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Article

Generating of Nonisospectral Integrable Hierarchies via the Lie-Algebraic Recursion Scheme

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Abstract: In the paper, we introduce an efficient method for generating non-isospectral integrable hierarchies, which can be used to derive a great many non-isospectral integrable hierarchies. Based on the scheme, we derive a non-isospectral integrable hierarchy by using Lie algebra and the corresponding loop algebra. It follows that some symmetries of the non-isospectral integrable hierarchy are also studied. Additionally, we also obtain a few conserved quantities of the isospectral integrable hierarchies.

Keywords: non-isospectral integrable hierarchy; Lie algebra; Hamiltonian structure; symmetry; conserved quantity

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1. Introduction

We have known that there exist two main approaches for constructing nonlinear systems integrable by the inverse scattering transform: the one of the Lax representation ($L_t = [A, L]$) and the one of the zero curvature representation ($U_t - V_x + [U, V] = 0$) [1,2]. In [3], the authors introduced the spectral transform technique to solve certain classes of nonlinear evolution equations, and gave a thorough account also of the non-isospectral deformations of KdV-like equations [4,5]. Magri once proposed one approach for generating integrable systems [6], which was called the Lax-pair method [7,8]. Based on it, Tu [9] proposed a method for generating integrable Hamiltonian hierarchies by making use of a trace identity, which was called the Tu scheme [10,11]. Through making use of the Tu scheme, some integrable systems and the corresponding Hamiltonian structures as well as other properties were obtained, such as the works in [12–16]. It is well known that many different methods for generating isospectral integrable equations have been proposed [17–19]. However, as far non-isospectral integrable equations are concerned, fewer works were presented, as far as we know. In [20,21], the author proposed a method of constructing its corresponding non-isospearal $\lambda_t = \lambda^n (n \geq 0)$ hierarchy of evolution equations closely related to τ-symmetries. Generally speaking, integrable systems correspond to the isospectral ($\lambda_t = 0$) case, and mastersymmetries of integrable systems correspond to the non-isospectral $\lambda_t = \lambda^n (n \geq 0)$ case. In [22], the author adopted the Lenard series method to obtain some non-isospectral integrable hierarchies under the case $\lambda_t = \lambda^{m+1} M$, and found that the same spectral problem can produce two different hierarchies of soliton evolution equations.

In this article, we apply an efficient scheme to generating non-isospectral integrable hierarchies of evolution equations under the case where $\lambda_t = \sum\limits_{j=0}^{n} k_j(t)\lambda^{n-j}$. Obviously, this case is a generalized expression for the case $\lambda_t = \lambda^n$ [23,24]. By taking different values of the parameters in the non-isospectral integrable hierarchies, we can obtain many integrable equations, such as the coupled equations. Under obtaining non-isospectral integrable systems, their properties including Darboux transformations, exact solutions, and so on, could be investigated; a lot of such work has been done, such as the papers [25–34].

2. A Non-Isospectral Integrable Hierarchy

In this section, we derive a non-isospectral integrable hierarchy by using the Lie algebra, and obtain a Hamiltonian construction of the hierarchy via the trace identity proposed by Tu [9]. In the following, the steps for generating non-isospectral integrable hierarchies of evolution equations present

Step 1: Introducing the spectral problems

$$\psi_x = U\psi,\ U = R + u_1 e_1(n) + \cdots + u_q e_q(n), \tag{1}$$

$$\psi_t = V\psi,\ V = A_1 e_1(n) + \cdots + A_p e_p(n), \tag{2}$$

$$\lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-N_i i}, \tag{3}$$

where the potential functions $u_1, \cdots, u_q \in S$(the Schwartz space), and $R(n)$, $e_1(n), \cdots, e_p(n) \in \widetilde{G}$ satisfy that

(a) $R, e_1, \cdots, e_p$ are linear independent,
(b) R is pseudoregular,
(c) $\deg(R(n)) \geq \deg(e_i(n)), i = 1, 2, \ldots, p.$

Step 2: Solving the following stationary zero curvature equation for $A_i, i = 1, 2, \ldots, p$:

$$V_x = \frac{\partial U}{\partial \lambda}\lambda_t + [U, V]. \tag{4}$$

It follows that one can get the compatibility condition of Equations (1) and (2)

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_t - V_x + [U, V] = 0. \tag{5}$$

Equation (4) can be broken down into

$$-V^{(n)}_{+,x} + \frac{\partial U}{\partial \lambda}\lambda^{(n)}_{t,+} + [U, V^{(n)}_+] = V^{(n)}_{-,x} - \frac{\partial U}{\partial \lambda}\lambda^{(n)}_{t,-} - [U, V^{(n)}_-], \tag{6}$$

where

$$\lambda^{(m)}_{t,+} = \lambda^{N_i m}\lambda_t - \lambda^{(m)}_{t,-} = \sum_{i=\mu}^{m} k_i(t)\lambda^{N_i m - N_i i + x},\ x = 0, 1, \cdots, N_i - 1; m < n.$$

Step 3: We search for a modified term $\triangle_n$ so that, for

$$V^{(n)} = (\lambda^{N_i n}V)_+ + \triangle_n =: V^{(n)}_+ + \triangle_n,$$

$$-V^{(n)}_x + \frac{\partial U}{\partial \lambda}\lambda^{(n)}_{t,+} + [U, V^{(n)}] = B_1 e_1 + \cdots + B_q e_q,$$

where $B_i(i = 1, 2, \ldots, q) \in C$.

Step 4: The non-isospectral integrable hierarchies of evolution equations could be deduced via the non-isospectral zero curvature equation

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda^{(n)}_{t,+} - V^{(n)}_x + [U, V^{(n)}] = 0. \tag{7}$$

Step 5: The Hamiltonian structures of the hierarchies Equation (7) are sought out according to the trace identity given by Tu [9]. We will show the specific calculation process in the following:

A basis of the Lie algebra A is given by

$$A = span\{h, e, f\}$$

with $h = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $e = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $f = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and the corresponding loop algebra is taken by

$$\widetilde{A} = span\{h(n), e(n), f(n)\},$$

where $h(n) = h\lambda^{2n}$, $e(n) = e\lambda^{2n-1}$, $f(n) = f\lambda^{2n-1}$. It is easy to find that the commutator of $\widetilde{A}$ is as follows:

$$[h(n), e(m)] = f\lambda^{2n+2m-1} = f(m+n),\ [h(n), f(m)] = e(m+n),\ [e(n), f(m)] = h(m+n-1),\ m, n \in Z,$$

where the gradations of $h(n)$, $e(n)$, and $f(n)$ are given by

$$\deg h(n) = 2n,\ \ \deg e(n) = 2n-1,\ \ \deg f(n) = 2n-1,\ n \in Z.$$

Consider the following non-isospectral problems based on $\widetilde{A}$

$$\psi_x = U\psi,\ \ U = h(1) + qe(1) + rf(1) = \frac{1}{2}\begin{pmatrix} \lambda^2 & (r+q)\lambda & 0 & 0 \\ (r-q)\lambda & -\lambda^2 & 0 & 0 \\ 0 & 0 & \lambda^2 & (r+q)\lambda \\ 0 & 0 & (r-q)\lambda & -\lambda^2 \end{pmatrix}, \tag{8}$$

$$\psi_t = V\psi,\ \ V = ah(0) + be(1) + cf(1) = \frac{1}{2}\begin{pmatrix} a & (b+c)\lambda & 0 & 0 \\ (-b+c)\lambda & -a & 0 & 0 \\ 0 & 0 & a & (b+c)\lambda \\ 0 & 0 & (-b+c)\lambda & -a \end{pmatrix}, \tag{9}$$

where $i^2 = -1$, $a = \sum\limits_{i\geq 0} a_i\lambda^{-2i}$, $b = \sum\limits_{i\geq 0} b_i\lambda^{-2i}$, $c = \sum\limits_{i\geq 0} c_i\lambda^{-2i}$.

It follows that we have

$$\begin{aligned} \frac{\partial U}{\partial \lambda}\lambda_t &= \frac{1}{2}\begin{pmatrix} 2\lambda & r+q & 0 & 0 \\ r-q & -2\lambda & 0 & 0 \\ 0 & 0 & 2\lambda & r+q \\ 0 & 0 & r-q & -2\lambda \end{pmatrix}\sum_{i\geq 0} k_i(t)\lambda^{-2i+1} \\ &= \sum_{i\geq 0} k_i(t)[2h(1-i) + qe(1-i) + rf(1-i)]. \end{aligned}$$

Furthermore, the following equation can be derived by taking $\lambda_t = \sum\limits_{i\geq 0} k_i(t)\lambda^{1-2i}$ with Equation (6),

$$\begin{cases} a_{ix} = qc_{i+1} - rb_{i+1} + 2k_{i+1}(t), \\ b_{ix} = c_{i+1} - ra_i + k_i(t)q, \\ c_{ix} = b_{i+1} - qa_i + k_i(t)r, \end{cases} \tag{10}$$

that is,

$$\begin{cases} a_{ix} = qb_{ix} - rc_{ix} - q^2k_i(t) + r^2k_i(t) + 2k_{i+1}(t), \\ c_{i+1} = b_{ix} + ra_i - qk_i(t), \\ b_{i+1} = c_{ix} + qa_i - rk_i(t). \end{cases} \tag{11}$$

We take the initial values

$$b_0 = k_0\partial^{-1}q, \;\; c_0 = k_0\partial^{-1}r.$$

Then, Equation (11) admits that

$$a_0 = 2k_1(t)x + \beta_0(t),$$
$$b_1 = 2k_1(t)qx, \quad c_1 = 2k_1(t)rx,$$
$$a_1 = k_1(t)x(q^2 - r^2) + 2k_2(t)x + \beta_1(t),$$
$$b_2 = k_1(t)(r + 2xr_x) + qx(k_1(t)q^2 - k_1(t)r^2 + 2k_2(t)),$$
$$c_2 = k_1(t)(q + 2xq_x) + rx(k_1(t)q^2 - k_1(t)r^2 + 2k_2(t)),$$
$$\cdots$$

where $\beta_0(t)\beta_1(t) = 0$ is an integral constant. Denoting that

$$V_+^{(n)} = \sum_{i=0}^{n}(a_ih(n-i) + b_ie(n+1-i) + c_if(n+1-i)),$$

$$V_-^{(n)} = \sum_{i=n+1}^{\infty}(a_ih(n-i) + b_ie(n+1-i) + c_if(n+1-i)),$$

$$\lambda_{t,+}^{(n)} = \sum_{i=0}^{n}K_i(t)\lambda^{2n-2i+1}, \quad \lambda_{t,-}^{(n)} = \sum_{i=n+1}^{\infty}K_i(t)\lambda^{2n-2i+1},$$

By using Equations (8) and (9), the gradations of the left-hand side of Equation (6) are derived as:

$$\deg V_+^{(n)} =: (0,1,1) \geq 0, \;\; \deg \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} =: (2,1,1) \geq 1, \;\; \deg([U, V_+^{(n)}]) =: (2,1,1;0,1,1) \geq 1,$$

which signifies that the minimum gradation of the left-hand side of Equation (6) is zero. Similarly, the gradations of the right-hand side of Equation (6) are also obtained as follows:

$$\deg V_-^{(n)} =: (-2,-1,-1) \leq -1, \;\; \deg \tfrac{\partial U}{\partial \lambda}\lambda_{t,-}^{(n)} =: (0,-1,-1) \leq 0, \; \deg([U, V_-^{(n)}]) =: (2,1,1;-2,-1,-1) \leq 1,$$

which indicates that the maximum gradation of the right-hand side of Equation (6) is 1. By taking these terms which have the gradations 0 and 1, one has

$$V_{-,x}^{(n)} - \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(n)} - [U, V_-^{(n)}] = -b_{n+1}f(1) - c_{n+1}e(1) - qc_{n+1}h(0) + rb_{n+1}h(0) - 2K_{n+1}(t)h(0),$$

that is,

$$-V_{+,x}^{(n)} + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} + [U, V_+^{(n)}] = -b_{n+1}f(1) - c_{n+1}e(1) - qc_{n+1}h(0) + rb_{n+1}h(0) - 2K_{n+1}(t)h(0). \tag{12}$$

In what follows, we takes modified term $\triangle_n = -a_nh(0)$ so that, for $V^{(n)} = V_+^{(n)} - a_nh(0)$ to obtain the non-isospectral integrable hierarchies, we have from Equation (13) that

$$-V_x^{(n)} + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(n)} + [U, V^{(n)}] = (-c_{n+1} + ra_n)e(1) + (-b_{n+1} + qa_n)f(1).$$

Thus, Equation (7) admits the non-isospectral integrable hierarchy

$$
\begin{aligned}
u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} &= \begin{pmatrix} c_{n+1} - ra_n \\ b_{n+1} - qa_n \end{pmatrix} = \begin{pmatrix} b_{nx} - K_n(t)q \\ c_{nx} - K_n(t)r \end{pmatrix} \\
&= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} c_n \\ b_n \end{pmatrix} - K_n(t) \begin{pmatrix} q \\ r \end{pmatrix} \\
&=: J_1 \begin{pmatrix} c_n \\ b_n \end{pmatrix} - K_n(t) \begin{pmatrix} q \\ r \end{pmatrix},
\end{aligned} \tag{13}
$$

or

$$
\begin{aligned}
u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} &= \begin{pmatrix} r\partial^{-1} r b_{n+1} + (1 - r\partial^{-1} q) c_{n+1} - 2rK_{n+1}(t)x \\ -q\partial^{-1} q c_{n+1} + (1 + q\partial^{-1} r) b_{n+1} - 2qK_{n+1}(t)x \end{pmatrix} \\
&= \begin{pmatrix} 1 - r\partial^{-1} q & r\partial^{-1} r \\ -q\partial^{-1} q & 1 + q\partial^{-1} r \end{pmatrix} \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} + 2K_{n+1}(t)x \begin{pmatrix} -r \\ -q \end{pmatrix} \\
&=: J_2 \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} + 2K_{n+1}(t)x \begin{pmatrix} -r \\ -q \end{pmatrix},
\end{aligned} \tag{14}
$$

where

$$
J_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 - r\partial^{-1} q & r\partial^{-1} r \\ -q\partial^{-1} q & 1 + q\partial^{-1} r \end{pmatrix}.
$$

From Equation (11), we infer that

$$
\begin{aligned}
\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} &= \begin{pmatrix} -r\partial^{-1} r\partial & \partial + r\partial^{-1} q\partial \\ \partial - q\partial^{-1} r\partial & q\partial^{-1} q\partial \end{pmatrix} \begin{pmatrix} c_n \\ b_n \end{pmatrix} + K_n(t) \begin{pmatrix} r\partial^{-1}(-q^2 + r^2) - q \\ q\partial^{-1}(-q^2 + r^2) - r \end{pmatrix} + 2K_{n+1}(t)x \begin{pmatrix} r \\ q \end{pmatrix} \\
&=: L \begin{pmatrix} c_n \\ b_n \end{pmatrix} + K_n(t)Q + 2K_{n+1}(t)xR,
\end{aligned} \tag{15}
$$

where

$$
L = \begin{pmatrix} -r\partial^{-1} r\partial & \partial + r\partial^{-1} q\partial \\ \partial - q\partial^{-1} r\partial & q\partial^{-1} q\partial \end{pmatrix}, \quad Q = \begin{pmatrix} r\partial^{-1}(-q^2 + r^2) - q \\ q\partial^{-1}(-q^2 + r^2) - r \end{pmatrix}, \quad R = \begin{pmatrix} r \\ q \end{pmatrix}.
$$

Therefore, Equation (13) can be written as

$$
\begin{aligned}
u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} &= J_1 L^n \begin{pmatrix} K_0 \partial^{-1} r \\ K_0 \partial^{-1} q \end{pmatrix} + J_1 \sum_{i=0}^{n-1} (L^i K_{n-1-i}(t) Q) + 2J_1 \sum_{i=0}^{n-1} L^i K_{n-i}(t) xR - K_n(t) \begin{pmatrix} q \\ r \end{pmatrix} \\
&= \Phi^n K_0 \begin{pmatrix} q \\ r \end{pmatrix} + \sum_{i=0}^{n-1} \Phi^i J_1 K_{n-1-i}(t) Q + 2 \sum_{i=0}^{n-1} K_{n-i}(t) \Phi^i \partial \begin{pmatrix} xq \\ xr \end{pmatrix} - K_n(t) \begin{pmatrix} q \\ r \end{pmatrix},
\end{aligned} \tag{16}
$$

where

$$
\Phi = J_1 L J_1^{-1} = \begin{pmatrix} q_x \partial^{-1} q + q^2 & \partial - q_x \partial^{-1} r - qr \\ \partial + r_x \partial^{-1} q + qr & -r_x \partial^{-1} r - r^2 \end{pmatrix}. \tag{17}
$$

When $n = 1$, the non-isospectral integrable hierarchy Equation (16) becomes

$$
\begin{cases} q_t = K_1(2xq_x + q), \\ r_t = K_1(2xr_x + r). \end{cases} \tag{18}
$$

When $n=2$, the non-isospectral integrable hierarchy Equation (16) reduces to

$$\begin{cases} q_t = K_1(q^3x - qr^2x + r + 2r_x x)_x + 2K_2(qx)_x + K_2(2xq_x + q), \\ r_t = K_1(-r^3x + rq^2x + q + 2q_x x)_x + K_2(2xr_x + r) \end{cases} \tag{19}$$

Furthermore, we focus on a format of Hamiltonian constructure of the hierarchy Equation (16) via the trace identity proposed by Tu [9]. Denoting the trace of the square matrices A and B by $< A, B >= tr(AB)$.

Equations (8) and (9) admit that

$$< V, \frac{\partial U}{\partial q} >= -b\lambda^2, \ < V, \frac{\partial U}{\partial r} >= c\lambda^2, \ < V, \frac{\partial U}{\partial \lambda} >= cr\lambda + 2a\lambda - bq\lambda,$$

which can be substituted into the trace identity

$$\frac{\delta}{\delta u}(< V, \frac{\partial U}{\partial \lambda} >) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} < V, \frac{\partial U}{\partial q} > \\ < V, \frac{\partial U}{\partial r} > \end{pmatrix}$$

gives rise to

$$\frac{\delta}{\delta u}(cr\lambda + 2a\lambda - bq\lambda) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\begin{pmatrix} -b\lambda^{2+\gamma} \\ c\lambda^{2+\gamma} \end{pmatrix}. \tag{20}$$

It follows that one can get the following equation by comparing the two sides of the above formula

$$\frac{\delta}{\delta u}(2a_n - qb_n + rc_n) = (2 - 2n + \gamma)\begin{pmatrix} -b_n \\ c_n \end{pmatrix}. \tag{21}$$

Inserting the initial values of Equations (11) into (21), we obtain $\gamma = 0$. Hence, we have

$$\begin{pmatrix} -b_n \\ c_n \end{pmatrix} = \frac{\delta H_n}{\delta u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_n \\ b_n \end{pmatrix} =: M_1\begin{pmatrix} c_n \\ b_n \end{pmatrix},$$

where

$$H_n = \frac{2\bar{i}a_n - qb_n - rc_n}{2n-2}, \ M_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, the hierarchy Equations (13) and (14) can be written as

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J_1 M_1^{-1}\frac{\delta H_n}{\delta u} - K_n(t)\begin{pmatrix} q \\ r \end{pmatrix} = J_2 M_1^{-1}\frac{\delta H_{n+1}}{\delta u} + 2K_{n+1}(t)x\begin{pmatrix} -r \\ -q \end{pmatrix}. \tag{22}$$

It is remarkable that, when $K_n(t) = K_{n+1}(t) = 0$, Equation (22) is the Hamiltonian structure of the corresponding isospectral integrable hierarchy of Equation (16).

3. Discussion on Symmetries and Conserved Quatities

In this section, we consider the K symmetries and τ symmetries of the hierarchy Equation (16), and obtain some conserved quantities of the hierarchy Equation (16) from the obtained symmetries. The way to find K symmetries and τ symmetries comes from Li and Zhu [14], who applied the isospectral and non-isospectral integrable AKNS hierarchy to construct K symmetries and τ symmetries which constitute an infinite-dimensional Lie algebra. In the following, we show the specific process.

One can find that the Φ presented in Equation (17) satisfies

$$\Phi'[\Phi f]g - \Phi'[\Phi g]f = \Phi\{\Phi'[f]g - \Phi'[g]f\},$$

for $\forall f, g \in S$. Therefore, Φ is the hereditary symmetry of Equation (16). In addition, we can also prove the following relation holding:

Proposition 1.

$$\Phi'[K_0] = [K_0', \Phi], \tag{23}$$

where $K_0 = \begin{pmatrix} q_x \\ r_x \end{pmatrix} = u_{t_0}$.

In fact,

$$\Phi'[K_0] = \partial \begin{pmatrix} q_x\partial^{-1}q + q\partial^{-1}q_x & -q_x\partial^{-1}r - q\partial^{-1}r_x \\ r_x\partial^{-1}q + r\partial^{-1}q_x & -r_x\partial^{-1}r - r\partial^{-1}r_x \end{pmatrix},$$

and thus

$$\Phi'[K_0] = \begin{pmatrix} q_{xx}\partial^{-1}q + (q^2)_x + q_x\partial^{-1}q_x & -q_{xx}\partial^{-1}r - (qr)_x - q_x\partial^{-1}r_x \\ r_{xx}\partial^{-1}q + (qr)_x + r_x\partial^{-1}q_x & -r_{xx}\partial^{-1}r - (r^2)_x - r_x\partial^{-1}r_x \end{pmatrix},$$

$$\begin{aligned} K_0'\Phi &= \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} q_x\partial^{-1}q + q^2 & \partial - q_x\partial^{-1}r - qr \\ \partial + r_x\partial^{-1}q + qr & -r_x\partial^{-1}r - r^2 \end{pmatrix} \\ &= \begin{pmatrix} q_{xx}\partial^{-1}q + 3qq_x + q^2\partial & \partial^2 - q_{xx}\partial^{-1}r - 2q_xr - qr_x - qr\partial \\ \partial^2 + r_{xx}\partial^{-1}q + 2r_xq + rq_x + qr\partial & -r_{xx}\partial^{-1}r - 3rr_x - r^2\partial \end{pmatrix}. \end{aligned}$$

$$\Phi K_0' = \begin{pmatrix} q_x\partial^{-1}q\partial + q^2\partial & \partial^2 - q_x\partial^{-1}r\partial - qr\partial \\ \partial^2 + r_x\partial^{-1}q\partial + qr\partial & -r_x\partial^{-1}r\partial - r^2\partial \end{pmatrix}$$

We therefore verified that Equation (23) is correct. Owing to the Φ is a hereditary symmetry, one finds

$$\Phi'[K_m] = [K_m', \Phi],$$

which means Φ is a strong symmetry, where $K_m = \Phi^m \begin{pmatrix} q_x \\ r_x \end{pmatrix}$.

Proposition 2.

$$\Phi'[xu] + \Phi(xu)' - (xu)'\Phi = HI, \tag{24}$$

where $u = \begin{pmatrix} q_x \\ r_x \end{pmatrix}$, $H = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$ *and I is an identity matrix.*

In fact,

$$\Phi'[xu] = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{cases} A = q_x\partial^{-1}q + xq_{xx}\partial^{-1}q + 2xq_xq + q_x\partial^{-1}xq_x, \\ B = -(q_x\partial^{-1}r + xq_{xx}\partial^{-1}r + xq_xr + xqr_x + q_x\partial^{-1}xr_x), \\ C = r_x\partial^{-1}q + xr_{xx}\partial^{-1}q + xr_xq + xrq_x + r_x\partial^{-1}xq_x, \\ D = -(r_x\partial^{-1}r + xr_{xx}\partial^{-1}r + 2xr_xr + r_x\partial^{-1}xr_x). \end{cases}$$

$$\Phi(xu)' = \begin{pmatrix} xq^2\partial + xqq_x - q_x\partial^{-1}(q + xq_x) & -xqr\partial + \partial + x\partial^2 - xrq_x + q_x\partial^{-1}(r + xr_x) \\ xqr\partial + \partial + x\partial^2 + xqr_x - r_x\partial^{-1}(q + xq_x) & -xr^2\partial - xrr_x + r_x\partial^{-1}(r + xr_x) \end{pmatrix},$$

$$(xu)'\Phi = \begin{pmatrix} xq_{xx}\partial^{-1}q + 3xqq_x + xq^2\partial & x\partial^2 - xq_{xx}\partial^{-1}r - 2xrq_x - xqr_x - xqr\partial \\ x\partial^2 + xr_{xx}\partial^{-1}q + 2xqr_x + xrq_x + xqr\partial & -xr_{xx}\partial^{-1}r - 3xrr_x - xr^2\partial \end{pmatrix},$$

where

$$(xu)'[\sigma] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \begin{pmatrix} x(q+\epsilon\sigma_1)_x \\ x(q+\epsilon\sigma_2)_x \end{pmatrix} = x\partial \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \implies (xu)' = \begin{pmatrix} x\partial & 0 \\ 0 & x\partial \end{pmatrix}.$$

Thus, Equation (24) holds.

Proposition 3.

$$[K_1, xu] = [\Phi u, xu] = Hu + K_1, \tag{25}$$

where $u = \begin{pmatrix} q_x \\ r_x \end{pmatrix}$, $H = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$ *and* $K_1 = \Phi u$.

In fact,

$$\Phi u = \begin{pmatrix} r_{xx} + \frac{1}{2}q_x(q^2 - r^2) - qrr_x + q^2q_x \\ q_{xx} + \frac{1}{2}r_x(q^2 - r^2) + qrq_x - r^2r_x \end{pmatrix},$$

$$(\Phi u)' = \begin{pmatrix} \frac{1}{2}(q^2 - r^2)\partial + 3qq_x + q^2\partial - rr_x & \partial^2 - qr\partial - (qr)_x \\ \partial^2 + qr\partial + (qr)_x & \frac{1}{2}(q^2 - r^2)\partial - 3rr_x - r^2\partial + qq_x \end{pmatrix},$$

$$(\Phi u)' \begin{pmatrix} xq_x \\ xr_x \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(q^2 - r^2)\partial(xq_x) + 3xqq_x^2 + q^2\partial(xq_x) - xrr_xq_x + \partial^2(xr_x) - qr\partial(xr_x) - xr_x(qr)_x \\ \partial^2(xq_x) + qr\partial(xq_x) + xq_x(qr)_x + \frac{1}{2}(q^2 - r^2)\partial(xr_x) - 3xrr_x^2 - r^2\partial(xr_x) + xr_xqq_x \end{pmatrix},$$

Then, we have

$$(xu)'[\Phi u] = \begin{pmatrix} x\partial(r_{xx} + \frac{1}{2}q_x(q^2 - r^2) - qrr_x + q^2q_x) \\ x\partial(q_{xx} + \frac{1}{2}r_x(q^2 - r^2) + qrq_x - r^2r_x) \end{pmatrix},$$

$$[\Phi u, xu] = (\Phi u)'[xu] - (xu)'[\Phi u] = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} q_x \\ r_x \end{pmatrix} + \begin{pmatrix} r_{xx} + \frac{1}{2}q_x(q^2 - r^2) - qrr_x + q^2q_x \\ q_{xx} + \frac{1}{2}r_x(q^2 - r^2) + qrq_x - r^2r_x \end{pmatrix} = Hu + K_1.$$

We therefore verified that Equation (25) is correct.

Proposition 4.

$$[K_m, K_n] = 0, \quad m, n = 0, 1, 2, \cdots \tag{26}$$

where $K_m = \Phi^m u$, $K_n = \Phi^n u$.

Proposition 5.

$$[\Phi^m xu, xu] = m\Phi^{m-1}(xu).$$

The proofs of Propositions 4 and 5 were presented in [23].

From the above results, one have

$$[\Phi^m xu, \Phi^n xu] = (m - n)\Phi^{m+n-1}(xu), \quad m = 0, 1, 2, \cdots; n = 0, 1, 2, \cdots.$$

We can find that $\{\Phi^n u, \Phi^m xu\}$ *can not constitute a Lie algebra from Equation (25). However,* $\{\Phi^n u, n = 0, 1, 2, \cdots\}$ *and* $\{\Phi^n xu, n = 0, 1, 2, \cdots\}$ *constitute the infinite-dimensional Lie algebra, respectively based on the above analysis.*

Now, we will derive some conserved quantities of Tu isospectral hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \Phi^n \begin{pmatrix} q_x \\ r_x \end{pmatrix}. \tag{27}$$

Definition 1 ([14,16]). *If we have known the integrable hierarchy $u_t = K_n(u)$, then the v satisfied the following equation:*

$$\frac{dv}{dt} + K'^{*}v = 0, \tag{28}$$

which is called the conserved covariance, where K' is the linearized operator of K, and K'^{*} denotes a conjugate operator of K'.

Proposition 6 ([13,16]). *If σ is a symmetry of Equation$u_t = K_n(u)$, v is its conserved covariance, then we have*

$$\int_{-\infty}^{\infty} v\sigma dx =< v, \sigma >,$$

which is independent of time t, that is, $\frac{d}{dt} < v, \sigma >= 0$.

Definition 2 ([13,14,16]). *If $F'f =< v, f >$, for $\forall f \in S$, then v is called the gradient of the functional F, which is denoted by $v = \frac{\delta F}{\delta u}$.*

Proposition 7 ([16]). *If $v' = v'^{*}$, then v is the gradient of the following functional*

$$F = \int_0^1 < v(\lambda u), u > d\lambda. \tag{29}$$

According to the symbols above, we can deduce:

Proposition 8 ([13,14]). *If I is a conserved quality of the hierarchy $u_t = K_n(u)$, and the conserved covariance v satisfies*

$$I'K_n =< v, K_n >,$$

then one has

$$\frac{\partial I}{\partial t} + < v, K_n >= 0,$$

that is,

$$\frac{\partial v}{\partial t} + K_n'^{*}v + v'K_n = 0.$$

Therefore, we deduce the following conserved quantities related to the integrable hierarchy $u_t = K_n(u)$

$$I_m = \int_0^1 < \partial_x^{-1}K_m(\lambda u), u > d\lambda. \tag{30}$$

Moreover, a few conserved quantities are also derived for the integrable hierarchy Equation (27) as follows:

$$K_0 = \Phi^0 u = \begin{pmatrix} q_x \\ r_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -r_x \\ q_x \end{pmatrix},$$

$$I_0 = \int_0^1 < \partial_x^{-1}K_0(\lambda u), u > d\lambda = \int_0^1 < [\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_x\lambda \\ r_x\lambda \end{pmatrix}]^T, \begin{pmatrix} q \\ r \end{pmatrix} > d\lambda = \int_{-\infty}^{\infty} (q_x r - r_x q) dx,$$

$$K_1 = \Phi u = \begin{pmatrix} r_{xx} + \frac{1}{2}q_x(q^2 - r^2) - qrr_x + q^2 q_x \\ q_{xx} + \frac{1}{2}r_x(q^2 - r^2) + qrq_x - r^2 r_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -q_{xx} - \frac{1}{2}r_x(q^2 - r^2) - qrq_x + r^2 r_x \\ r_{xx} + \frac{1}{2}q_x(q^2 - r^2) - qrr_x + q^2 q_x \end{pmatrix},$$

$$\begin{aligned} I_1 &= \int_0^1 < [\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{xx}\lambda + \frac{1}{2}q_x(q^2 - r^2)\lambda^3 - qrr_x\lambda^3 + q^2 q_x \lambda^3 \\ q_{xx}\lambda + \frac{1}{2}r_x(q^2 - r^2)\lambda^3 + qrq_x\lambda^3 - r^2 r_x \lambda^3 \end{pmatrix}]^T, \begin{pmatrix} q \\ r \end{pmatrix} > d\lambda \\ &= \int_{-\infty}^{\infty} [\frac{1}{2}(-qq_{xx} + rr_{xx}) + \frac{1}{8}(q^2 - r^2)(q_x r - r_x q)] dx, \end{aligned}$$

$$\vdots$$

$$I_k = \int_{-\infty}^{\infty} < \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K_k(\lambda u), \begin{pmatrix} q \\ r \end{pmatrix} > d\lambda.$$

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