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Almost Sure Convergence for the Maximum and Minimum of Normal Vector Sequences

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Abstract: In this paper, we prove the almost sure convergences for the maximum and minimum of nonstationary and stationary standardized normal vector sequences under some suitable conditions.

Keywords: extreme value; almost sure central limit theorem; multivariate vectors; maximum and minimum; nonstationary and stationary normal sequences

1. Introduction

The extreme phenomena in nature and human society can be explored by the classical extreme value theory [1–3]. Almost sure convergence shows a nice behavior of the various ways of convergences [4–6]. Brosamler and Schatte firstly put forward the almost sure central limit theorem (ASCLT) on partial sums for independent identically distributed (i.i.d.) random variables [7,8]. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E(X_n) = 0, Var(X_n) = 1$ and $T_n = \sum_{k=1}^{n} X_k$. Under some regularity conditions, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left(\frac{T_k}{\sqrt{k}} \le x\right) = \Phi(x) \;\; a.s., \tag{1}$$

for any x, where I denote the indicator function and $\Phi(x)$ stands for the standard normal distribution function. Later, Ibragimov and Lifshits extend Equation (1) to the functional form [9]. Cheng et al. [10], Fahrnar and Stadtmüller [6] and Berkes and Csáki [11] respectively consider the ASCLT on maximum of i.i.d random variables. Csáki and Gondigdanzan investigate the ASCLT for the maximum of a stationary weakly dependent Gaussian sequences [12]. Chen and Lin extend the ASCLT to nonstationary Gaussian sequences [13]. Chen et al. provide an ASCLT for the maxima of multivariate stationary Gaussian sequences under some mild conditions [14]. Zhao et al. explore the ASCLT for the maxima and sum of a nonstationary Gaussian vector sequence [15]. Weng et al. put forward an ASCLT for the maxima and minima of a strongly dependent stationary Gaussian vector sequence [16].

The purpose of this paper is to extend the result of the ASCLT for the maximum and minimum to multivariate general normal vector sequences, which include the two cases of nonstationary and stationary, under some suitable conditions. Throughout this paper, $\{X_1, X_2, \ldots\}$ is a standardized nonstationary Gaussian sequence of d-dimensional random vectors (i.e., each component of the random vectors has a zero mean and a unit standard deviation). The covariance matrix is denoted by

$$r_{ij}(p) = Cov(X_i(p), X_j(p)), \;\; r_{ij}(p,q) = Cov(X_i(p), X_j(q))$$

such that $|r_{ij}(p)| \leq \rho_{|i-j|}(p)$ and $|r_{ij}(p,q)| \leq \rho_{|i-j|}(p,q)$ where

$$\sup_{1\leq p\leq d} \rho_n(p) < 1, \quad \sup_{1\leq p\neq q\leq d} \rho_n(p,q) < 1$$

for $n \geq 1$.

We set

$$M_{k,n} = \big(M_{k,n}(1), \ldots, M_{k,n}(d)\big), \ \ M_{k,n}(p) = \max_{k+1\leq i\leq n} X_i(p),$$

especially

$$M_n = M_{0,n}, \ \ M_n(p) = M_{0,n}(p)$$

for $p = 1, \ldots, d$. The level $u_n = \big(u_n(1), \ldots, u_n(d)\big)$ and $v_n = \big(v_n(1), \ldots, v_n(d)\big)$ are two real vectors. The expression $u_n > v_n$ implies $u_n(p) > v_n(p)$ for all $p = 1, \ldots, d$ and $a \ll b$ stands for $a = O(b)$. Finally, we write $a_n = (2\log n)^{\frac{1}{2}}$ and $b_n = a_n - \frac{1}{2}a_n^{-1}\log(4\pi\log n)$.

2. Results

Theorem 1. *Let $\{X_n\}_{n=1}^{\infty}$ be a standardized nonstationary normal d-dimensional vector sequence satisfying*

(a) $\delta = \max_{p\neq q}\big(\sup_{n\geq 1}\big(|r_n(p)|, |r_n(p,q)|\big)\big) < 1$;

(b) there exists $\gamma \geq \frac{2(1+\delta)}{1-\delta}$, such that

$$\frac{1}{n^2}\sum_{p=1}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p)|\exp\big(\gamma|r_{ij}(p)|\log(j-i)\big) \ll (\log\log n)^{-(1+\varepsilon)}, \tag{2}$$

$$\frac{1}{n^2}\sum_{1\leq p\neq q\leq n}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p,q)|\exp\big(\gamma|r_{ij}(p,q)|\log(j-i)\big) \ll (\log\log n)^{-(1+\varepsilon)} \tag{3}$$

where $\varepsilon > 0$.

Suppose that the levels $u_n(p)$ and $v_n(p)$ satisfy $n\,(1-\Phi(u_n(p))) \to \tau_p$, $n\Phi(v_n(p)) \to \eta_p$ for $0 \leq \tau_p, \eta_p < \infty$ and $p = 1, 2, \ldots, d$, then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I(v_k < m_k \leq M_k \leq u_k) = \prod_{p=1}^{d}\exp(-(\tau_p+\eta_p)) \quad a.s. \tag{4}$$

Especially, let $u_n(p) = a_n^{-1}x_p + b_n$ and $v_n(p) = -a_n^{-1}y_p - b_n$, where x_p and y_p are real numbers for $p = 1, 2, \ldots, d$, then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I(v_k < m_k \leq M_k \leq u_k) = \prod_{p=1}^{d}\exp\big(-(e^{-x_p}+e^{-y_p})\big) \quad a.s. \tag{5}$$

Corollary 1. *Under the conditions of Theorem 1, if the levels $u_n(p)$ satisfies $n\,(1-\Phi(u_n(p))) \to \tau_p$ as $n \to \infty$, then*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left(|M_k| \leq u_k\right) = \prod_{p=1}^{d}\exp(-2\tau_p). \tag{6}$$

Especially, the level $u_n(p)$ satisfies $u_n(p) = a_n^{-1}x_p + b_n$ for $p = 1, 2, \ldots, d$, then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left(|M_k| \leq u_k\right) = \prod_{p=1}^{d}\exp(-2e^{-x_p}). \tag{7}$$

Theorem 2. *Let $\{X_n\}_{n=1}^{\infty}$ be a standardized nonstationary normal d-dimensional vector sequence satisfying*

$$\rho_n(p)\log n(\log\log n)^{-(1+\varepsilon)} = O(1), \quad \rho_n(p,q)\log n(\log\log n)^{-(1+\varepsilon)} = O(1). \tag{8}$$

If $n\left(1-\Phi(u_n(p))\right) \to \tau_p$ and $n\Phi\left(v_n(p)\right) \to \eta_p$ as $n \to \infty$ for $0 \le \tau_p, \eta_p < \infty$ and $\varepsilon > 0$, then (4) holds.

Especially, set $u_n(p) = a_n^{-1}x_p + b_n$ and $v_n(p) = -a_n^{-1}y_p - b_n$, where x_p and y_p are real numbers for $p = 1, 2, \ldots, d$, then (5) holds.

Theorem 3. *Let $Z_1, Z_2, \ldots$ be a standardized stationary normal sequence of d-dimensional random vectors satisfying*

(a) $r_n(p,q) \to 0$ and $r_n(p) \to 0$ for $1 \le p \ne q \le d$ as $n \to \infty$,

(b) there exists $\gamma \ge \frac{2(1+\delta)}{1-\delta}$ with $\delta = \max\limits_{p\ne q}\left(\sup\limits_{n\ge 1}\left(|r_n(p)|, |r_n(p,q)|\right)\right) < 1$, such that

$$\frac{1}{n}\sum_{p=1}^{d}\sum_{k=1}^{n}|r_k(p)|\log k\exp(\gamma|r_k(p)|\log k) \ll (\log\log n)^{-(1+\varepsilon)}, \tag{9}$$

$$\frac{1}{n}\sum_{1\le p\ne q\le d}\sum_{k=1}^{n}|r_k(p,q)|\log k\exp(\gamma|r_k(p,q)|\log k) \ll (\log\log n)^{-(1+\varepsilon)}. \tag{10}$$

If $n\left(1-\Phi(u_n(p))\right) \to \tau_p$ and $n\Phi(v_n(p)) \to \eta_p$ as $n \to \infty$ for $0 \le \tau_p, \eta_p < \infty$ and $\varepsilon > 0$, then (4) holds.

Especially, set $u_n(p) = a_n^{-1}x_p + b_n$ and $v_n(p) = -a_n^{-1}y_p - b_n$, where x_p and y_p are real numbers for $p = 1, \ldots, d$, then (5) holds.

Theorem 4. *Let $Z_1, Z_2, \ldots$ be a standardized stationary normal sequence d-dimensional random vectors satisfying*

$$r_n(p)\log n(\log\log n)^{1+\varepsilon} = O(1), \ \ r_n(p,q)\log n(\log\log n)^{1+\varepsilon} = O(1), \ \ 1 \le p \ne q \le d. \tag{11}$$

If $n\left(1-\Phi(u_n(p))\right) \to \tau_p$ and $n\Phi\left(v_n(p)\right) \to \eta_p$ as $n \to \infty$ for $0 \le \tau_p, \eta_p < \infty$ and $\varepsilon > 0$, then (4) holds.

Especially, set $u_n(p) = a_n^{-1}x_p + b_n$ and $v_n(p) = -a_n^{-1}y_p - b_n$, where x_p and y_p are real numbers for $p = 1, \ldots, d$, then (5) holds.

Notice: We replace the nonstationary sequence $\{X_n\}_{n=1}^{\infty}$ with the stationary sequence $\{Z_n\}_{n=1}^{\infty}$ in Theorem 3 and 4. The symbols of $\{X_n\}_{n=1}^{\infty}$ are used to denote the random vector sequence $\{Z_n\}_{n=1}^{\infty}$ in the two theorems without ambiguities.

3. Proofs of the Main Results

In the section, we present and prove some lemmas which are useful in the proofs of the main results.

Lemma 1. *Let $\{\xi_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ be standardized nonstationary normal sequences of d-dimensional random vectors with $r_{ij}^0(p) = Cov(\xi_i(p), \xi_j(p))$, $r_{ij}^0(p,q) = Cov(\xi_i(p), \xi_j(q))$ and $r_{ij}^*(p) = Cov(\eta_i(p), \eta_j(p))$, $r_{ij}^*(p,q) = Cov(\eta_i(p), \eta_j(q))$. Denote $\rho_{ij}(p) = \max(|r_{ij}^0(p)|, |r_{ij}^*(p)|)$, $\rho_{ij}(p,q) =$*

$\max\left(|r_{ij}^0(p,q)|, |r_{ij}^*(p,q)|\right)$ *and let* $\{u_n\}, \{v_n\}$ *be real vectors. If* $\max_{p\neq q}\sup_{n\geq 1}\left(|r_n(p)|, |r_n(p,q)|\right) = \delta < 1$ *and* $\omega_{ni}(p) = \min\left(|u_{ni}(p)|, |v_{ni}(p)|\right)$*, then*

$$\begin{aligned}
&\Big|\mathbb{P}\big(\bigcap_{j=1}^{n}(-v_{nj} < \xi_j \leq u_{nj})\big) - \mathbb{P}\big(\bigcap_{j=1}^{n}(-v_{nj} < \eta_j \leq u_{nj})\big)\Big| \\
&\leq K_1 \sum_{p=1}^{d}\sum_{1\leq i<j\leq n}|r_{ij}^0(p) - r_{ij}^*(p)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(p)}{2\left(1+\rho_{ij}(p)\right)}\Big) \\
&\quad + K_2 \sum_{1\leq p\neq q\leq d}\sum_{1\leq i<j\leq n}|r_{ij}^0(p,q) - r_{ij}^*(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2\left(1+\rho_{ij}(p,q)\right)}\Big)
\end{aligned}$$

with the positive constants K_1, K_2 *which depend on* δ.

Proof. It follows from Theorem 11.1.2 in Leadbetter et al. [17]. □

Lemma 2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a standardized nonstationary normal d-dimensional vector sequence satisfying the conditions (a) and (b) of Theorem 1, then*

$$\sum_{1\leq p\neq q\leq d}\sum_{1\leq i<j\leq n}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+r_{ij}(p,q))}\Big) \ll (\log\log n)^{-(1+\varepsilon)}, \tag{12}$$

$$\sum_{p=1}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(p)}{2(1+r_{ij}(p))}\Big) \ll (\log\log n)^{-(1+\varepsilon)}. \tag{13}$$

Proof. Firstly, we peove Equation (12). This sum can be divided into two terms T_1 and T_2,

$$\begin{aligned}
&\sum_{1\leq p\neq q\leq d}\sum_{1\leq i<j\leq n}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+|r_{ij}(p,q)|)}\Big) \\
= &\sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n \\ j-i\leq n^{\frac{2}{\gamma}}}}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+|r_{ij}(p,q)|)}\Big) \\
&+\sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n \\ j-i> n^{\frac{2}{\gamma}}}}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+|r_{ij}(p,q)|)}\Big) \\
\triangleq\ & T_1 + T_2.
\end{aligned}$$

Since $\exp\left(\frac{u_n^2(p)}{2}\right) \sim \frac{\sqrt{\log n}}{n}$, we have $\omega_{ni}(p) = \min\left(|u_{ni}(p)|, |v_{ni}(p)|\right) \sim \frac{\sqrt{\log n}}{n}$. Let $\beta = \frac{2}{\gamma}$, that is $0 < \beta < \frac{1-\delta}{1+\delta}$, then the first term T_1

$$\begin{aligned}
T_1 &\leq \sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n \\ j-i\leq n^{\beta}}}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+\delta)}\Big) \\
&\ll n^{1+\beta}\left(n^{-2}\log n\right)^{\frac{1}{1+\delta}} \\
&= n^{1+\beta-\frac{2}{1+\delta}}(\log n)^{\frac{1}{1+\delta}}.
\end{aligned}$$

As $1+\beta-\frac{2}{1+\delta} < 0$, we get

$$T_1 \ll (\log\log n)^{-(1+\varepsilon)}. \tag{14}$$

Note that $j-i>n^{\beta}$, we have $\log n<\log(j-i)/\beta$. Then, we consider the second part T_2,

$$\begin{aligned}
T_2 &\leq \sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n\\ j-i>n^{\beta}}}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+|r_{ij}(p,q)|)}\Big)\\
&\ll \sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n\\ j-i>n^{\beta}}}|r_{ij}(p,q)|\big(n^{-2}\log n\big)^{\frac{1}{1+|r_{ij}(p,q)|}}\\
&= n^{-2}\sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n\\ j-i>n^{\beta}}}|r_{ij}(p,q)|n^{\frac{2|r_{ij}(p,q)|}{1+|r_{ij}(p,q)|}}\log n^{\frac{1}{1+|r_{ij}(p,q)|}}\\
&\leq n^{-2}\sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n\\ j-i>n^{\beta}}}|r_{ij}(p,q)|(j-i)^{\frac{2|r_{ij}(p,q)|}{\beta}}\log(j-i)\\
&\leq n^{-2}\sum_{1\leq p\neq q\leq d}\sum_{\substack{1\leq i<j\leq n\\ j-i>n^{\beta}}}|r_{ij}(p,q)|\exp\big(\gamma|r_{ij}(p,q)|\log(j-i)\big)\log(j-i).
\end{aligned}$$

By the condition (a) of Theorem 1, we get

$$T_2\ll(\log\log n)^{-(1+\varepsilon)}. \tag{15}$$

Combining Equation (14) and Equation (15) induces that Equation (12) holds. Equation (13) can be proved in the similar way. □

Lemma 3. *Let $\{X_n\}_{n=1}^{\infty}$ be a standardized nonstationary normal sequence of d-dimensional random vectors satisfying (a) of Theorem 1 and*

(c) there exists $\gamma\geq\frac{2(1+\delta)}{1-\delta}$, as $n\to\infty$

$$\frac{1}{n^2}\sum_{p=1}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p)|\exp\left(\gamma|r_{ij}(p)|\log(j-i)\right)\to 0, \tag{16}$$

$$\frac{1}{n^2}\sum_{1\leq p\neq q\leq n}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p,q)|\exp\left(\gamma|r_{ij}(p,q)|\log(j-i)\right)\to 0. \tag{17}$$

We have

$$\sum_{p=1}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(p)}{2(1+|r_{ij}(p)|)}\Big)\stackrel{n\to\infty}{\longrightarrow}0, \tag{18}$$

$$\sum_{1\leq p\neq q\leq n}^{d}\sum_{1\leq i<j\leq n}|r_{ij}(p,q)|\exp\Big(-\frac{\omega_{ni}^2(p)+\omega_{nj}^2(q)}{2(1+|r_{ij}(p,q))|}\Big)\stackrel{n\to\infty}{\longrightarrow}0. \tag{19}$$

Proof. The proof of Lemma 3 is similar to Lemma 2. □

Lemma 4. *Suppose that $\{X_n\}_{n=1}^{\infty}$ is a standardized nonstationary normal sequence of d-dimensional random vectors satisfying the conditions (a) and (b) of Theorem 1.*

Let $u_n(p)$ and $v_n(p)$ be such that $n(1-\Phi(u_n(p)))\to\tau_p$ and $n\Phi(v_n(p))\to\eta_p$ as $n\to\infty$ for all $p=1,2,\ldots,d$, then

$$\mathbb{P}(v_k<m_k\leq M_k\leq u_k)\to\prod_{p=1}^{d}\exp(-(\tau_p+\eta_p)). \tag{20}$$

Especially, let $u_n(p) = \frac{1}{a_n}x_p + b_n$ *and* $v_n(p) = -\frac{1}{a_n}y_p - b_n$ *with* $x_p, y_p \in R$ *for all* $p = 1, 2, \dots, d$, *then*

$$\mathbb{P}(v_k < m_k \leq M_k \leq u_k) \to \prod_{p=1}^{d} \exp(-(e^{-x_p} + e^{-y_p})). \tag{21}$$

Proof. We consider the joint distribution of the maximum M_n and the minimum m_n of $\{X_n\}_{n=1}^{\infty}$

$$\begin{aligned}
&\Big|\mathbb{P}(v_n < m_n \leq M_n \leq u_n) - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&\leq \Big|\mathbb{P}(v_n < m_n \leq M_n \leq u_n) - \prod_{p=1}^{d} (\Phi(u_p) - \Phi(v_p))^n\Big| \\
&\quad + \Big|\prod_{p=1}^{d} (\Phi(u_p) - \Phi(v_p))^n - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&\triangleq L_1 + L_2.
\end{aligned}$$

By Lemmas 1 and 3, we have

$$\begin{aligned}
L_1 &= \Big|\mathbb{P}(v_n < m_n \leq M_n \leq u_n) - \prod_{p=1}^{d} (\Phi(u_p) - \Phi(v_p))^n\Big| \\
&\leq K_1 \sum_{p=1}^{d} \sum_{1 \leq i < j \leq n} |r_{ij}^0(p) - r_{ij}^*(p)| \exp\Big(-\frac{\omega_{ni}^2(p) + \omega_{nj}^2(p)}{2(1+\rho_{ij}(p))}\Big) \\
&+ K_2 \sum_{1 \leq p \neq q \leq d} \sum_{1 \leq i < j \leq n} |r_{ij}^0(p,q) - r_{ij}^*(p,q)| \exp\Big(-\frac{\omega_{ni}^2(p) + \omega_{nj}^2(q)}{2(1+\rho_{ij}(p,q))}\Big) \\
&\stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{22}$$

Based on the definition of u_n and v_n, we get

$$\begin{aligned}
L_2 &= \Big|\prod_{p=1}^{d} (\Phi(u_p) - \Phi(v_p))^n - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&= \Big|\prod_{p=1}^{d} [1 - (1 - \Phi(u_p)) - \Phi(v_p)]^n - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&= \Big|\prod_{p=1}^{d} \Big(1 - \frac{\eta_p}{n} - \frac{\tau_p}{n} + o(\tfrac{1}{n})\Big)^n - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&\stackrel{n\to\infty}{\longrightarrow} \Big|\prod_{p=1}^{d} \exp(-(\tau_p + \eta_p)) - \prod_{p=1}^{d} \exp(-(\tau_p + \eta_p))\Big| \\
&= 0.
\end{aligned} \tag{23}$$

Combining Equation (22) and Equation (23) induces that Equation (20) hold. Equation (21) is a special case of Equation (20). Then Lemma 4 holds. □

Lemma 5. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a standardized nonstationary normal d-dimensional vector sequence satisfying the conditions (a) and (b) of Theorem 1, then*

$$\mathbb{E}\Big|I\{M_n \leq u_n\} - I\{M_{k,n} \leq u_n\}\Big| \ll \frac{k}{n} + (\log\log n)^{-(1+\varepsilon)}, \tag{24}$$

$$\mathbb{E}\Big|I\{m_n > v_n\} - I\{m_{k,n} > v_n\}\Big| \ll \frac{k}{n} + (\log\log n)^{-(1+\varepsilon)}. \tag{25}$$

Proof. We firstly consider Equation (24),

$$\begin{aligned}
\mathbb{E}\Big|I\{M_n \le u_n\} - I\{M_{k,n} \le u_n\}\Big| &= \mathbb{E}\Big|I(X_1 \le u_{n1}, \ldots, X_n \le u_{nn}) - I(X_{k+1} \le u_{n(k+1)}, \ldots, X_n \le u_{nn})\Big| \\
&= \mathbb{P}(X_{k+1} \le u_{n(k+1)}, \ldots, X_n \le u_{nn}) - \mathbb{P}(X_1 \le u_{n1}, \ldots, X_n \le u_{nn}) \\
&\le \Big|\mathbb{P}(X_{k+1} \le u_{n(k+1)}, \ldots, X_n \le u_{nn}) - \prod_{p=1}^{d}\prod_{j=k+1}^{n}\Phi(u_{nj}(p))\Big| \\
&\quad + \Big|\mathbb{P}(X_1 \le u_{n1}, \ldots, X_n \le u_{nn}) - \prod_{p=1}^{d}\prod_{j=1}^{n}\Phi(u_{nj}(p))\Big| \\
&\quad + \Big|\prod_{p=1}^{d}\prod_{j=k+1}^{n}\Phi(u_{nj}(p)) - \prod_{p=1}^{d}\prod_{j=1}^{n}\Phi(u_{nj}(p))\Big| \\
&\triangleq A + B + C.
\end{aligned}$$

By Theorem 4.2.1 in Leadbetter et al. [17] and Lemma 2, we obtain

$$A \ll (\log\log n)^{-(1+\varepsilon)}, \tag{26}$$

$$B \ll (\log\log n)^{-(1+\varepsilon)}. \tag{27}$$

As $\lambda_n(p) = \min_{1\le i\le n} u_{ni}(p) \ge c(\log n)^{\frac{1}{2}}$, then $u_{ni}(p) \ge c(\log n)^{\frac{1}{2}}$ for $p = 1, 2, \ldots, d$. Define u_n by $1 - \Phi(u_n) = \frac{1}{n}$, then we have $u_{ni}(p) \ge u_n$ for some c as $p = 1, 2, \ldots, d$. The third part C can be controled as below,

$$\begin{aligned}
C &= \prod_{j=k+1}^{n}\prod_{p=1}^{d}\Phi(u_{nj}(p)) - \prod_{j=1}^{n}\prod_{p=1}^{d}\Phi(u_{nj}(p)) \\
&\le 1 - \prod_{j=1}^{k}\prod_{p=1}^{d}\Phi(u_{nj}(p)) \\
&\le \sum_{p=1}^{d}\Big(1 - \prod_{j=1}^{k}\Phi(u_{nj}(p))\Big) \\
&\le \sum_{p=1}^{d}(1 - \Phi^k(u_n)) \\
&\le \sum_{p=1}^{d}\Big(1 - (1 - \frac{1}{n})^k\Big) \\
&\ll \frac{k}{n}.
\end{aligned} \tag{28}$$

Using Equations (26)–(28), Equation (24) can be proved.

Next, we prove Equation (25). As $m_{k,n} = \min_{k+1 \le i \le n} X_i$, then $-m_{k,n} = \max_{k+1 \le i \le n} (-X_i)$.

$$\begin{aligned}
&\mathbb{E}\Big|I\{m_n > v_n\} - I\{m_{k,n} > v_n\}\Big| \\
&\quad= \mathbb{P}(m_{k,n} > v_n) - \mathbb{P}(m_n > v_n) \\
&\quad= \mathbb{P}(-m_{k,n} < -v_n) - \mathbb{P}(-m_n < -v_n) \\
&\quad\le \Big|\mathbb{P}(-m_{k,n} < -v_n) - \prod_{p=1}^{d} \Phi^{n-k}(-v_n(p)))\Big| \\
&\qquad\quad + \Big|\mathbb{P}(-m_n < -v_n) - \prod_{p=1}^{d} \Phi^{n}(-v_n(p)))\Big| \\
&\qquad\quad + \Big|\prod_{p=1}^{d} \Phi^{n-k}(-v_n(p))) - \prod_{p=1}^{d} \Phi^{n}(-v_n(p)))\Big| \\
&\quad\triangleq A_1 + A_2 + A_3.
\end{aligned}$$

Since

$$x^{n-k} - x^n \le \frac{k}{n}, \;\; 0 \le x \le 1,$$

we have

$$A_3 \le \frac{k}{n}. \tag{29}$$

By Theorem 4.2.1 in Leadbetter et al. [17] and Lemma 2, we get

$$A_k \ll (\log\log n)^{-(1+\varepsilon)}, \;\; k = 1, 2. \tag{30}$$

Using Equations (29) and (30), Equation (25) can be obtained. Then Lemma 5 holds. □

Lemma 6. *Let $\{X_n\}_{n=1}^{\infty}$ be a standardized nonstationary normal d-dimensional vector sequence satisfying the conditions (a) and (b) of Theorem 1, then*

$$\Big|Cov\Big(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\}\Big)\Big| \ll (\log\log n)^{-(1+\varepsilon)}. \tag{31}$$

Proof. By Lemmas 1 and 2, we have

$$\begin{aligned}
&\Big|Cov\Big(I(M_k \le u_k, m_k > v_k), I(M_{k,n} \le u_n, m_{k,n} > v_n)\Big)\Big| \\
&\quad= \mathbb{P}(v_k < m_k \le M_k \le u_k, v_n < m_{k,n} \le M_{k,n} \le u_n) \\
&\qquad - \mathbb{P}\big(v_k < m_k \le M_k \le u_k\big)\mathbb{P}\big(v_n < m_{k,n} \le M_{k,n} \le u_n\big) \\
&\quad\ll \sum_{p=1}^{d} \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r_{ij}(p)| \exp\Big(-\frac{\acute{w}^2(p)}{1 + r_{ij}(p)}\Big) \\
&\qquad + \sum_{1 \le p \ne q \le d} \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r_{ij}(p,q)| \exp\Big(-\frac{\acute{w}^2(p) + \acute{w}^2(q)}{2(1 + r_{ij}(p,q))}\Big) \\
&\quad\ll (\log\log n)^{-(1+\varepsilon)},
\end{aligned}$$

where $\acute{w}(p) = min\big(|v_k(p)|, |v_n(p)|, |u_k(p)|, |u_n(p)|\big), p = 1, 2, \ldots, d.$ □

Lemma 7. *Let* $Y_1, Y_2 \ldots$ *be a sequence of bounded random variables. If*

$$Var(\sum_{k=1}^{n} \frac{1}{k} Y_k) \ll (\log n)^2 (\log\log n)^{-(1+\epsilon)}, \tag{32}$$

for some $\varepsilon > 0$*, then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(Y_k - EY_k) = 0 \quad a.s. \tag{33}$$

Proof. The proof can be found in Lemma 3.1 [18]. □

Proof Theorem 1. Let $\chi_k = I(v_k < m_k \le M_k \le u_k)$, then

$$\begin{aligned} Var\Big(\sum_{k=1}^{n} \frac{1}{k}\chi_k\Big) &= \sum_{k=1}^{n} \frac{1}{k^2} Var(\chi_k) + 2 \sum_{1\le k<l\le n} \frac{cov(\chi_k, \chi_l)}{kl} \\ &\le \sum_{k=1}^{n} \frac{1}{k^2} + 2 \sum_{1\le k<l\le n} \frac{cov(\chi_k, \chi_l)}{kl} \\ &\triangleq A + B. \end{aligned}$$

Note that for $k < l$, the absolute value of the numerator of the second term B can be expressed as below,

$$\begin{aligned} \Big|cov(\chi_k, \chi_l)\Big| &= \Big|cov\big(I(v_k < m_k \le M_k \le u_k), I(v_l < m_l \le M_l \le u_l)\big)\Big| \\ &\le \Big|cov\big(I(v_k < m_k \le M_k \le u_k), I(v_l < m_l \le M_l \le u_l) \\ &\quad -I(v_l < m_l \le M_{k,l} \le u_l)\big)\Big| + \Big|cov\big(I(v_k < m_k \le M_k \le u_k), \\ &\quad I(v_l < m_l \le M_{k,l} \le u_l) - I(v_l < m_{k,l} \le M_{k,l} \le u_l)\big)\Big| \\ &\quad +\Big|cov\big(I(v_l < m_l \le M_{k,l} \le u_l), I(v_l < m_{k,l} \le M_{k,l} \le u_l)\big)\Big| \\ &\quad \triangleq B_1 + B_2 + B_3. \end{aligned}$$

By Lemma 5, we get

$$\begin{aligned} B_1 &\le 2\mathbb{E}\Big|I(v_l < m_l \le M_l \le u_l) - I(v_l < m_l \le M_{k,l} \le u_l)\Big| \\ &\le 2\mathbb{E}\Big|I(M_l \le u_l) - I(M_{k,l} \le u_l)\Big| \\ &\ll \frac{k}{l} + (\log\log n)^{-(1+\varepsilon)}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} B_2 &\le 2\mathbb{E}\Big|I(v_l < m_l \le M_{k,l} \le u_l) - I(v_l < m_{k,l} \le M_{k,l} \le u_l)\Big| \\ &\le 2\mathbb{E}\Big|I(m_l > v_l) - I(m_{k,l} > v_l)\Big| \\ &\ll \frac{k}{l} + (\log\log n)^{-(1+\varepsilon)}. \end{aligned} \tag{35}$$

By Lemma 6, we obtain

$$B_3 \le (\log\log l)^{-(1+\varepsilon)}. \tag{36}$$

Combining Equations (34)–(36), we can estimate B,

$$\begin{aligned} B &\ll \sum_{1\le k<l\le n} \frac{1}{kl}\Big(\frac{k}{l} + (\log\log n)^{-(1+\varepsilon)}\Big) \\ &\ll \sum_{1\le k<l\le n} \frac{1}{l^2} + \sum_{1\le k<l\le n} \frac{1}{kl}(\log\log n)^{-(1+\varepsilon)} \\ &\ll \log n + (\log n)^2(\log\log n)^{-(1+\varepsilon)}. \end{aligned}$$

Lastly, we can draw the conclusion

$$Var\Big(\sum_{k=1}^{n} \frac{1}{k}\chi_k\Big) \ll (\log n)^2(\log\log n)^{-(1+\varepsilon)}.$$

By Lemma 7, Theorem 1 is proved. □

Proof Theorem 2. If we use Equation (8) instead of the conditions (a) and (b) of Theorem 1, Lemma 2, Lemma 3, Lemma 5 and Lemma 6 still hold. Theorem 2 can be proved. □

Proof Theorem 3. Replace (a) and (b) of Theorem 1 with (a) and (b) of Theorem 3, then Equations (4) and (5) still hold. □

Proof Theorem 4. If we use Equation (11) instead of Equation (8), Theorem 4 can be completed. □

4. Conclusions

The almost sure central limit theorems for the maxima and minimum of general normal vector sequences under suitable conditions are put forward. We note that $\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}$ is greater than 1 and converges to 1 as $N \to \infty$. The convergence rate is mainly decided by the $\log n$ and the rate is not so fast. The extreme value theory deals with extreme phenomena which are less likely to occur, but more harmful [1–3]. The maximum and minimum can be used to depict the extreme risk in the economy and natural disaster (such as floods, hurricane, stock market crash, megaseism and so on), and then their joint limiting distribution computes the probability of the controllable risk in an interval.

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References

1. Galambos, J. *Asymptotic Theory of Extreme Order Statistics*; John Wiley & Sons: Hoboken, NJ, USA, 1978.
2. Beirlant, J.; Goegebeur, Y.; Segers, J.; Teugels, J.L. *Statistics of Extremes: Theory and Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2006.
3. Resnick, S.I. *Extreme Values, Regular Variation, and Point Processes*; Springer: Berlin/Heidelberg, Germany, 1989; Volume 84.
4. Bremaud, P. *Discrete Probability Models and Methods*; Springer: Berlin/Heidelberg, Germany, 2017.
5. Oliveira, P.E. *Almost Sure Convergence*; Springer: Berlin/Heidelberg, Germany, 2012.

6. Cheng, S.; Peng, L.; Qi, Y. Almost Sure Convergence in Extreme Value Theory. *Math. Nachrichten* **2006**, *190*, 43–50. [CrossRef]
7. Brosamler, G.A. An almost everywhere central limit theorem. *Math. Proc. Camb. Philos. Soc.* **1998**, *104*, 561. [CrossRef]
8. Hüsler, J.; Schüpbach, M. Limit results for maxima in non-stationary multivariate Gaussian sequences. *Stoch. Process. Their Appl.* **1988**, *28*, 91–99. [CrossRef]
9. Ibragimov, I.; Lifshits, M. On the convergence of generalized moments in almost sure central limit theorem. *Stat. Probab. Lett.* **1998**, *40*, 343–351. [CrossRef]
10. Fahrner, I.; Stadtmüller, U. On almost sure max-limit theorems. *Stat. Probab. Lett.* **1998**, *37*, 229–236. [CrossRef]
11. Berkes, I.; Csáki, E. A universal result in almost sure central limit theory. *Stoch. Process. Their Appl.* **2001**, *94*, 105–134. [CrossRef]
12. Csáki, E.; Gonchigdanzan, K. Almost sure limit theorems for the maximum of stationary Gaussian sequences. *Stat. Probab. Lett.* **2002**, *58*, 195–203. [CrossRef]
13. Chen, S.; Lin, Z. Almost sure max-limits for nonstationary Gaussian sequence. *Stat. Probab. Lett.* **2006**, *76*, 1175–1184. [CrossRef]
14. Chen, Z.; Peng, Z.; Zhang, H. An almost sure limit theorem for the maxima of multivariate stationary gaussian sequences. *J. Aust. Math. Soc.* **2009**, *86*, 315. [CrossRef]
15. Zhao, S.; Peng, Z.; Wu, S. Almost Sure Convergence for the Maximum and the Sum of Nonstationary Guassian Sequences. *J. Inequalities Appl.* **2010**, *2010*, 856495. [CrossRef]
16. Weng, Z.; Peng, Z.; Nadarajah, S. The almost sure limit theorem for the maxima and minima of strongly dependent Gaussian vector sequences. *Extremes* **2012**, *15*, 389–406. [CrossRef]
17. Leadbetter, M.R.; Lindgren, G.; Rootzén, H. *Extremes and Related Properties of Random Sequences and Processes*; Springer: Berlin/Heidelberg, Germany, 1983.
18. Gonchigdanzan, K. An almost sure limit theorem for the product of partial sums with stable distribution. *Stat. Probab. Lett.* **2008**, *78*, 3170–3175. [CrossRef]