2$ be the free group with two generators. Then there exists a $p\in(2,\infty)$ such that*

$$C^*(\mathbb{F}_2)\overset{max}{\otimes}C^*(\mathbb{F}_2)\neq C^*_{\ell_p}(\mathbb{F}_2)\overset{p}{\otimes}C^*_{\ell_p}(\mathbb{F}_2)\neq C^*_\lambda(\mathbb{F}_2)\overset{min}{\otimes}C^*_\lambda(\mathbb{F}_2).$$

Proof. Since $\mathbb{F}_2\times\mathbb{F}_2$ is not amenable, by Prop 2.12 in [5] $C^*(\mathbb{F}_2\times\mathbb{F}_2)\neq C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)$ for any $p\in[1,+\infty)$. Since $C^*(\mathbb{F}_2)\overset{max}{\otimes}C^*(\mathbb{F}_2)=C^*(\mathbb{F}_2\times\mathbb{F}_2)$ and $C^*_{\ell_p}(\mathbb{F}_2)\overset{p}{\otimes}C^*_{\ell_p}(\mathbb{F}_2)=C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)$, we have for any $p\in[1,+\infty)$ $C^*(\mathbb{F}_2)\overset{max}{\otimes}C^*(\mathbb{F}_2)\neq C^*_{\ell_p}(\mathbb{F}_2)\overset{p}{\otimes}C^*_{\ell_p}(\mathbb{F}_2)$.

Since $C^*_{\ell_p}(\mathbb{F}_2)\overset{p}{\otimes}C^*_{\ell_p}(\mathbb{F}_2)=C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)$ and $C^*_\lambda(\mathbb{F}_2\times\mathbb{F}_2)=C^*_\lambda(\mathbb{F}_2)\overset{min}{\otimes}C^*_\lambda(\mathbb{F}_2)$, we only need to find some $p\in(2,+\infty)$ with $C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)\neq C^*_\lambda(\mathbb{F}_2\times\mathbb{F}_2)$. Let $S=\{a,b,a^{-1},b^{-1}\}\subseteq\mathbb{F}_2$ be the standard generating set and let $|\cdot|$ denote the corresponding word length. A well known result of [7] states that for every $n\in\mathbb{N}$,

$$h_n(s):=e^{-\frac{|s|}{n}}$$

is positive definite function on $\mathbb{F}_2$ and clearly $h_n \to 1$ pointwise. Now for $(s,t) \in \mathbb{F}_2 \times \mathbb{F}_2$, we define

$$\varphi_n\left((s,t)\right) := h_n(s) = e^{-\frac{|s|}{n}}$$

and

$$\psi_n\left((s,t)\right) := h_n(t) = e^{-\frac{|t|}{n}}.$$

For any $\alpha_i \in \mathbb{C}$ and $(s_i,t_i) \in \mathbb{F}_2 \times \mathbb{F}_2$,$1 \le i \le n$, we have

$$\begin{aligned}
&\sum_{i,j} \alpha_i \bar{\alpha}_j \varphi_n\left(\left(s_j,t_j\right)^{-1}(s_i,t_i)\right)\\
&= \sum_{i,j} \alpha_i \bar{\alpha}_j \varphi_n\left(\left(s_j^{-1}s_i, t_j^{-1}t_i\right)\right)\\
&= \sum_{i,j} \alpha_i \bar{\alpha}_j h_n\left(s_j^{-1}s_i\right) \ge 0.
\end{aligned}$$

So each φ_n is a positive definite function on $\mathbb{F}_2 \times \mathbb{F}_2$, (Similarly ψ_n is a positive definite function). Fixing n, we have $\varphi_n \in \ell_{p_n}(\mathbb{F}_2 \times \mathbb{F}_2)$ for sufficiently large p_n. Let $\pi_n : C^*_{\ell_{p_n}}(\mathbb{F}_2 \times \mathbb{F}_2) \to B(\mathcal{H}_n)$ be the GNS presentations related to φ_n, and let $\xi_n \in \mathcal{H}_n$ be the canonical cyclic vector. Since $\varphi_n((s,t)) \to 1$, we see that $\|\pi_n((s,t))\xi_n - \xi_n\| \to 0$ for all $(s,t) \in \mathbb{F}_2 \times \mathbb{F}_2$. Hence the trivial representation is contained in the direct sum representation $\oplus\pi_n$weakly. If for each n, $C^*_{\ell_{p_n}}(\mathbb{F}_2 \times \mathbb{F}_2) = C^*_\lambda(\mathbb{F}_2 \times \mathbb{F}_2)$, $\oplus\pi_n$ would be defined on $C^*_\lambda(\mathbb{F}_2 \times \mathbb{F}_2)$. Since $\mathbb{F}_2 \times \mathbb{F}_2$ is not amenable, the trivial representation cannot be contained in any representation of $C^*_\lambda(\mathbb{F}_2 \times \mathbb{F}_2)$ weakly. This is a contradiction. Therefore for some n, $C^*_{\ell_{p_n}}(\mathbb{F}_2 \times \mathbb{F}_2) \neq C^*_\lambda(\mathbb{F}_2 \times \mathbb{F}_2)$ $\square$

In the paper [8], Okayasu give a characterization of positive definite functions on a free group with finite generators, which can be extended to the positive linear functionals on the free group C^*-algebras associated with the ideal ℓ_p. This is a generalization of Haagerup's famous characterization for the case of the reduced free group C^*-algebra. The strategy in these two papers also works for the group $\mathbb{F}_2 \times \mathbb{F}_2$.

For non negative integers k_1, k_2, we define

$$W_{(k_1,k_2)} = \{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2| \quad |s_1| = k_1 \quad \text{and} \quad |s_2| = k_2\}.$$

$\chi_{(k_1,k_2)}$ denotes the characteristic function on $W_{(k_1,k_2)}$.

Lemma 1. *Let $q \in [1,2]$. Let k_i, l_i and $m_i (i = 1,2)$ be non-negative integers. Let f and g be functions on $\mathbb{F}_2 \times \mathbb{F}_2$ such that $suppf \subseteq W_{(k_1,k_2)}$ and $suppg \subseteq W_{(l_1,l_2)}$ respectively. If $|k_i - l_i| \le m_i \le k_i + l_i$ and $k_i + l_i - m_i$ is even, then*

$$|(f * g) \cdot \chi_{(m_1,m_2)}|_q \le |f|_q \cdot |g|_q$$

and if (m_1,m_2) is any other values, then

$$|(f * g) \cdot \chi_{(m_1,m_2)}|_q = 0.$$

Proof. Note that

$$\begin{aligned}
(f*g)\left((s_1,s_2)\right) &= \sum_{\substack{(t_1,t_2),(u_1,u_2) \in \mathbb{F}_2 \times \mathbb{F}_2 \\ s_i = t_i u_i}} f\left((t_1,t_2)\right) \cdot g\left((u_1,u_2)\right)\\
&= \sum_{\substack{|t_i| = k_i, |u_i| = l_i \\ s_i = t_i u_i}} f\left((t_1,t_2)\right) \cdot g\left((u_1,u_2)\right).
\end{aligned}$$

Since the possible values of $|t_i u_i|$ are $|k_i - l_i|, |k_i - l_i| + 2, \ldots, k_i + l_i$, we have

$$\left|(f * g)\, \chi_{(m_1,m_2)}\right|_q = 0$$

for any other (m_1, m_2). We only consider the $q \neq 1$ ($q = 1$ is similar and trivial). First, we assume that $m_i = k_i + l_i (i = 1, 2)$. In this case, if $|s_i| = m_i$, then s_i can be uniquely written as a product $t_i u_i$ with $t_i = |k_i|$ and $u_i = |l_i|$. Hence

$$(f * g)\,((s_1, s_2)) = f((t_1, t_2)) \cdot g((u_1, u_2)).$$

Therefore

$$\begin{aligned}
\left|(f * g)\, \chi_{(m_1,m_2)}\right|_q^q &= \sum_{(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2} |\,(f * g)\,((s_1, s_2)) \cdot \chi_{(m_1,m_2)}\,((s_1, s_2))\,|^q \\
&= \sum_{\substack{|t_i u_i| = k_i + l_i \\ |t_i| = k_i, |u_i| = l_i}} |f\,((t_1, t_2))|^q \cdot |g\,((u_1, u_2))|^q \\
&\leq \sum_{|t_i| = k_i, |u_i| = l_i} |f\,((t_1, t_2))|^q \cdot |g\,((u_1, u_2))|^q \\
&= |f|_q^q \cdot |g|_q^q .
\end{aligned}$$

Next we assume that $m_i = |k_i - l_i|, \ldots, k_i + l_i - 2$. In these cases, we have $m_i = k_i + l_i - 2j_i$, for $1 \leq j_i \leq \min(k_i, l_i), (i = 1, 2)$. Let $s_i = t_i u_i$ with $|s_i| = m_i, |t_i| = k_i$, and $|u_i| = l_i$. Then s_i can be uniquely written as a product $t'_i u'_i$ such that $t_i = t'_i v_i, u_i = v_i^{-1} u'_i$ with $|t'_i| = k_i - j_i, |u'_i| = l_i - j_i$, and $|v_i| = |v_i^{-1}| = j_i$. We define

$$f'\,((t_1, t_2)) = \left(\sum_{|v_i| = j_i} |f\,((t_1 v_1, t_2 v_2))|^q\right)^{\frac{1}{q}}, \; if \;\; |t_i| = k_i - j_i,$$

and $f'\,((t_1, t_2)) = 0$ otherwise. Similarly, we define

$$g'\,((u_1, u_2)) = \left(\sum_{|v_i| = j_i} \left|g\left(\left(v_1^{-1} u_1, v_2^{-1} u_2\right)\right)\right|^q\right)^{\frac{1}{q}}, if \;\; |u_i| = l_i - j_i,$$

and $g'\,((u_1, u_2)) = 0$, otherwise. Note that supp$f' \subseteq W_{(k_1 - j_1, k_2 - j_2)}$, and supp$g' \subseteq W_{(l_1 - j_1, l_2 - j_2)}$. Moreover,

$$|f'|_q^q = \sum_{|t_i| = k_i - j_i} \left(\sum_{|v_i| = j_i} |f\,((t_1 v_1, t_2 v_2))|^q\right) = |f|_q^q,$$

and $|g'|_q^q = |g|_q^q$. Take a real number p with $\frac{1}{p} + \frac{1}{q} = 1$. Since $1 < q \leq 2, 2 \leq p < +\infty$, so $q \leq p$ in general. Owing to Hölder inequality, we have

$$\begin{aligned}
|(f*g)((s_1,s_2))| &= \left|\sum_{\substack{|t_i|=k_i,|u_i|=l_i\\ s_i=t_iu_i}} f((t_1,t_2))\cdot g((u_1,u_2))\right| \\
&= \left|\sum_{|v_i|=j_i} f((t'_1v_1,t'_2v_2))\cdot g\left(\left(v_1^{-1}u'_1,v_2^{-1}u'_2\right)\right)\right| \\
&\le \left|\sum_{|v_i|=j_i} |f((t'_1v_1,t'_2v_2))|^q\right|^{\frac{1}{q}}\cdot\left|\sum_{|v_i|=j_i}\left|g\left(\left(v_1^{-1}u'_1,v_2^{-1}u'_2\right)\right)\right|^p\right|^{\frac{1}{p}} \\
&\le \left|\sum_{|v_i|=j_i} |f((t'_1v_1,t'_2v_2))|^q\right|^{\frac{1}{q}}\cdot\left|\sum_{|v_i|=j_i}\left|g\left(\left(v_1^{-1}u'_1,v_2^{-1}u'_2\right)\right)\right|^q\right|^{\frac{1}{q}} \\
&= f'(t'_1,t'_2)\cdot g(u'_1,u'_2) = (f'*g')((s_1,s_2)),
\end{aligned}$$

where $s_i = t'_iu'_i$ and $|s_i| = k_i + l_i - 2j_i = |t'_i| + |u'_i|$. Therefore, $\left|(f*g)\cdot\chi_{(m_1,m_2)}\right| \le (f'*g')\cdot\chi_{(m_1,m_2)}$. Since $(k_i - j_i) + (l_i - j_i) = m_i$, it follows from the first part of the proof that

$$\begin{aligned}
\left|(f*g)\,\chi_{(m_1,m_2)}\right| &\le \left|(f'*g')\cdot\chi_{(m_1,m_2)}\right| \\
&\le |f'|_q\cdot|g'|_q \\
&= |f|_q\cdot|g|_q.
\end{aligned}$$

At last, we assume that $m_1 = k_1 + l_1$ and $m_2 = |k_2 - l_2|,\ldots,k_2 + l_2 - 2$; or $m_1 = |k_1 - l_1|,\ldots,k_1 + l_1 - 2$ and $m_2 = k_2 + l_2$. We only need to consider the first case.In this case, $m_1 = k_1 + l_1$, and $m_2 = k_2 + l_2 - 2j_2$ for $1 \le j_2 \le \min(k_2,l_2)$. Then s_1 can be uniquely written as a product t_1u_1 with $|t_1| = k_1$ and $|u_1| = l_1$. Let $s_2 = t_2u_2$ with $|s_2| = m_2$, $|t_2| = k_2$, $|u_2| = l_2$. Then s_2 can be uniquely written as a product $t'_2u'_2$ such that $t_2 = t'_2v_2$, $u_2 = v_2^{-1}u'_2$, with $|t'_2| = k_2 - j_2$, $|u'_2| = l_2 - j_2$ and $|v_2| = |v_2^{-1}| = j_2$. The following proof is almost the same as the second part with $j_1 = 0$. □

Lemma 2. *Let k_1,k_2 be non-negative integers. Let $1 \le q \le p \le \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. If a unitary representation $\pi : \mathbb{F}_2\times\mathbb{F}_2 \to \mathcal{U}(\mathcal{H})$ has a cyclic vector ξ such that $\pi_{\xi,\xi} \in \ell_p(\mathbb{F}_2\times\mathbb{F}_2)$ then*

$$\|\ \pi(f)\ \| \le (k_1 + k_2 + 2)\cdot|f|_q,$$

for $f \in C_c(\mathbb{F}_2\times\mathbb{F}_2)$ with $supp f \subseteq W_{(k_1,k_2)}$.

Proof. We only consider $1 \le q \le 2$ and $2 \le p \le +\infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. We consider the norm $\left|(f^* * f)^{(*2n)}\right|_q$. Write $f_{2j-1} = f^*$ and $f_{2j} = f$ for $j = 1,\ldots,2n$. Then

$$(f^* * f)^{(*2n)} = f_1 * f_2 * \cdots * f_{4n}.$$

We also denote $g = f_2 * \cdots * f_{4n}$. So we have

$$(f^* * f)^{(*2n)} = f_1 * g.$$

Since $f^*((s_1,s_2)) = \bar{f}((s_1^{-1},s_2^{-1}))$, $\operatorname{supp} f_j \subseteq W_{(k_1,k_2)}$, for $j = 1,2,\ldots,4n$ and $g \in c_c(\mathbb{F}_2\times\mathbb{F}_2)$. Put $g_{(l_1,l_2)} = g\mathcal{X}_{(l_1,l_2)}$. Then $\operatorname{supp} g_{(l_1,l_2)} \subseteq W_{(l_1,l_2)}$ and

$$|g|_q^q = \sum_{l_1,l_2=0}^{+\infty}\left|g_{(l_1,l_2)}\right|_q^q.$$

Clearly, $\left|g_{(l_1,l_2)}\right|_q = 0$ for all but finitely many l_1, l_2. Moreover set

$$h = f_1 * g = \sum_{l_1,l_2=0}^{+\infty} f_1 * g_{(l_1,l_2)},$$

and $h_{(m_1,m_2)} = h\chi_{(m_1,m_2)}$. Then $h \in c_c(\mathbb{F}_2 \times \mathbb{F}_2)$ and

$$|h|_q^q = \sum_{m_1,m_2=0}^{+\infty} \left|h_{(m_1,m_2)}\right|_q^q.$$

$\left|h_{(m_1,m_2)}\right|_q = 0$ for all but finitely many m_1, m_2. By Lemma 1,

$$\left|\left(f_1 * g_{(l_1,l_2)}\right) \cdot \chi_{(m_1,m_2)}\right|_q \leq |f_1|_q \cdot \left|g_{(l_1,l_2)}\right|_q$$

in the case where $|k_i - l_i| \leq m_i \leq k_i + l_i$, and $k_i + l_i - m_i$ is even, and $|(f_1 * g) \cdot \chi_{(m_1,m_2)}|_q = 0$ for any other values of m_i. Hence,

$$\begin{aligned}
\left|h_{(m_1,m_2)}\right|_q &= \left|\sum_{l_1,l_2=0}^{+\infty} \left(f_1 * g_{(l_1,l_2)}\right) \cdot \chi_{(m_1,m_2)}\right|_q \\
&\leq \sum_{l_1,l_2=0}^{+\infty} \left|\left(f_1 * g_{(l_1,l_2)}\right) \cdot \chi_{(m_1,m_2)}\right|_q \\
&\leq |f_1|_q \cdot \sum_{\substack{l_i=|m_i-k_i| \\ m_i+k_i-l_i \ even}}^{m_i+k_i} \left|g_{(l_1,l_2)}\right|_q.
\end{aligned}$$

By writing $l_i = m_i + k_i - 2j_i$, we have

$$\begin{aligned}
|h\,(m_1,m_2)|_q &\leq |f_1|_q \cdot \sum_{j_1=0}^{\min(m_1,k_1)} \sum_{j_2=0}^{\min(m_2,k_2)} \left|g_{(m_1+k_1-2j_1,m_2+k_2-2j_2)}\right|_q \\
&\leq |f_1|_q \cdot \left(\sum_{j_1,j_2} \left|g_{(m_1+k_1-2j_1,m_2+k_2-2j_2)}\right|_q^q\right)^{\frac{1}{q}} \cdot \left(\sum_{j_1,j_2} 1^p\right)^{\frac{1}{p}} \\
&\leq (k_1+k_2+2)^{\frac{1}{p}} \cdot |f_1|_q \cdot \left(\sum_{j_1,j_2} \left|g_{(m_1+k_1-2j_1,m_2+k_2-2j_2)}\right|_q^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|h|_q^q &= \sum_{m_1,m_2=0}^{+\infty} |h\,(m_1,m_2)|_q^q \\
&\leq (k_1+k_2+2)^{\frac{q}{p}} \cdot |f_1|_q^q \cdot \sum_{m_1,m_2=0}^{+\infty} \sum_{j_1=0}^{\min(m_1,k_1)} \sum_{j_2=0}^{\min(m_2,k_2)} \left|g_{(m_1+k_1-2j_1,m_2+k_2-2j_2)}\right|_q^q \\
&= (k_1+k_2+2)^{\frac{q}{p}} \cdot |f_1|_q^q \cdot \sum_{j_1=0}^{k_1} \sum_{m_1=j_1}^{+\infty} \sum_{j_2=0}^{k_2} \sum_{m_2=j_2}^{+\infty} \left|g_{(m_1+k_1-2j_1,m_2+k_2-2j_2)}\right|_q^q \\
&= (k_1+k_2+2)^{\frac{q}{p}} \cdot |f_1|_q^q \cdot \sum_{j_1=0}^{k_1} \sum_{l_1=k_1-j_1}^{+\infty} \sum_{j_2=0}^{+\infty} \sum_{l_2=k_2-j_2}^{+\infty} \left|g_{(l_1 l_2)}\right|_q^q \\
&\leq (k_1+k_2+2)^{\frac{q}{p}} \cdot |f_1|_q^q \cdot \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} |g|_q^q \\
&\leq (k_1+k_2+2)^{\frac{q}{p}+1} \cdot |f_1|_q^q \cdot |g|_q^q.
\end{aligned}$$

Hence $|f_1 * g|_q \leq (k_1+k_2+2)\cdot|f_1|_q\cdot|g|_q$, i.e.

$$|f_1 * f_2 * \cdots * f_{4n}|_q \leq (k_1+k_2+2)\cdot|f_1|_q\cdot|f_2 * \cdots * f_{4n}|_q.$$

Inductively we have

$$\left|(f^* * f)^{*2n}\right|_q \leq (k_1+k_2+2)^{4n-1}\cdot|f|_q^{4n}.$$

Therefore, it follows from Lemma 3.2 in [8] that

$$\|\pi(f)\| \leq \liminf_{n\to+\infty}\left|(f^* * f)^{*2n}\right|_q^{\frac{1}{4n}} \leq (k_1+k_2+2)\cdot|f|_q.$$

□

Theorem 4. *Let $2 \leq p < \infty$. Let φ be a positve definite function on $\mathbb{F}_2\times\mathbb{F}_2$. Then the following conditions are equivalent:*

(1) φ can be extended to the positive linear functional on $C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)$;

(2) $\sup\limits_{k_1,k_2}|\varphi\cdot\chi_{(k_1,k_2)}|_p\cdot(k_1+k_2+2)^{-1}<\infty$;

(3) The function $(s_1,s_2)\to\varphi(s_1,s_2)\cdot(2+|s_1|+|s_2|)^{-1-\frac{2}{p}}$ belongs to $\ell_p(\mathbb{F}_2\times\mathbb{F}_2)$;

(4) For any $\alpha\in(0,1)$, the function $(s_1,s_2)\to\varphi(s_1,s_2)\cdot\alpha^{|s_1|+|s_2|}$ belongs to $\ell_p(\mathbb{F}_2\times\mathbb{F}_2)$.

Proof. We assume that $\varphi((e,e))=1$.

(1)⇒(2) It follows from (1) that w_φ extends to the station $C^*_{\ell_p}(\mathbb{F}_2\times\mathbb{F}_2)$, where

$$\omega_\varphi(f)=\sum_{(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2} f((s_1,s_2))\cdot\varphi((s_1,s_2)) \quad \text{for} \quad f\in c_c(\mathbb{F}_2\times\mathbb{F}_2)$$

Hence, for $f\in c_c(\mathbb{F}_2\times\mathbb{F}_2)$, we have

$$\left|\omega_\varphi(f)\right| \leq \|f\|_{\ell_p}.$$

Set $f=|\varphi|^{p-2}\cdot\bar{\varphi}\cdot\chi_{(k_1,k_2)}$.
Then

$$\begin{aligned}
\left|\omega_\varphi(f)\right| &= \sum_{(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2}\varphi((s_1,s_2))\cdot f((s_1,s_2))\\
&= \sum_{(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2}\varphi((s_1,s_2))\cdot|\varphi|^{p-2}((s_1,s_2))\cdot\bar{\varphi}((s_1,s_2))\cdot\chi_{(k_1,k_2)}((s_1,s_2))\\
&= \sum_{(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2}|\varphi|^{p}((s_1,s_2))\cdot\chi_{(k_1,k_2)}((s_1,s_2))\\
&= \left|\varphi\cdot\chi_{(k_1,k_2)}\right|_p^p.
\end{aligned}$$

Let $\pi:\mathbb{F}_2\times\mathbb{F}_2\to U(\mathcal{H})$ be an ℓ_p-representation with a dense subspace $\mathcal{H}_0$, then

$$\|\pi(f)\|^2=\sup_{\xi\in H_0,\|\xi\|=1}\langle\pi(f^* * f)\,\xi\,|\xi\rangle_{\mathcal{H}}.$$

Fix $\xi\in\mathcal{H}_0$ with $\|\xi\|=1$. We denote by σ the restriction of π onto the subspace

$$\mathcal{H}_\sigma=\overline{span}\{\pi((s_1,s_2))\,\xi\,|(s_1,s_2)\in\mathbb{F}_2\times\mathbb{F}_2\}\subseteq\mathcal{H}.$$

Then

$$\langle \pi (f^* * f) \xi | \xi \rangle_{\mathcal{H}} = \langle \sigma (f^* * f) \xi | \xi \rangle_{\mathcal{H}_\sigma}.$$

Note that ξ is cyclic for σ such that $\sigma_{\xi,\xi} \in \ell_p (\mathbb{F}_2 \times \mathbb{F}_2)$. Take a real number q with $\frac{1}{p} + \frac{1}{q} = 1$. Since $2 \le p < +\infty$, we have $1 < q \le 2$. Since supp $f \subseteq W_{(k_1,k_2)}$, it follows the Lemma 2 that

$$\|\sigma (f)\| \le (k_1 + k_2 + 2) \cdot |f|_q.$$

Hence

$$\|\sigma (f^* * f)\| = \|\sigma (f)\|^2 \le (k_1 + k_2 + 2)^2 \cdot |f|_q^2.$$

Therefore,

$$\begin{aligned}
\|f\|_{l_p}^2 &= \sup \{ \|\pi (f)\| | \, \pi \, is \, an \, \ell_p - representation \} \\
&\le (k_1 + k_2 + 2) \cdot |f|_q \\
&= (k_1 + k_2 + 2) \cdot \left(\sum_{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2} |f ((s_1, s_2))|^q \right)^{\frac{1}{q}} \\
&= (k_1 + k_2 + 2) \cdot \left(\sum_{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2} |\varphi|^{(p-1)q} ((s_1, s_2)) \cdot \chi_{(k_1,k_2)} ((s_1, s_2)) \right)^{\frac{1}{q}} \\
&= (k_1 + k_2 + 2) \cdot \left(\sum_{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2} |\varphi|^p ((s_1, s_2)) \cdot \chi_{(k_1,k_2)} ((s_1, s_2)) \right)^{\frac{1}{q}} \\
&= (k_1 + k_2 + 2) \cdot \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p^{\frac{p}{q}} \\
&= (k_1 + k_2 + 2) \cdot \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p^{p-1}.
\end{aligned}$$

Since $f = |\varphi|^{p-2} \cdot \bar{\varphi} \cdot \chi_{(k_1,k_2)}$, we have

$$\begin{aligned}
|\omega_\varphi (f)| &= \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p^p \\
&\le \|f\|_{\ell_p} \\
&\le (k_1 + k_2 + 2) \cdot \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p^{p-1}.
\end{aligned}$$

Consequently,

$$\left| \varphi \cdot \chi_{(k_1,k_2)} \right| \le k_1 + k_2 + 2.$$

(2)$\Rightarrow$(3)

$$\begin{aligned}
&\sum_{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2} |\varphi ((s_1, s_2))|^p \cdot (2 + |s_1| + |s_2|)^{-p-2} \\
&= \sum_{k_1,k_2=0}^{+\infty} \sum_{\substack{|s_1|=k_1 \\ |s_2|=k_2}} |\varphi ((s_1, s_2))|^p \cdot (2 + k_1 + k_2)^{-p-2} \\
&= \sum_{k_1,k_2=0}^{+\infty} \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p^p \cdot (2 + k_1 + k_2)^{-p} \cdot (2 + k_1 + k_2)^{-2} \\
&\le \left\{ \sup_{k_1,k_2} \left| \varphi \cdot \chi_{(k_1,k_2)} \right|_p \cdot (2 + k_1 + k_2)^{-1} \right\}^p \cdot \sum_{k_1,k_2=0}^{+\infty} (2 + k_1 + k_2)^{-2} < +\infty.
\end{aligned}$$

(3)$\Rightarrow$(4) Obviously.
(4)$\Rightarrow$(1) Set

$$\varphi_\alpha : (s_1, s_2) \mapsto \alpha^{|s_1|}, \psi_\alpha : (s_1, s_2) \mapsto \alpha^{|s_2|} \quad \text{from} \quad \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{R}.$$

For any $a_i \in \mathbb{C}$ and $(s_{1i}, s_{2i}) \in \mathbb{F}_2 \times \mathbb{F}_2$, we have

$$\begin{aligned} \sum a_i \bar{a}_j \varphi_\alpha \left((s_{1j}, s_{2j})^{-1} \cdot (s_{1i}, s_{2i}) \right) &= \sum a_i \bar{a}_j \varphi_\alpha \left(s_{1j}^{-1} s_{1i}, s_{2j}^{-1} s_{2i} \right) \\ &= \sum a_i \bar{a}_j \alpha^{\left| s_{1j}^{-1} s_{1i} \right|} \geq 0. \end{aligned}$$

So φ_α and similarly ψ_α are positive definite functions on $\mathbb{F}_2 \times \mathbb{F}_2$. This implies that the function

$$\Phi_\alpha \left((s_1, s_2) \right) \triangleq \alpha^{|s_1|+|s_2|} = \alpha^{|s_1|} \alpha^{|s_2|} = \varphi_\alpha \left((s_1, s_2) \right) \psi_\alpha \left((s_1, s_2) \right)$$

is positive definite and $\Psi_\alpha \left((s_1, s_2) \right) = \varphi \left((s_1, s_2) \right) \alpha^{|s_1|+|s_2|}$ is also positive definite on $\mathbb{F}_2 \times \mathbb{F}_2$. By the GNS construction(The unitary representation via GNS approach refers to the conclusions of appendix C in reference [9]), we obtain the unitary representation σ_α of $\mathbb{F}_2 \times \mathbb{F}_2$ with the cyclic vector ξ_α such that

$$\Psi_\alpha \left((s_1, s_2) \right) = \langle \sigma_\alpha \left((s_1, s_2) \right) \xi_\alpha \,| \xi_\alpha \,\rangle .$$

Since σ_α is an ℓ_p- representation ,Ψ_α can be considered as a state on $C^*_{\ell_p} \left(\mathbb{F}_2 \times \mathbb{F}_2 \right)$. By taking the w^*-limit of Ψ_α as $\alpha \uparrow 1$, we obtain that φ can be extended to the state of ${C_{l_p}}^* \left(\mathbb{F}_2 \times \mathbb{F}_2 \right)$. □

Corollary 3. *Let $p \in [2, \infty)$ and $\alpha \in (0,1)$. The positive definite function $\Phi_\alpha(s_1, s_2) = \alpha^{|s_1|+|s_2|}$ can be extended to the state on $C^*_{\ell_p}(\mathbb{F}_2 \times \mathbb{F}_2)$ if and only if $\alpha < 3^{-\frac{1}{p}}$.*

Proof. Since

$$\begin{aligned} \sum_{k_1,k_2=1}^{+\infty} 3^{k_1+k_2-2} \alpha^{p(k_1+k_2)} &= 3^{-2} \sum_{k_1,k_2=1}^{+\infty} \left(3 \cdot \alpha^p \right)^{(k_1+k_2)} \\ &= 3^{-2} \left[\sum_{k_1=1}^{+\infty} \left(3 \cdot \alpha^p \right)^{k_1} \right] \left[\sum_{k_2=1}^{+\infty} \left(3 \cdot \alpha^p \right)^{k_2} \right], \end{aligned}$$

it follows from Theorem 4 (4) that we have

$$\begin{aligned} \Phi_\alpha \in l_p \left(\mathbb{F}_2 \times \mathbb{F}_2 \right) &\Leftrightarrow \forall \beta \in (0,1), (\alpha\beta)^{|s_1|+|s_2|} \in \ell_p(\mathbb{F}_2 \times \mathbb{F}_2) \\ &\Leftrightarrow \sum_{(s_1,s_2) \in \mathbb{F}_2 \times \mathbb{F}_2} (\alpha\beta)^{p(|s_1|+|s_2|)} < +\infty \\ &\Leftrightarrow \sum_{k_1,k_2=1}^{+\infty} \sum_{|s_1|=k_1,|s_2|=k_2} (\alpha\beta)^{p(|s_1|+|s_2|)} < +\infty \\ &\Leftrightarrow \sum_{k_1,k_2=1}^{+\infty} 3^{k_1+k_2-2} \alpha^{p(k_1+k_2)} < +\infty \\ &\Leftrightarrow 3\alpha^p < 1 \\ &\Leftrightarrow \alpha < 3^{-\frac{1}{p}}. \end{aligned}$$

□

Corollary 4. *For $2 \leq q < p \leq \infty$, the canonical quotient map from $C^*_{\ell_p}(\mathbb{F}_2 \times \mathbb{F}_2) \overset{onto}{\longrightarrow} C^*_{\ell_q}(\mathbb{F}_2 \times \mathbb{F}_2)$ is not injective. So*

$$C^*_{\ell_p}(\mathbb{F}_2) \overset{p}{\otimes} C^*_{\ell_p}(\mathbb{F}_2) \ncong C^*_{\ell_q}(\mathbb{F}_2) \overset{q}{\otimes} C^*_{\ell_q}(\mathbb{F}_2).$$

Proof. If $p = +\infty$ and $C^* \left(\mathbb{F}_2 \times \mathbb{F}_2 \right) = {C_q}^* \left(\mathbb{F}_2 \times \mathbb{F}_2 \right)$, we obtain that $\mathbb{F}_2 \times \mathbb{F}_2$ is amenable by Prop2.12 in [5]. This is a contradiction.

In the following, we consider $2 \leq q < p < +\infty$. Suppose that the canonical map from $C^*_{l_p}(\mathbb{F}_2 \times \mathbb{F}_2)$ onto $C^*_{l_q}(\mathbb{F}_2 \times \mathbb{F}_2)$ is injective from some $q < p$. Take a real number α with

$$3^{-\frac{1}{q}} < \alpha < 3^{-\frac{1}{p}}.$$

For $\Phi_\alpha(s_1, s_2) = \alpha^{|s_1|+|s_2|}$, by Corollary 3 we have

$$|\omega_{\Phi_\alpha}(f)| \leq \|f\|_{\ell_p} = \|f\|_{\ell_q}, for f \in c_c(\mathbb{F}_2 \times \mathbb{F}_2).$$

Therefore, it follows again that Φ_α can be extended to the state on $C^*_{l_q}(\mathbb{F}_2 \times \mathbb{F}_2)$, but it contradicts to the choice of α and Corollary 3. □

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References

1. Junge, M.; Pisier, G. Bilinear forms on exact operator spaces and $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$. *Geom. Funct. Anal.* **1995**, *5*, 329–363. [CrossRef]
2. Ozawa, N.; Pisier, G. A continuum of C*-norms on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\mathcal{H})$ and related tensor products. *preprint* **2014**. [CrossRef]
3. Wiersma, M. C*-norms for tensor products of discrete group C*-algebras. *Bull. Lond. Math. Soc.* **2015**, *47*, 219–224. [CrossRef]
4. Kirchberg, E. On nonsemisimple extensions, tensor products and exactness of group C*-algebras. *Invent. Math.* **1994**, *452*, 449–489.
5. Brown, N.P.; Guentner, E. New C*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.* **2013**, *45*, 1181–1193. [CrossRef]
6. Brown, N.P.; Ozawa, N. *C*-algebras and Finite Dimensional Approximations*; Graduate Studies in Math; American Mathematical Society: Providence, RI, USA, 2008; Volume 88.
7. Haagerup, U. An example of a nonnuclear C*-algebra which has the metric approximation property. *Invent. Math.* **1978**, *50*, 279–293. [CrossRef]
8. Okayasu, R. Free group C*-algebras associated with ℓ_p. *Int. J. Math.* **2014**, *25*, 65–66. [CrossRef]
9. Bekka, M.B.; Harpe, P.D.; Valette, A. *Kazhdan's Property (T), New Mathematical Monographs 11*; Cambridge University Press: Cambridge, UK, 2008.