mathematics

MDPI

Article

Inertial Iterative Schemes with Variable Step Sizes for Variational Inequality Problem Involving Pseudomonotone Operator

Jamilu Abubakar [1,2,†], Poom Kumam [1,3,4,*], Habib ur Rehman [1] and Abdulkarim Hassan Ibrahim [1]

1 Department of Mathematics, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand; abubakar.jamilu@udusok.edu.ng (J.A.); hrehman.hed@gmail.com (H.u.R.); ibrahimkarym@gmail.com (A.H.I.)
2 Department of Mathematics, Usmanu Danfodiyo University, Sokoto 840001, Nigeria
3 Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand
4 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: poom.kumam@mail.kmutt.ac.th
† Current address: Department of Mathematics, King Mongkut's University of Technology Thonburi, 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand.

Received: 18 March 2020; Accepted: 8 April 2020; Published: 16 April 2020



Abstract: Two inertial subgradient extragradient algorithms for solving variational inequality problems involving pseudomonotone operator are proposed in this article. The iterative schemes use self-adaptive step sizes which do not require the prior knowledge of the Lipschitz constant of the underlying operator. Furthermore, under mild assumptions, we show the weak and strong convergence of the sequences generated by the proposed algorithms. The strong convergence in the second algorithm follows from the use of viscosity method. Numerical experiments both in finite- and infinite-dimensional spaces are reported to illustrate the inertial effect and the computational performance of the proposed algorithms in comparison with the existing state of the art algorithms.

Keywords: variational inequality problem; Lipschitz-type conditions; viscosity method; subgradient extragradient method; pseudomonotone operator

MSC: 47H05; 47J20; 47J25; 65K15

1. Introduction

This paper considers the problem of finding a point $\breve{u} \in \mathbb{E}$ such that

$$\langle \mathbb{A}\breve{u}, u - \breve{u} \rangle \geq 0, \qquad \forall\, u \in \mathbb{E}, \tag{1}$$

where $\mathbb{E}$ is a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $\mathbb{A}$ is an operator on $\mathbb{H}$. The Variational Inequality (VI) problem Equation (1) is a fundamental problem in optimization theory which is applied in many areas of study, such as transportation problems, equilibrium, economics, engineering and so on (Refs. [1–14]).

There are many approaches to the VI problem, the basic one being the regularization and projection method. Many studies have been carried out and several algorithms have been considered and proposed (Refs. [5,15–24]). In this study, we are interested in the projection method.

The main iterative scheme in this study is given by

$$\begin{cases} t_n = u_n + \varrho_n(u_n - u_{n-1}), \\ s_n = P_{\mathbb{E}}\left(t_n - \lambda_n \mathbb{A} t_n\right), \\ T_n = \{u \in \mathbb{H} : \langle t_n - \lambda_n \mathbb{A} t_n - s_n, u - s_n \rangle \leq 0\} \\ u_{n+1} = P_{T_n}\left(t_n - \lambda_n \mathbb{A} s_n\right) \qquad n \geq 1. \end{cases} \tag{2}$$

where ϱ_n is the inertial parameter with two different updating rules. The step size λ_n is define to be self adaptively updated according to a new step size rule.

We now discuss the relationship between our scheme and the existing algorithms. The iterative scheme Equation (2) with a fixed step size is exactly the scheme proposed in [25] and without the inertial step, it reduces to the original subgradient extragradient scheme proposed by Censor et al. [26] where the underlying operator is monotone. The subgradient extragradient method has been studied, modified and improved by several researchers to produce variant methods. Most of these modifications used a fixed step size which depends on the factorials of the underlying operator such as strongly (inverse) modulus and Lipschitz constant. Therefore, algorithms with fixed step size require the prior knowledge of such factorials to be implemented. In a situation where such constants are difficult to compute or do not exist, such algorithms may be impossible to implement.

Recently, Yang et al. [27] proposed a modified subgradient algorithm with variable step size which does not require the knowledge of the Lipschitz constant for solving Equation (1). However, they considered the underlying operator to be monotone and the sequence generated converges weakly. Interested readers may refer to some recent articles that propose algorithms with variable step sizes which are independent of factorials of the underlying operator and Lipschitz constant (Refs. [28–31]). The question now is, can we have an iterative scheme involving a more general class of operators with a strong convergence and self-adaptive step size? We hence provide a positive answer to this question.

Inspired and motivated by [26,27], we proposed a self-adaptive subgradient extragradient algorithm by incorporating the inertial extrapolation step with the subgradient extragradient scheme. The aim of this modification is to obtain a self-adaptive scheme with fast convergence properties involving a more general class of operators. Moreover, we present a strong convergence version of the proposed algorithm by incorporating a viscosity method. The proposed schemes do not require prior knowledge of the Lipschitz constant of the operator. Furthermore, we present numerical experiments in finite- and infinite-dimensional spaces to illustrate the performance and the effect of the inertial step when compared to the existing algorithms in the literature.

The outline of this work is as follows: We give some definitions and Lemmas which we will use in our convergence analysis in the next section. We present the convergence analysis of our proposed schemes in Section 3 and lastly, in Section 4, we illustrate the inertial effect and the computational performance of our algorithms by giving some examples.

2. Preliminaries

This section recalls some known facts and necessary tools that we need for the convergence analysis of our method. Throughout this article, $\mathbb{H}$ is a real Hilbert space with inner product and norm denoted respectively as $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, $\mathbb{E}$ is a nonempty closed and convex subset of $\mathbb{H}$. The notation $u_j \rightharpoonup u$ (resp $u_j \longrightarrow u$) is used to indicate that, respectively, the sequence $\{u_j\}$ converges weakly (strongly) to u. The following is known to hold in a Hilbert space:

$$\|t \pm s\|^2 = \|t\|^2 + \|s\|^2 \pm 2\langle t, s \rangle, \tag{3}$$

for every $t, s \in \mathbb{H}$ [32].

Definition 1. *Let $\mathbb{A} : \mathbb{H} \longrightarrow \mathbb{H}$ be a mapping defined on a real Hilbert space $\mathbb{H}$. $\mathbb{A}$ is said to be:*

(1) Pseudomonotone if

$$\langle \mathbb{A}u, v-u\rangle \geq 0, \quad \Longrightarrow \langle \mathbb{A}v, u-v\rangle \geq 0 \quad \forall\, u, v \in \mathbb{E}.$$

(2) sequentially weakly continuous on $\mathbb{H}$ if $u_j \rightharpoonup u \in \mathbb{H}$, then $\mathbb{A}u_j \rightharpoonup \mathbb{A}u$.

Lemma 1. *[32] For any $\mathbb{E} \subset \mathbb{H}$ closed and convex and a metric projection from $\mathbb{H}$ onto $\mathbb{A}$. $v = P_{\mathbb{E}}u$ for any $u \in \mathbb{H}$, $\Leftrightarrow \langle u - v, \omega - v\rangle \leq 0, \quad \forall\, \omega \in \mathbb{E}$.*

Lemma 2. *[33] For any $\mathbb{E} \subset \mathbb{H}$ closed and convex, for every u in $\mathbb{H}$, the following holds*

i. $\|P_{\mathbb{E}}u - P_{\mathbb{E}}v\|^2 \leq \langle P_{\mathbb{E}}u - P_{\mathbb{E}}v, u - v\rangle$ *for all* $v \in \mathbb{H}$,
ii. $\|P_{\mathbb{E}}u - v\|^2 \leq \|u - v\|^2 - \|u - P_{\mathbb{E}}v\|^2$ *for all* $v \in \mathbb{E}$.

For more properties of projection interested reader should see [33].

Lemma 3. *[34] Let $\mathbb{A} : \mathbb{E} \longrightarrow \mathbb{H}$ be a continuous and pseudomonotone mapping. Then $\breve{u} \in \Gamma$ if and only if $\breve{u}$ is a solution of the problem of finding $u \in \mathbb{E}$ such that $\langle \mathbb{A}v, v - u\rangle \geq 0, \quad \forall\, v \in \mathbb{E}$.*

Lemma 4. *[35] Suppose $\{\gamma_n\}$, $\{\phi_n\}$ and $\{\varrho_n\}$ be sequences in $[0, \infty)$ such that for all $n \geq 1$,*

$$\gamma_{n+1} \leq \gamma_n + \varrho_n(\gamma_n - \gamma_{n-1}) + \phi_n, \quad \sum_{n=1}^{\infty} \phi_n < \infty$$

and there exists $\varrho \in \mathbb{R}$ with $0 \leq \varrho_n \leq \varrho \leq 1$ for all $n \geq 1$. Then the following are satisfied:

(i) $\sum_{n=1}^{\infty}[\gamma_n - \gamma_{n-1}]_+ < \infty$, *where* $[a]_+ = \max\{a, 0\}$;
(ii) there exists $\gamma^* \in [0, \infty)$ *with* $\lim \gamma_n = \gamma^*$.

Lemma 5. *[36] Let $\mathbb{E} \subset \mathbb{H}$ be a nonempty set and a sequence $\{u_j\}$ in $\mathbb{H}$ such that the following are satisfied:*

(a) for every $u \in \mathbb{E}$, $\lim_{j\to\infty} \|u_j - u\|$ exists;
(b) every sequentially weak cluster point of $\{u_j\}$ is in E.

Therefore, $\{u_j\}$ converges weakly in $\mathbb{E}$.

Lemma 6. *Let $\{\varrho_n\}$ be a nonnegative real number sequence, $\{\gamma_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{\delta_n\}$ be a sequence of real numbers satisfying*

$$\varrho_{n+1} \leq (1 - \gamma_n)\varrho_n + \gamma_n\delta_n, \quad \textit{for all} \quad n \geq 1.$$

If $\limsup_{i\to\infty} \delta_{n_j} \leq 0$ for every subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ satisfying $(\varrho_{n_j+1} - \varrho_{n_j}) \geq 0$, then $\lim_{n\to\infty} \varrho_n = 0$.

3. A Self-Adaptive Subgradient Extragradient Scheme for Variational Inequality Problem

In this section, we give a detail description of our proposed algorithms. First, we present weak convergence analysis of the iterates generated by the algorithm to the solution of the VI problem Equation (1) involving pseudomonotone operator. We suppose the following assumptions for the analysis of our method.

Assumption 1.
A1 The feasible set of Equation (1) is nonempty closed and convex subset of $\mathbb{H}$.
A2 The solution set Γ of Equation (1) is nonempty.
A3 $\mathbb{A} : \mathbb{H} \longrightarrow \mathbb{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathbb{H}$ and sequentially weakly continuous on $\mathbb{E}$.

Remark 1. *For $0 \leq \varrho_n \leq \varrho$, it can be observed from Equation (5) that $\varrho_n \|u_n - u_{n-1}\|^2 \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$. This implies that*

$$\sum_{n=1}^{\infty} \varrho_n \|u_n - u_{n-1}\|^2 < \infty.$$

Algorithm 1 Adaptive Subgradient Extragradient Algorithm for Pseudomonotone Operator.

Initialization: Choose $u_{-1}, u_0 \in \mathbb{H}$, $\mu \in (0,1)$, $\varrho > 0$ and $\lambda_0 > 0$.

Iterative Steps: Given the current iterates u_{n-1} and $u_n \in \mathbb{H}$.

Step 1. Set t_n as:

$$t_n := u_n + \varrho_n(u_n - u_{n-1}), \tag{4}$$

where

$$\varrho_n := \begin{cases} min\left\{\varrho, \frac{1}{n^2\|u_n - u_{n-1}\|^2}\right\}, & \text{if } u_n \neq u_{n+1}, \\ \varrho, & \text{otherwise .} \end{cases} \tag{5}$$

Step 2. Compute

$$s_n = P_{\mathbb{E}}\left(t_n - \lambda_n \mathbb{A} t_n\right).$$

If $t_n = s_n$ or $\mathbb{A}t_n = 0$, stop. Else, go to step 3.

Step 3. Construct

$$T_n := \left\{x \in \mathbb{H} : \langle t_n - \lambda_n \mathbb{A} t_n - s_n, x - s_n\rangle \leq 0\right\}.$$

Compute

$$u_{n+1} = P_{T_n}\left(t_n - \lambda_n \mathbb{A} s_n\right).$$

where the stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} := \begin{cases} \min\left\{\lambda_n, \Lambda_n\right\}, & \text{if } \langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle > 0 \\ \lambda_n, & \text{otherwise } 0. \end{cases} \tag{6}$$

where

$$\Lambda_n := \frac{\mu(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle}$$

Set $n := n + 1$ and go back to **Step 1.**

Lemma 7. *The generated sequence $\{\lambda_n\}$ by Equation (6) is monotonically decreasing and bounded from below by $\min\{\frac{\mu}{L}, \lambda_0\}$*

Proof. It can be observed that, the sequence $\{\lambda_n\}$ is monotonically decreasing. Since $\mathbb{A}$ is a Lipschitz function with Lipschitz's constant L, for $\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle > 0$, we have

$$\begin{aligned} \frac{\mu(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle} &\geq \frac{\mu(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2\|\mathbb{A}t_n - \mathbb{A}s_n\|\|u_{n+1} - s_n\|}, \\ &\geq \frac{\mu(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2L\|t_n - s_n\|\|u_{n+1} - s_n\|} \\ &\geq \frac{2\mu(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2L\|t_n - s_n\|^2 + \|u_{n+1} - s_n\|^2}, \\ &\geq \frac{\mu}{L} \end{aligned}$$

□

Remark 2. *By Lemma 7, Equation (6) is well-defined and*

$$\lambda_{n+1}\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle \leq \frac{\mu}{2}(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2).$$

Next, the following lemma and its proof is crucial for the convergence analysis of the sequence generated by Algorithm 1.

Lemma 8. *Let $\mathbb{A}$ be an operator satisfying the Assumption 1 (A3). Then, for all $p \in \Gamma \neq \emptyset$, we have*

$$\|u_{n+1} - \breve{u}\|^2 \leq \|t_n - \breve{u}\|^2 - \left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n - s_n\|^2 - \left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|u_{n+1} - s_n\|^2.$$

Proof. By Lemma 2 and the definition of u_{n+1}, we get

$$\begin{aligned}
\|u_{n+1} - \breve{u}\|^2 &= \|P_{T_n}(t_n - \lambda_n\mathbb{A}s_n) - \breve{u}\|^2,\\
&\leq \|(t_n - \lambda_n\mathbb{A}s_n) - \breve{u}\|^2 - \|(t_n - \lambda_n\mathbb{A}s_n) - u_{n+1}\|^2,\\
&= \|t_n - \breve{u}\|^2 - \|t_n - u_{n+1}\|^2 + 2\lambda_n\langle \breve{u} - u_{n+1}, \mathbb{A}s_n\rangle,\\
&= \|t_n - \breve{u}\|^2 - \|t_n - u_{n+1}\|^2 + 2\lambda_n\langle \breve{u} - s_n, \mathbb{A}s_n - \mathbb{A}\breve{u}\rangle\\
&\quad + 2\lambda_n\langle \breve{u} - s_n, \mathbb{A}\breve{u}\rangle + 2\lambda_n\langle s_n - u_{n+1}, \mathbb{A}s_n\rangle,\\
&\leq \|t_n - \breve{u}\|^2 - \|t_n - u_{n+1}\|^2 + 2\lambda_n\langle s_n - u_{n+1}, \mathbb{A}s_n\rangle,\\
&= \|t_n - \breve{u}\|^2 - \|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2 + 2\lambda_n\langle s_n - u_{n+1}, \mathbb{A}s_n\rangle\\
&\quad - 2\langle t_n - s_n, s_n - u_{n+1}\rangle,\\
&= \|t_n - \breve{u}\|^2 - \|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2\\
&\quad + 2\langle t_n - \lambda_n\mathbb{A}s_n - s_n, u_{n+1} - s_n\rangle. \qquad (7)
\end{aligned}$$

Using the definition of s_n and the fact that $u_{n+1} \in T_n$, we have

$$\begin{aligned}
\langle t_n - \lambda_n\mathbb{A}s_n - s_n, u_{n+1} - s_n\rangle &= \langle t_n - \lambda_n\mathbb{A}t_n - s_n, u_{n+1} - s_n\rangle\\
&\quad + \langle \lambda_n\mathbb{A}t_n - \lambda_n\mathbb{A}s_n, u_{n+1} - s_n\rangle,\\
&\leq \lambda_n\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle. \qquad (8)
\end{aligned}$$

Now, from the definition of λ_{n+1}, we have,

$$\langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle \leq \frac{1}{\lambda_{n+1}}\frac{\mu}{2}(\|t_n - s_n\|^2 - \|u_{n+1} - s_n\|^2). \qquad (9)$$

Combining Equations (7)–(9), we obtain the required result. □

Lemma 9. *Let $\{t_n\}$ be a sequence generated by Algorithm 1 and Assumption $(A_1 - A_3)$ is satisfied. If there exists a subsequence $\{t_{n_i}\}$ weakly convergent to $q \in \mathbb{H}$ with $\lim_{i\to\infty}\|t_{n_i} - s_{n_i}\| = 0$. Then $q \in \Gamma$.*

Proof. Since,

$$\langle t_{n_i} - \lambda_{n_i}\mathbb{A}t_{n_i} - s_{n_i}, u - s_{n_i}\rangle \leq 0 \quad \text{for all } u \in \mathbb{E},$$

this implies that

$$\frac{1}{\lambda_{n_i}}\langle t_{n_i} - s_{n_i}, u - s_{n_i}\rangle \leq \langle \mathbb{A}t_{n_i}, u - s_{n_i}\rangle \quad \text{for all } u \in \mathbb{E}.$$

Furthermore,

$$\frac{1}{\lambda_{n_i}}\langle t_{n_i} - s_{n_i}, u - s_{n_i}\rangle + \langle \mathbb{A}t_{n_i}, s_{n_i} - t_{n_i}\rangle \leq \langle \mathbb{A}t_{n_i}, u - t_{n_i}\rangle \quad \text{for all } u \in \mathbb{E}. \tag{10}$$

By the hypothesis, $\{t_{n_i}\}$ is bounded, consequently, $\{s_{n_i}\}$ is also bounded, and from the Lipschitz continuity of $\mathbb{A}$, $\{\mathbb{A}t_{n_i}\}$ is bounded as well. Passing limit as $i \to \infty$ to Equation (10), we obtain

$$\liminf_{i\to\infty} \langle \mathbb{A}t_{n_i}, u - t_{n_i}\rangle \geq 0 \quad \text{for all } u \in \mathbb{E}. \tag{11}$$

Furthermore, we have

$$\langle \mathbb{A}s_{n_i}, u - s_{n_i}\rangle = \langle \mathbb{A}s_{n_i} - \mathbb{A}t_{n_i}, u - t_{n_i}\rangle + \langle \mathbb{A}s_{n_i}, t_{n_i} - s_{n_i}\rangle \quad \text{for all } u \in \mathbb{E}. \tag{12}$$

$$\liminf_{i\to\infty} \langle \mathbb{A}s_{n_i}, u - s_{n_i}\rangle \geq 0.$$

The last inequality follows from Equations (11) and (12) and $\lim_{i\to\infty} \|\mathbb{A}t_{n_i} - \mathbb{A}s_{n_i}\| = 0$ (this follows from the fact that $\mathbb{A}$ is Lipschitz on $\mathbb{H}$, and $\lim_{i\to\infty} \|t_{n_i} - s_{n_i}\| = 0$.). Let $\{\eta_i\}$ be a decreasing sequence of positive numbers tending to 0 and for each i, N_i a smallest positive integer such that

$$\left\langle \mathbb{A}s_{n_j}, u - s_{n_j}\right\rangle + \eta_i \geq 0 \quad \text{for all } i \geq N_i. \tag{13}$$

It can be observed that the sequence $\{N_i\}$ is increasing. Moreover, assume $\mathbb{A}s_{n_i} \neq 0$ (otherwise s_{n_i} is a solution), set

$$\rho_{N_i} = \frac{\mathbb{A}s_{n_i}}{\|\mathbb{A}s_{n_i}\|^2},$$

therefore, for each i, we get $\langle \mathbb{A}s_{n_i}, \rho_{N_i}\rangle = 1$.

It can be inferred from Equation (13) that for any i

$$\langle \mathbb{A}s_{n_i}, u + \eta_i\rho_{N_i} - s_{n_i}\rangle \geq 0.$$

By the pseudomonotonicity of $\mathbb{A}$, we have

$$\langle \mathbb{A}(u + \eta_i\rho_{N_i}), u + \eta_i\rho_{N_i} - s_{n_i}\rangle \geq 0.$$

This implies that

$$\langle \mathbb{A}u, u - s_{n_i}\rangle \geq \langle \mathbb{A}u - \mathbb{A}(u + \eta_i\rho_{N_i}), u + \eta_i\rho_{N_i} - s_{n_i}\rangle - \eta_i \langle \mathbb{A}u, \rho_{N_i}\rangle. \tag{14}$$

To show that $q \in \Gamma$, to that end, we show that $\lim_{i\to\infty} \eta_i\rho_{N_i} = 0$. By the sequentially weakly lower semicontinuity of the norm mapping, we get

$$0 < \|\mathbb{A}q\| \leq \liminf_{i\to\infty} \|\mathbb{A}s_{n_i}\|.$$

Since, $\{s_{N_i}\} \subset \{s_{n_i}\}$ and $\eta_i \to 0$ as $i \to \infty$, we get

$$0 \leq \limsup_{i\to\infty} \|\eta_i\rho_{N_i}\| = \limsup_{i\to\infty} \left(\frac{\eta_i}{\|\mathbb{A}s_{n_i}\|}\right) \leq \limsup_{i\to\infty} \frac{\eta_i}{\|\mathbb{A}s_{n_i}\|} = 0.$$

Therefore, $\lim_{i\to\infty} \eta_i\rho_{N_i} = 0$. Now, from the fact that $\mathbb{A}$ is uniformly bounded, $\{t_{n_i}\}, \{\rho_{N_i}\}$, are bounded and $\lim_{i\to\infty} \eta_i\rho_{N_i} = 0$, we obtain

$$\liminf_{i\to\infty} \langle \mathbb{A}u, u - s_{n_i}\rangle \geq 0$$

Therefore, for all $u \in \mathbb{E}$, we have

$$\langle \mathbb{A}u, u-q\rangle = \lim_{i\to\infty}\langle \mathbb{A}u, u-s_{n_i}\rangle = \liminf_{i\to\infty}\langle \mathbb{A}u, u-s_{n_i}\rangle \geq 0.$$

By Lemma 3, $q \in \Gamma$ and hence the proof is complete. □

Remark 3. *As noted in [37,38], when the operator $\mathbb{A}$ is monotone, the sequentially weak- weak assumption is not needed in Lemma 9.*

Theorem 1. *Let A be an operator satisfying the Assumption 1 (A3). Then, for all $p \in \Gamma \neq \emptyset$, the sequence $\{u_n\}$ generated by Algorithm 1, converges weakly to $\breve{u}$.*

Proof. By Lemma 8, we have

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 \leq \|t_n-\breve{u}\|^2 &- \Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|t_n-s_n\|^2 \\ &- \Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|u_{n+1}-s_n\|^2.\end{aligned} \tag{15}$$

Moreover, from Equation (4), we have

$$\begin{aligned}\|t_n-\breve{u}\|^2 &= \|u_n+\varrho_n(u_n-u_{n-1})-\breve{u}\|^2, \\ &= \|(1+\varrho)(u_n-p)-\varrho_n(u_{n-1}-\breve{u})\|^2, \\ &= (1+\varrho_n)\|u_n-\breve{u}\|^2-\varrho_n\|u_{n-1}-\breve{u}\|^2+\varrho_n(1+\varrho_n)\|u_n-u_{n-1}\|^2.\end{aligned} \tag{16}$$

Substituting the last equation in Equation (15), we have

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 &\leq (1+\varrho_n)\|u_n-\breve{u}\|^2-\varrho_n\|u_{n-1}-\breve{u}\|^2+\varrho_n(1+\varrho_n)\|u_n-u_{n-1}\|^2, \\ &\quad -\Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|t_n-s_n\|^2-\Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|u_{n+1}-s_n\|^2. \\ &\leq \|u_n-\breve{u}\|^2+\varrho_n(\|u_n-\breve{u}\|^2-\|u_{n-1}-\breve{u}\|^2), \\ &\quad +\varrho_n(1+\varrho_n)\|u_n-u_{n-1}\|^2-\Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|t_n-s_n\|^2, \\ &\quad -\Big(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\Big)\|u_{n+1}-s_n\|^2.\end{aligned} \tag{17}$$

Observe that $\lim_{n\to\infty}\lambda_n\frac{\mu}{\lambda_{n+1}}=\mu$ with $0<\mu<1$, thus, there exists $\aleph \geq 0$, such that for all $n \geq \aleph$ $0<\lambda_n\frac{\mu}{\lambda_{n+1}}<1$. Consequently, we have

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 \leq \|u_n-\breve{u}\|^2 &+ \varrho_n(\|u_n-p\|^2-\|u_{n-1}-\breve{u}\|^2), \\ &+ \varrho_n(1+\varrho_n)\|u_n-u_{n-1}\|^2.\end{aligned}$$

Now, comparing the last equation with Lemma 4, we have

$$\Gamma_n = \|u_n-\breve{u}\|^2 \quad \text{and} \quad \varphi_n = \varrho_n(1+\varrho_n)\|u_n-u_{n-1}\|^2.$$

Therefore, we get

$$\sum_{n=1}^{\infty}\Big[\|u_n-\breve{u}\|^2-\|u_{n-1}-\breve{u}\|^2\Big]_+ \leq +\infty.$$

Consequently,

$$\lim_{n\to\infty}\Big[\|u_n-\breve{u}\|^2-\|u_{n-1}-\breve{u}\|^2\Big]_+ = 0,$$

and $\lim_{n\to\infty}\|u_n - \breve{u}\|^2$ exists. On the other hand, from Equation (17) we have

$$\begin{aligned}
&\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n - s_n\|^2 + \left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|u_{n+1} - s_n\|^2\\
&\leq \|u_n - \breve{u}\|^2 - \|u_{n+1} - \breve{u}\|^2 + \varrho_n(\|u_n - \breve{u}\|^2 - \|u_{n-1} - \breve{u}\|^2)\\
&\quad + \varrho_n(1+\varrho_n)\|u_n - u_{n-1}\|^2,\\
&\leq \|u_n - \breve{u}\|^2 - \|u_{n+1} - \breve{u}\|^2 + \varrho_n\left[\|u_n - \breve{u}\|^2 - \|u_{n-1} - \breve{u}\|^2\right]_+\\
&\quad + \varrho_n(1+\varrho_n)\|u_n - u_{n-1}\|^2 \quad \text{for all } n \geq N.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|t_n - s_n\| = 0.$$

Moreover, we obtain from Equation (4)

$$\|t_n - u_n\|^2 = \varrho_n^2\|u_n - u_{n-1}\|^2 \leq \alpha\cdot\varrho_n\|u_n - u_{n-1}\|^2 \to 0 \quad \text{as } n\to\infty,$$

hence,

$$\lim_{n\to\infty}\|t_n - u_n\| = 0. \tag{18}$$

Since $\lim_{n\to\infty}\|u_n - \breve{u}\|^2$ exists, therefore, the sequence $\{u_n\}$ is bounded. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup z$, then from Equation (18) we have $t_{n_i} \rightharpoonup z$. Now, since $\lim_{n\to\infty}\|t_n - s_n\| = 0$, by Lemma 9 we get $z \in \Gamma$. Consequently by Lemma 5, the sequence $\{u_n\}$ converges weakly to the solution of the VI problem. □

Notice that when $\varrho_n = 0$ in Algorithm 1, then we have the following as a corollary.

Corollary 1. *Suppose that $\mathbb{A}:\mathbb{E}\to\mathbb{H}$ satisfies (A_3) in Assumption 1. Let $\{u_n\}$ and $\{s_n\}$ be the sequences generated as follows:*

i. Choose $u_0 \in \mathbb{H}$, $\mu \in (0,1)$, and $\lambda_0 > 0$.

$$\begin{cases} s_n = P_{\mathbb{E}}(u_n - \lambda_n\mathbb{A}u_n),\\ u_{n+1} = P_{T_n}(u_n - \lambda_n\mathbb{A}s_n), \end{cases}$$

where $T_n = \{z \in \mathbb{H} : \langle u_n - \lambda_n\mathbb{A}u_n - s_n, z - s_n\rangle \leq 0\}$. Moreover, the stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu(\|u_n - s_n\|^2 - \|u_{n+1} - s_n\|^2)}{2\langle \mathbb{A}_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle}, \lambda_n\right\}, & \text{if } \langle \mathbb{A}_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle > 0\\ \lambda_n, & \text{otherwise .} \end{cases}$$

Then, the sequences $\{u_n\}$ and $\{s_n\}$ weakly converge to the solution $\breve{u} \in \Gamma$ of the VI problem.

Strong Convergence Analysis

In this subsection we present a modified version of Algorithm 1 by using viscosity method to establish a strong convergence. The modified algorithm similar to Algorithm 1 has adaptive step size, that is, the step size does not depend on the Lipschitz constant. Moreover, for the convergence analysis of the modified algorithm, the Lipschitz constant of the operator is not required.

For the convergence analysis, we suppose $g : \mathbb{H} \longrightarrow \mathbb{H}$ to be a contraction mapping with $\tau \in (0,1]$ as the contraction parameter. $\{\zeta_n\}$ is a positive sequence such that $\zeta_n = \circ(\gamma_n)$ where $\{\gamma_n\} \subset (0,1)$ with:

$$\lim_{n\to\infty} \gamma_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$$

The modified algorithm is as follows:

Algorithm 2 Adaptive Subgradient Extragradient Algorithm for Pseudomonotone Operator.

Initialization: Choose $u_{-1}, u_0 \in \mathbb{H}$, $\mu \in (0,1)$, $\varrho > 0$ and $\lambda_0 > 0$.

Iterative Steps: Given the current iterates u_{n-1} and $u_n \in \mathbb{H}$, choose ϱ_n such that $0 \leq \varrho_n \leq \varrho_n^*$, where

$$\varrho_n^* := \begin{cases} min\left\{\frac{\zeta_n}{\|u_n - x_{n+1}\|}, \varrho\right\}, & \text{if } u_n \neq u_{n+1}, \\ \varrho, & \text{otherwise .} \end{cases} \tag{19}$$

Step 1. Set t_n as:

$$t_n := u_n + \varrho_n(u_n - u_{n-1}), \tag{20}$$

Step 2. Compute

$$s_n = P_{\mathbb{E}}\left(t_n - \lambda_n \mathbb{A} t_n\right).$$

If $t_n = s_n$ or $\mathbb{A}t_n = 0$, stop. Else, go to step 3.

Step 3. Construct

$$T_n := \left\{x \in \mathbb{H} : \langle t_n - \lambda_n \mathbb{A} t_n - s_n, x - s_n\rangle \leq 0\right\}.$$

Compute

$$u_{n+1} = \gamma_n g(z_n) + (1 - \gamma_n) z_n,$$

where

$$z_n = P_{T_n}\left(t_n - \lambda_n \mathbb{A} s_n\right).$$

where the stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} := \begin{cases} \min\left\{\lambda_n, \Lambda_n\right\}, & \text{if } \langle \mathbb{A}t_n - \mathbb{A}s_n, u_{n+1} - s_n\rangle > 0 \\ \lambda_n, & \text{otherwise .} \end{cases} \tag{21}$$

where

$$\Lambda_n := \frac{\mu(\|t_n - s_n\|^2 - \|z_n - s_n\|^2)}{2\langle \mathbb{A}t_n - \mathbb{A}s_n, z_n - s_n\rangle}$$

Set $n := n + 1$ and go back to **Step 1.**

Remark 4. *It can be observed that from Equation (20) and $\zeta_n = \circ(\gamma_n)$ that is $\lim_{n\to\infty} \frac{\zeta_n}{\gamma_n} = 0$, we have for all $n \geq 1$*

$$\lim_{n\to\infty} \frac{\zeta_n}{\gamma_n}\|u_n - u_{n-1}\| \leq \lim_{n\to\infty} \frac{\zeta_n}{\gamma_n} = 0.$$

Theorem 2. *Let $\mathbb{A}$ be an operator satisfying the Assumption 1 (A3). Then, for all $p \in \Gamma \neq \emptyset$, the sequence $\{u_n\}$ generated by Algorithm 2, converges strongly to $\breve{u}$.*

Proof. To show that iterates generated by Algorithm 2 converges strongly to $\breve{u} \in \Gamma$, we consider four claims.

Claim I: The sequence $\{u_n\}$ generated by Algorithm 2 is bounded.

In fact, from Lemma 9,

$$\begin{aligned} \|z_n - \breve{u}\|^2 \leq \|t_n - \breve{u}\|^2 &- \left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n - s_n\|^2 \\ &- \left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|z_n - s_n\|^2. \end{aligned} \tag{22}$$

Recall that there exists $\aleph \in \mathbb{N}$, such that for all $n \geq \aleph$ $0 < \lambda_n \frac{\mu}{\lambda_{n+1}} < 1$. This implies that $1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu > 0$ for all $n \geq \aleph$. Thus, from Equation (22) we get

$$\|z_n - \breve{u}\|^2 \leq \|t_n - \breve{u}\|^2 \quad \text{for all} \quad n \geq N. \tag{23}$$

On the other hand,

$$\begin{aligned} \|t_n - \breve{u}\| &= \|u_n + \varrho_n(u_n - u_{n-1}) - \breve{u}\|, \\ &\leq \|u_n - \breve{u}\| + \|\varrho_n(u_n - u_{n-1})\|, \\ &= \|u_n - \breve{u}\| + \gamma_n \cdot \frac{\varrho_n}{\gamma_n}\|(u_n - u_{n-1})\|. \end{aligned} \tag{24}$$

It follows from Remark 4 that for all $n \in \mathbb{N}$, there exists a constant C_1 such that

$$\frac{\varrho_n}{\gamma_n}\|(u_n - u_{n-1})\| \leq C_1.$$

Putting Equation (23), Equation (24) and the last equation together, we obtain

$$\|z_n - \breve{u}\| \leq \|t_n - \breve{u}\| \leq \|u_n - \breve{u}\| + \gamma_n C_1 \quad \text{for all} \quad n \geq \aleph. \tag{25}$$

On the other hand, we have

$$\begin{aligned} \|u_{n+1} - \breve{u}\| &= \|\gamma_n g(z_n) + (1 - \gamma_n)z_n - \breve{u}\|, \\ &= \|\gamma_n(g(z_n) - \breve{u}) + (1 - \gamma_n)(z_n - \breve{u})\|, \\ &\leq \gamma_n\|g(z_n) - \breve{u}\| + (1 - \gamma_n)\|z_n - \breve{u}\|, \\ &\leq \gamma_n\|g(z_n) - g(\breve{u})\| + \gamma_n\|g(\breve{u}) - \breve{u}\| + (1 - \gamma_n)\|z_n - \breve{u}\|, \\ &\leq \gamma_n\tau\|z_n - \breve{u}\| + \gamma_n\|g(\breve{u}) - \breve{u}\| + (1 - \gamma_n)\|z_n - \breve{u}\|, \\ &\leq (1 - (1 - \tau)\gamma_n)\|z_n - \breve{u}\| + \gamma_n\|g(\breve{u}) - \breve{u}\|. \end{aligned} \tag{26}$$

Putting Equation (13) in Equation (26), we get

$$\begin{aligned} \|u_{n+1} - \breve{u}\| &\leq (1 - (1 - \tau)\gamma_n)\|u_n - \breve{u}\| + \gamma_n C_1 + \gamma_n\|g(\breve{u}) - \breve{u}\| \quad \text{for all} \quad n \geq \aleph, \\ &\leq (1 - (1 - \tau)\gamma_n)\|u_n - \breve{u}\| + (1 - \tau)\gamma_n \frac{C_1 + \gamma_n\|g(\breve{u}) - \breve{u}\|}{1 - \tau} \quad \text{for all} \quad n \geq \aleph, \\ &\leq \max\left\{\|u_n - \breve{u}\|, \frac{C_1 + \gamma_n\|g(\breve{u}) - \breve{u}\|}{1 - \tau}\right\} \quad \text{for all} \quad n \geq \aleph, \\ &\vdots \\ &\leq \max\left\{\|u_n - \breve{u}\|, \frac{C_1 + \gamma_n\|g(\breve{u}) - \breve{u}\|}{1 - \tau}\right\} \quad \text{for all} \quad n \geq \aleph, \end{aligned}$$

Therefore, $\{u_n\}$ is bounded and thus, $\{t_n\}, \{g(u_n)\}$ and $\{z_n\}$ are bounded.
Claim II:

$$\begin{aligned} &\left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n - s_n\|^2 + \left(1 - \frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|u_{n+1} - s_n\|^2 \\ &\leq \|u_n - \breve{u}\|^2 - \|u_{n+1} - \breve{u}\|^2 + \gamma_n C_4. \end{aligned} \tag{27}$$

In fact, from Equation (26), we have

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 &\leq \gamma_n\|g(z_n)-\breve{u}\|^2+(1-\gamma_n)\|z_n-\breve{u}\|^2,\\ &\leq \gamma_n\left(\|g(z_n)-g(\breve{u})\|+\|g(\breve{u})-\breve{u}\|\right)^2+(1-\gamma_n)\|z_n-\breve{u}\|^2,\\ &\leq \gamma_n\left(\tau\|z_n-\breve{u}\|+\|g(\breve{u})-\breve{u}\|\right)^2+(1-\gamma_n)\|z_n-\breve{u}\|^2,\\ &= \gamma_n\|z_n-\breve{u}\|^2+\gamma_n\left(2\|z_n-\breve{u}\|\cdot\|g(\breve{u})-\breve{u}\|+\|g(\breve{u})-\breve{u}\|^2\right)\\ &\quad+(1-\gamma_n)\|z_n-\breve{u}\|^2,\\ &\leq \gamma_n\|z_n-\breve{u}\|^2+(1-\gamma_n)\|z_n-\breve{u}\|^2+\gamma_nC_2,\quad \text{for some}\quad C_2>0,\\ &= \|z_n-\breve{u}\|^2+\gamma_nC_2. \end{aligned}\tag{28}$$

Substituting Equation (22) into Equation (28), we obtain

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 \leq \|t_n-\breve{u}\|^2&-\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n-s_n\|^2\\ &-\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|z_n-s_n\|^2+\gamma_nC_2.\end{aligned}\tag{29}$$

From Equation (25), we get

$$\begin{aligned}\|t_n-\breve{u}\|^2 &\leq \left(\|u_n-\breve{u}\|+\gamma_nC_1\right)^2,\\ &= \|u_n-\breve{u}\|^2+\gamma_n\left(2C_1\|u_n-\breve{u}\|+\gamma_nC_1^2\right),\\ &\leq \|u_n-\breve{u}\|^2+\gamma_nC_3,\quad \text{for some}\quad C_3>0.\end{aligned}\tag{30}$$

Putting Equations (29) and (30), we have

$$\begin{aligned}\|u_{n+1}-\breve{u}\|^2 \leq \|u_n-\breve{u}\|^2+\gamma_nC_3&-\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n-s_n\|^2\\ &-\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|z_n-s_n\|^2+\gamma_nC_2.\end{aligned}$$

Thus,

$$\begin{aligned}&\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|t_n-s_n\|^2+\left(1-\frac{\lambda_n}{2\lambda_{n+1}}\mu\right)\|u_{n+1}-s_n\|^2\\ &\leq \|u_n-\breve{u}\|^2-\|u_{n+1}-\breve{u}\|^2+\gamma_nC_4,\end{aligned}$$

where $C_4=\gamma_nC_3+\gamma_nC_2$.

Claim III:

$$\begin{aligned}\|u_{n+1}-\breve{u}\| &\leq (1-(1-\tau)\gamma_n)\|z_n-\breve{u}\|\\ &\quad+(1-\tau)\gamma_n\Big[\frac{2}{1-\tau}\left\langle g(\breve{u})-\breve{u},u_{n+1}-\breve{u}\right\rangle+\frac{3C}{1-\tau}\cdot\frac{\varrho_n}{\gamma_n}\|u_n-u_{n-1}\|\Big].\end{aligned}\tag{31}$$

In fact, we have

$$\begin{aligned}\|t_n-p\|^2 &= \|u_n+\varrho_n(u_n-u_{n-1})-\breve{u}\|^2,\\ &\leq \|u_n-\breve{u}\|^2+\varrho_n^2\|u_n-u_{n-1}\|^2+2\varrho_n\left\langle u_n-\breve{u},u_n-u_{n-1}\right\rangle,\\ &\leq \|u_n-\breve{u}\|^2+\varrho_n^2\|u_n-u_{n-1}\|^2+2\varrho_n\|u_n-\breve{u}\|\|u_n-u_{n-1}\|.\end{aligned}\tag{32}$$

Using the inequality $\|t+s\| \leq \|t\|^2 + 2\langle s, t+s\rangle$, we get

$$\begin{aligned}
\|u_{n+1}-\breve{u}\|^2 &= \|\gamma_n g(z_n) + (1-\gamma_n)z_n - \breve{u}\|^2, \\
&= \|\gamma_n(g(z_n)-g(\breve{u})) + (1-\gamma_n)(z_n-\breve{u}) + \gamma_n(g(\breve{u})-\breve{u})\|^2, \\
&\leq \|\gamma_n(g(z_n)-g(\breve{u})) + (1-\gamma_n)(z_n-\breve{u})\|^2 + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \\
&\leq \gamma_n\|g(z_n)-g(\breve{u})\|^2 + (1-\gamma_n)\|z_n-\breve{u}\|^2 + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \\
&\leq \gamma_n\tau^2\|z_n-\breve{u}\|^2 + (1-\gamma_n)\|z_n-\breve{u}\|^2 + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \\
&\leq (1-(1-\tau)\gamma_n)\|z_n-\breve{u}\|^2 + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \\
&\leq (1-(1-\tau)\gamma_n)\|t_n-\breve{u}\|^2 + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \qquad (33)
\end{aligned}$$

where the last inequality follows from Equation (23). Substituting Equation (32) into Equation (33), we get

$$\begin{aligned}
&\|u_{n+1}-\breve{u}\|^2 \\
&\quad\leq (1-(1-\tau)\gamma_n)\|u_n-\breve{u}\|^2 + \varrho_n^2\|u_n-u_{n-1}\|^2 \\
&\qquad + 2\varrho_n\|u_n-\breve{u}\|\|u_n-u_{n-1}\| + 2\gamma_n\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle, \\
&\quad\leq (1-(1-\tau)\gamma_n)\|u_n-\breve{u}\|^2 + (1-\tau)\gamma_n\frac{2}{1-\tau}\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle \\
&\qquad + \varrho_n\|u_n-u_{n-1}\|\,(2\|u_n-\breve{u}\| + \varrho_n\|u_n-u_{n-1}\|) \quad \text{for all} \quad n \geq \aleph, \\
&\quad\leq (1-(1-\tau)\gamma_n) + (1-\tau)\gamma_n\frac{2}{1-\tau}\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle \\
&\qquad + 3C\varrho_n\|u_n-u_{n-1}\| \quad \text{for all} \quad n \geq \aleph, \\
&\quad\leq (1-(1-\tau)\gamma_n) + (1-\tau)\gamma_n\frac{2}{1-\tau}\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle \\
&\qquad + \frac{3C}{1-\tau}\cdot\frac{\varrho_n}{\gamma_n}\|u_n-u_{n-1}\| \quad \text{for all} \quad n \geq \aleph,
\end{aligned}$$

where $M = \sup_{n\in\mathbb{N}}\{\|u_n-\breve{u}\|, \varrho_n\|u_n-u_{n-1}\|\} > 0$.

Claim IV: The sequence $\{\|u_n-\breve{u}\|\}$ converges to zero. From Lemma 6 it suffices to show that $\limsup_{k\to\infty}\langle g(\breve{u})-\breve{u}, u_{n+1}-\breve{u}\rangle \leq 0$ for every subsequence $\{\|u_{n_k}-\breve{u}\|\}$ of $\{\|u_n-\breve{u}\|\}$ such that

$$\liminf_{k\to\infty}\Big(\|u_{n_k+1}-\breve{u}\| - \|u_{n_k}-\breve{u}\|\Big). \qquad (34)$$

To this end, let $\{\|u_{n_k}-\breve{u}\|\}$ be a subsequence of $\{\|u_n-\breve{u}\|\}$ such that Equation (34) is satisfied, then by claim II we get

$$\begin{aligned}
&\limsup_{k\to\infty}\Big[\Big(1-\frac{\lambda_{n_k}}{2\lambda_{n_k+1}}\mu\Big)\|t_{n_k}-v_{n_k}\|^2 + \Big(1-\frac{\lambda_{n_k}}{2\lambda_{n_k+1}}\mu\Big)\|u_{n_k+1}-v_{n_k}\|^2\Big] \\
&\qquad\leq \limsup_{k\to\infty}\Big[\|u_{n_k}-\breve{u}\|^2 - \|u_{n_k+1}-\breve{u}\|^2 + \gamma_{n_k}C_4\Big], \\
&\qquad\leq \limsup_{k\to\infty}\Big[\|u_{n_k}-\breve{u}\|^2 - \|u_{n_k+1}-\breve{u}\|^2\Big] + + \limsup_{k\to\infty}\gamma_{n_k}C_4, \\
&\qquad\leq -\liminf_{k\to\infty}\Big[\|u_{n_k+1}-\breve{u}\|^2 - \|u_{n_k}-\breve{u}\|^2\Big], \\
&\qquad\leq 0.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\| = 0 \qquad (35)$$

and

$$\lim_{k\to\infty} \|z_{n_k} - v_{n_k}\| = 0,$$

Next, we show that

$$\|u_{n_k+1} - u_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{36}$$

In fact, we have

$$\|u_{n_k+1} - z_{n_k}\| = \gamma_{n_k}\|z_{n_k} - g(z_{n_k})\| \to 0 \quad \text{as} \quad k \to \infty,$$

and

$$\|u_{n_k} - t_{n_k}\| = \alpha_{n_i}\|u_{n_k+1} - u_{n_k}\| = \gamma_{n_k}\frac{\alpha_{n_i}}{\gamma_{n_k}}\|u_{n_k+1} - u_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{37}$$

It now follows that, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ such that, for some $z \in \mathbb{H}$, $u_{n_{k_j}} \rightharpoonup z$, and

$$\limsup_{k\to\infty} \langle g(\breve{u}) - \breve{u}, u_{n_k} - \breve{u}\rangle = \lim_{k\to\infty} \left\langle g(\breve{u}) - \breve{u}, u_{n_{k_j}} - \breve{u}\right\rangle = \langle g(\breve{u}) - \breve{u}, z - \breve{u}\rangle \tag{38}$$

From Equation (37), we have $t_{n_k} \rightharpoonup z \quad \text{as} \quad k \to \infty$, which it follows from Equation (35) and Lemma 9 that $z \in \Gamma$. From Equation (38) and the fact that $\breve{u} = P_\Gamma \circ g(\breve{u})$, we have

$$\limsup_{k\to\infty} \langle g(\breve{u}) - \breve{u}, u_{n_k} - \breve{u}\rangle = \langle g(\breve{u}) - \breve{u}, z - \breve{u}\rangle \leq 0. \tag{39}$$

Putting Equations (36) and (39) together, we get

$$\begin{aligned} \limsup_{k\to\infty} \langle g(\breve{u}) - \breve{u}, u_{n_k+1} - \breve{u}\rangle &\leq \limsup_{k\to\infty} \langle g(\breve{u}) - \breve{u}, u_{n_k} - \breve{u}\rangle, \\ &= \langle g(\breve{u}) - \breve{u}, z - \tilde{u}\rangle, \\ &\leq 0. \end{aligned} \tag{40}$$

Thus, the desired result follows from $\lim_{n\to\infty} \frac{\alpha_{n_i}}{\gamma_{n_k}}\|u_{n_k+1} - u_{n_k}\| = 0$, Equation (40), Claim III and Lemma 6. □

4. Computational Experiments

Numerical experiments will be presented in this section to demonstrate the performance of our proposed methods. The Codes were run on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB, MATLAB version 9.5 (R2018b). Throughout these examples y-axis and x-axis represent $D_n = \|u_{n+1} - u_n\|$ and number of iterations or execution time (in seconds) respectively. The following examples were considered for the numerical experiments of the two proposed algorithms.

Example 1. *Let an operator $\mathbb{A}$ be define on $\mathbb{E} \subset \mathbb{R}^n$ as follows*

$$\mathbb{A} = (BB^T + S + D)u + q$$

where $q \in \mathbb{R}^n$, B is an $n \times n$ matrix, S is an $n \times n$ skew-symmetric matrix, D is an $n \times n$ diagonal matrix, whose diagonal entries are nonnegative. These matrices and the vector q are randomly generated in $(0, 1)$. The set $\mathbb{E} \subset \mathbb{R}^n$ is closed and convex and defined as

$$\mathbb{E} = \{u \in \mathbb{R}^n : Cu \leq d\}$$

where C is an $100 \times n$ matrix and d is a nonnegative vector. It is clear that $\mathbb{A}$ is monotone and L-Lipschitz continuous with $L = \max(eig(BB^T + S + D))$.

Example 2. *The following problem which was considered by Sun [39] (page 9, example 5), with an operator defined as*

$$\mathbb{A}u = Gu + Hu,$$

where

$$Gu = \left(g_1u, g_2u, \cdots, g_nu\right),$$

$$Hu = Ex + d,$$

$$g_iu = u_{i-1}^2 + u_i^2 + u_{i-1}u_i + u_iu_{i+1}, \quad i = 1, 2, \ldots, n \quad u_0 = u_{n+1} = 0.$$

Here E is an $n \times n$ square matrix with entries defined as

$$e_{l,m} = \begin{cases} 4 & m = l \\ 1 & l - m = 1 \\ -2 & l - m = -1 \\ 0 & otherwise, \end{cases}$$

while $d = (-1, -1, \cdots, -1)$.

Example 3. *Let $\mathbb{H} = l_2$ and $\mathbb{E} = \{u \in \mathbb{H} : \|u\| \leq 5\}$. Define an operator $\mathbb{A}$ as follows:*

$$\mathbb{A}u = (7 - \|u\|)u \quad \forall u \in \mathbb{H},$$

where $\|u\| = \sqrt{\sum_i |u_i|^2}$. Observe that the solution set $\Gamma \neq \emptyset$ and the operator is pseudomonotone but not monotone with Lipschitz constant $L = 11$. The projection onto $\mathbb{E}$ is explicitly computed as

$$P_{\mathbb{E}}u = \begin{cases} u & if \quad \|u\| \leq 5, \\ \frac{5u}{\|u\|}, & otherwise. \end{cases}$$

4.1. Comparative Analysis for Algorithm 1

Here, we give the numerical experiment of Algorithm 1 in comparison with some existing algorithms in the literature. We use the following algorithms and their corresponding control parameters details.

1. Dong et al. [19] on page 3 (shortly, iEgA) with $\alpha_n = 0.40$, $\lambda_n = 0.80$ and $\tau = \frac{1}{1.5L}$.
2. Thong et al. [25] on page 5 (shortly, ISEGM), with $\alpha = 0.20$ and $\tau = \frac{0.5 - 2\alpha - 0.5\alpha^2}{2L(0.5 - \alpha + 0.5\alpha^2)}$.
3. Dong et al. [17] Algorithm 3.1 (shortly, iPCA), with $\alpha_n = 0.40$, $\gamma = 1.5$ and $\tau = \frac{1}{2L}$.
4. Yang et al. [40] Algorithm 3.1 (shortly, Alg3.1), with $\alpha_n = 0.15$, $\mu = 0.5$ and $\lambda_1 = \frac{0.7}{\sqrt{26}}$.
5. Thong et al. [41] Algorithm 3.1 (shortly, WISEGM), $\alpha = 0.80$, $\mu = 0.9$ and $\tau_1 = 1$.

For Algorithm 1 (shortly, Algo1) we chose $\varrho = 0.50$, $\mu = 0.80$ and $\lambda_0 = 0.60$. The results are shown in Figures 1–8.

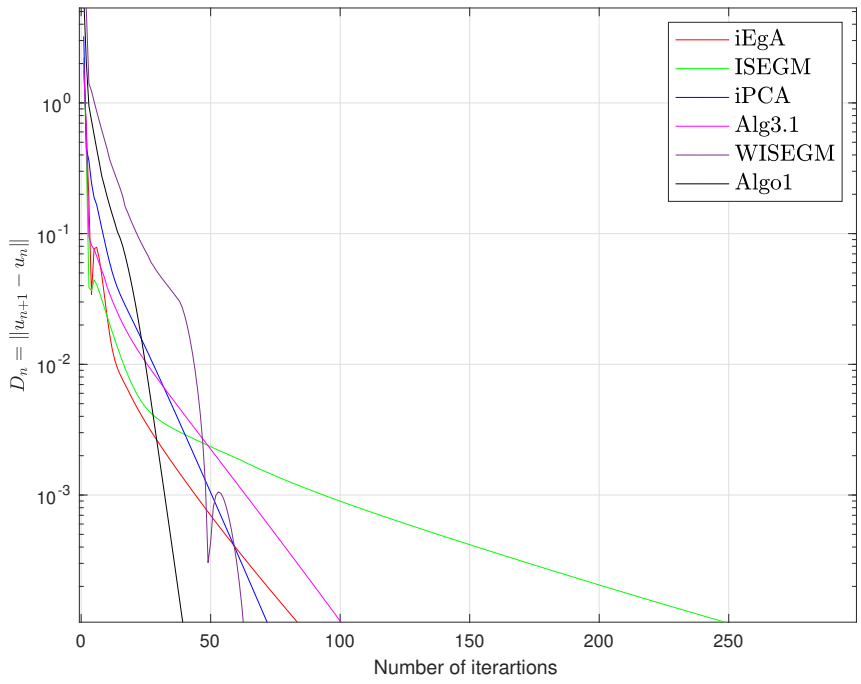

Figure 1. Example 1 in $\mathbb{R}^5$: Numbers of iterations are 85, 254, 73, 102, 63, 40 respectively.

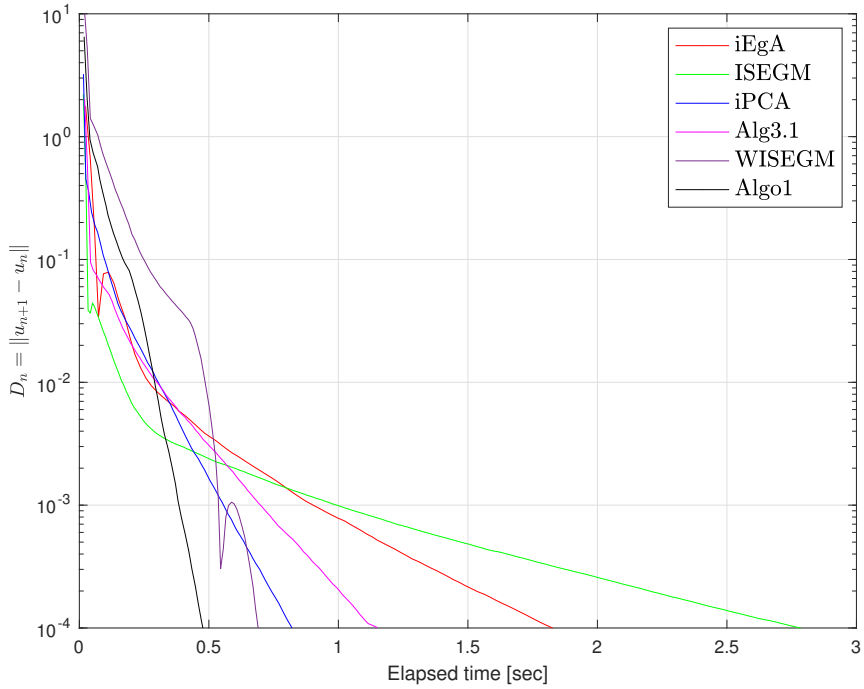

Figure 2. Example 1 in $\mathbb{R}^5$: Elapsed time 1.83, 2.78, 0.82, 1.25, 0.69, 0.48 respectively.

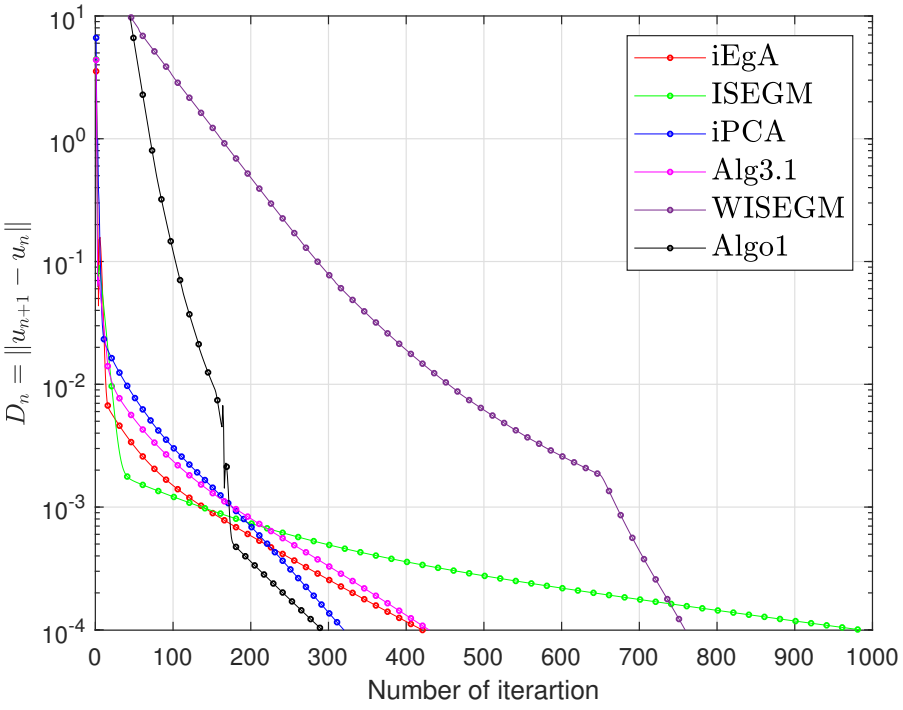

Figure 3. Example 1 in $\mathbb{R}^{50}$: Numbers of iterations 421, 985, 320, 430, 759, 292 respectively.

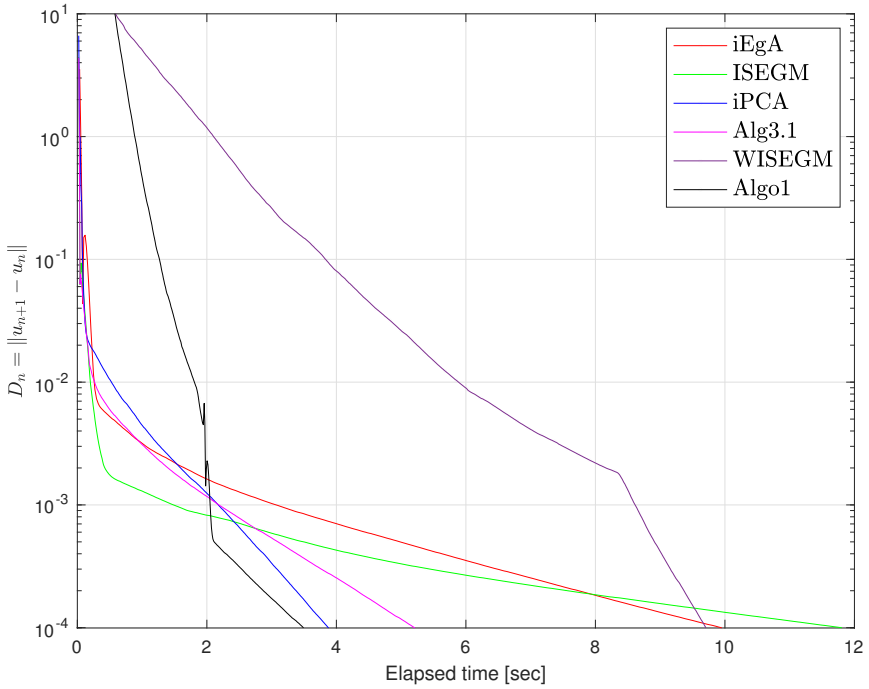

Figure 4. Example 1 in $\mathbb{R}^{50}$: Elapsed time 9.95, 11.79, 3.87, 5.19, 9.70, 3.41 respectively.

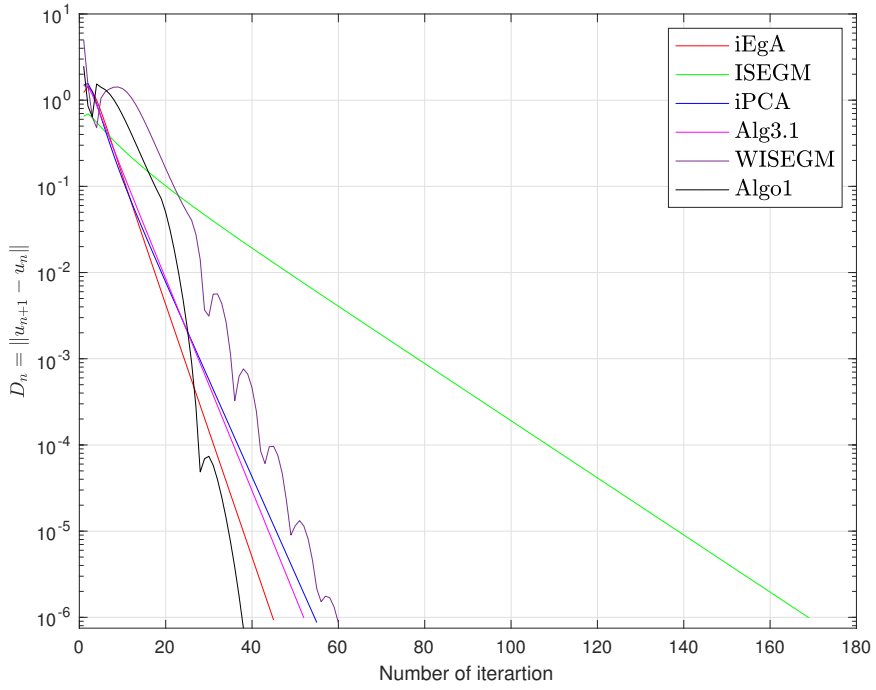

Figure 5. Example 2: Numbers of iterations are 45, 169, 155, 52, 60, 38 respectively.

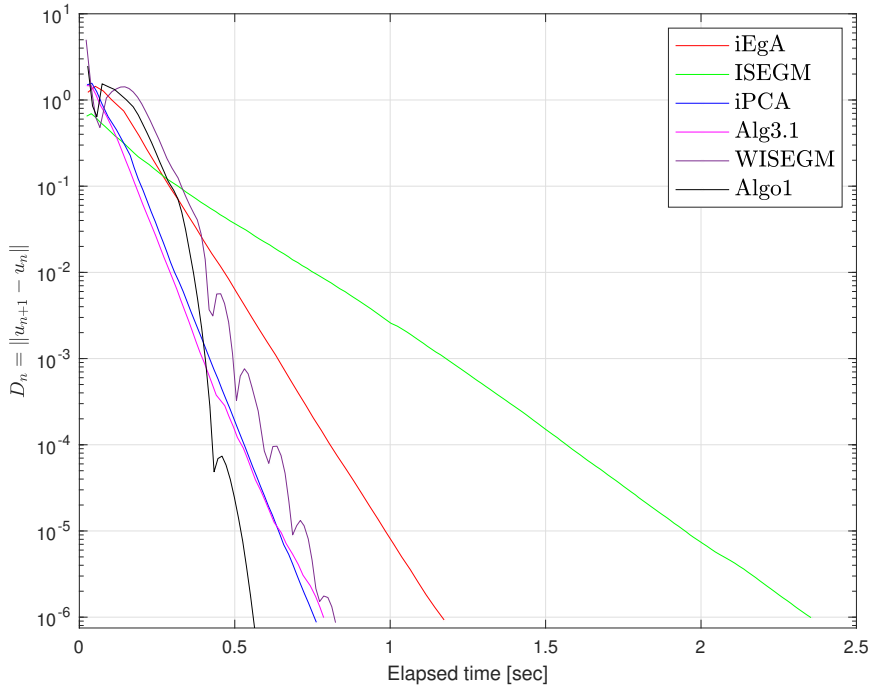

Figure 6. Example 2: Elapsed time are 1.17, 2.35, 0.76, 0.78, 0.82, 0.56 respectively.

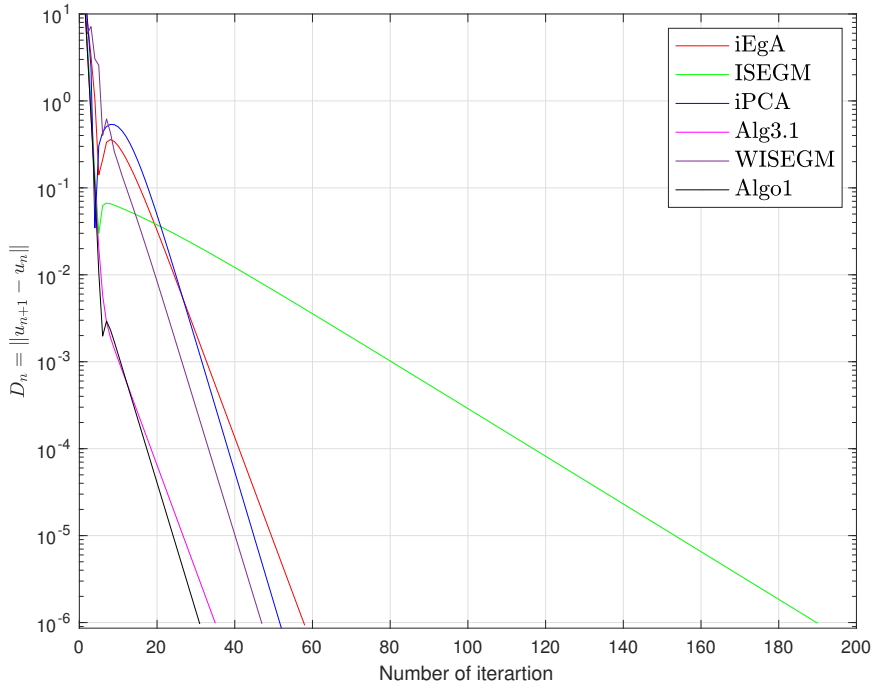

Figure 7. Example 3: Numbers of iterations are 58, 190, 52, 35, 47, 31 respectively.

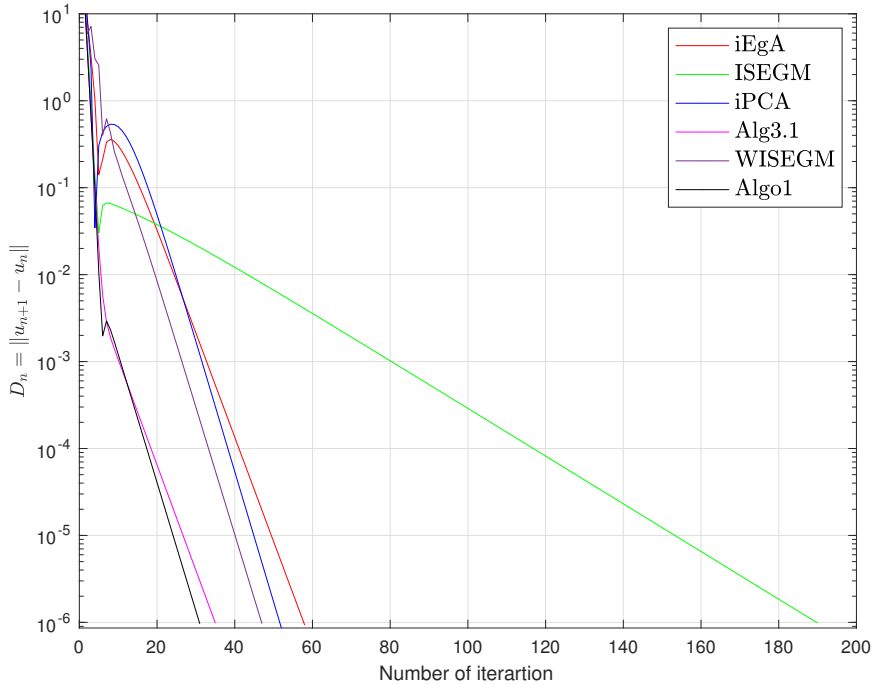

Figure 8. Example 3: Elapsed time are 2.94, 19.59, 2.73, 3.83, 4.76, 3.04 respectively.

It can be clearly seen from Figures 1–4, Figures 5 and 6 and Figures 7 and 8 for Example 1, 2 and 3 respectively that the proposed algorithm has fewer iterations and less CPU time than the compared algorithms.

4.2. Comparative Results of Algorithm 2

For the numerical results of Algorithm 2, we used the following algorithms with strong convergence.

1. Kraikaew et al. [42] on page 6 (shortly, hEgA), with $\alpha_k = \frac{1}{100(k+2)}$ and $\tau = \frac{1}{1.5L}$.
2. Yang et al. [27] Algorithm 2 (shortly, Alg.2), with $\alpha_k = \frac{1}{100(k+2)}$, $\mu = 0.4$ and $\lambda_0 = \frac{0.7}{L}$.
3. Shehu et al. [29] Algorithm 4.3 (shortly, Algorithm4.3), with $\alpha_n = \frac{1}{13n+2}$, $\gamma = 1.99$ and $\lambda_0 = \frac{0.9}{L}$.
4. Thong et al. [41] Algorithm 3.2 (shortly, SISEGM), with $\alpha = 1$, $\mu = 0.9$, $f(x) = \frac{1}{2x}$, $\beta_n = \frac{1}{n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$ and $\tau_1 = 1$.

For Algorithm 2 (shortly, Algo2), we take $\varrho = 0.8$, $\mu = 0.9$, $g(x) = \frac{1}{2x}$, $\gamma_n = \frac{1}{13n+2}$, $\epsilon_n = \frac{1}{(n+1)^2}$ and $\lambda_0 = 0.5$. The results are shown in Figures 9–16.

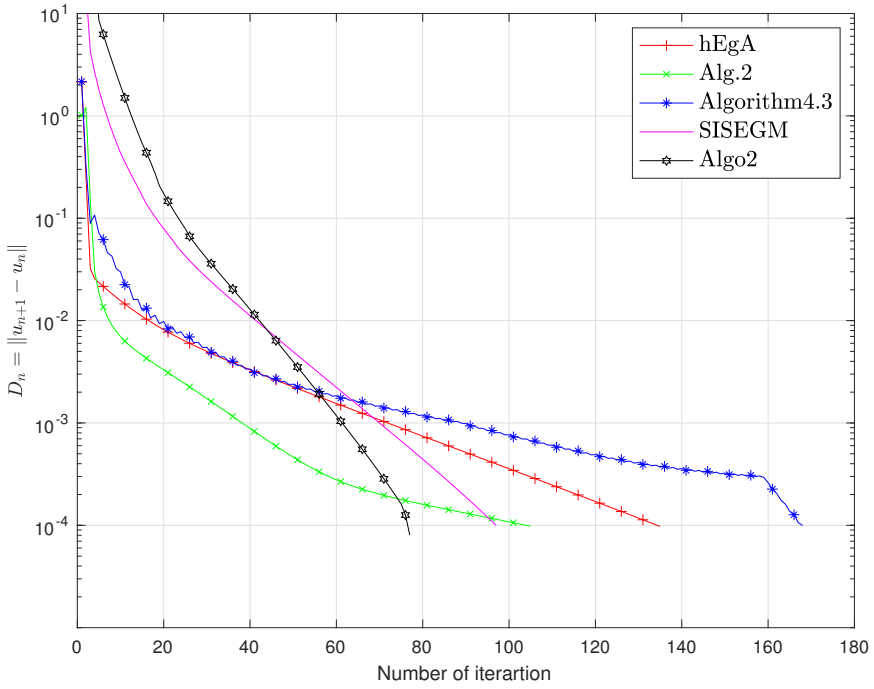

Figure 9. Example 1 in $\mathbb{R}^5$: Numbers of iterations are 135, 105, 168, 97, 77 respectively.

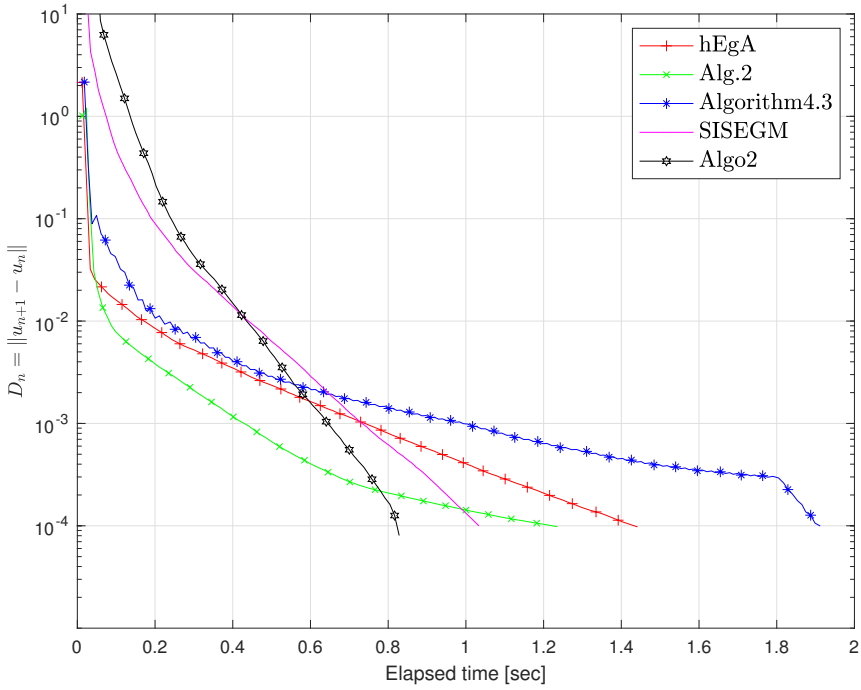

Figure 10. Example 1 in $\mathbb{R}^5$: Elapsed time 1.44, 1.23, 1.97, 1.03, 0.84 respectively.

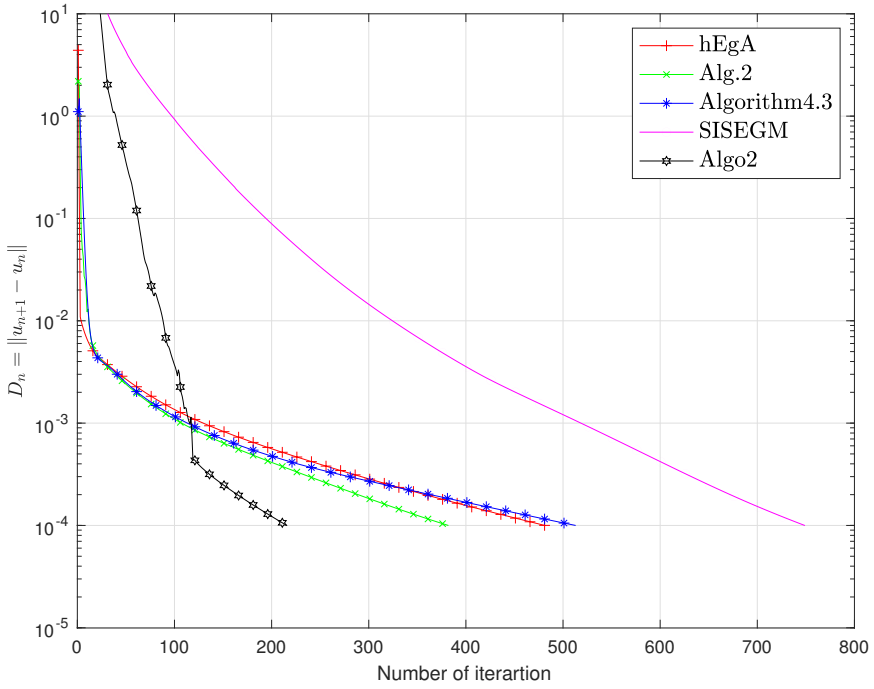

Figure 11. Example 1 in $\mathbb{R}^5$: Numbers of iterations are 482, 382, 513, 749, 216 respectively.

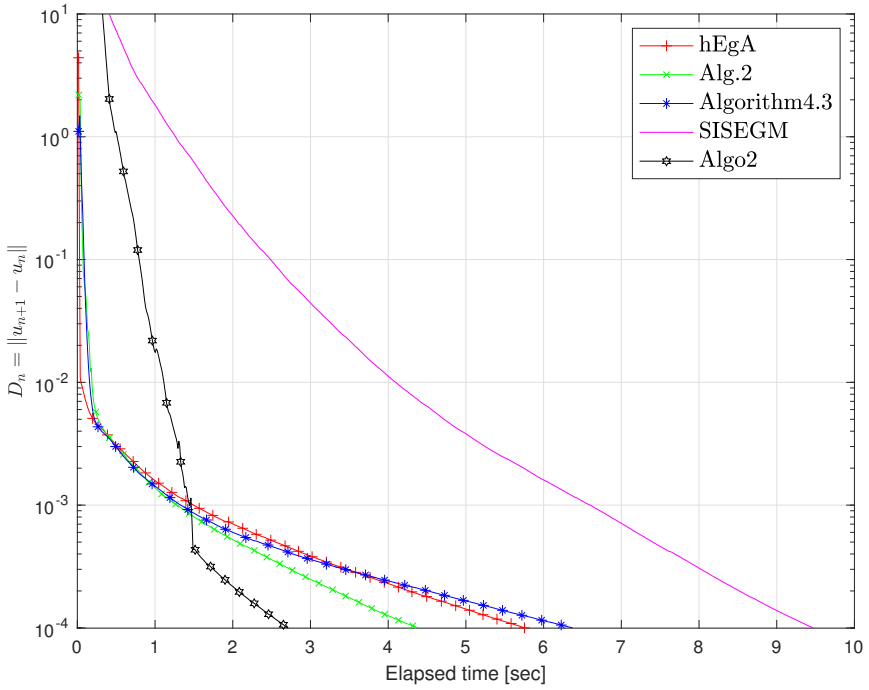

Figure 12. Example 1 in $\mathbb{R}^5$: Elapsed time 5.76, 4.38, 6.37, 9.47, 2.71 respectively.

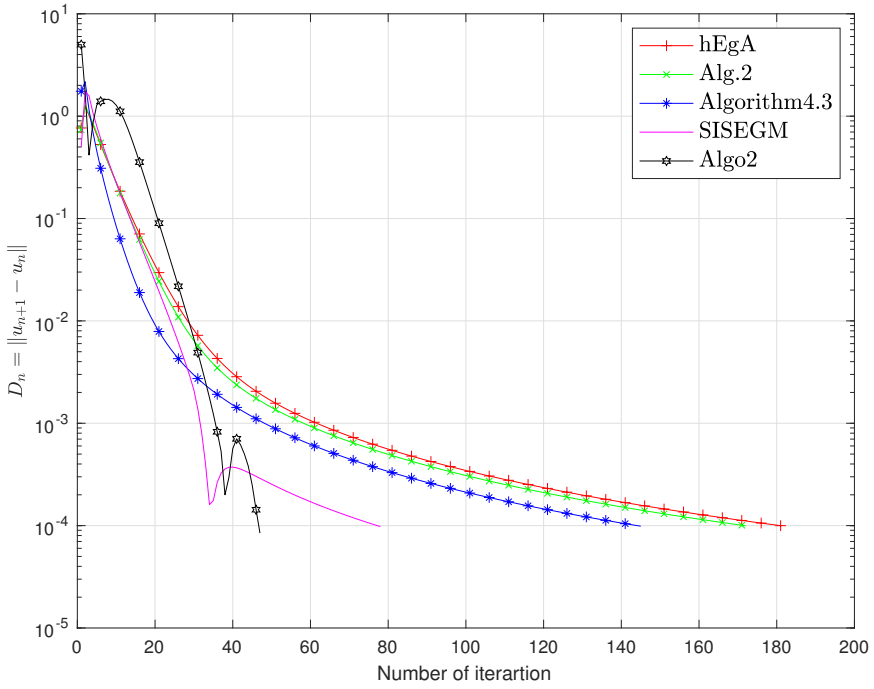

Figure 13. Example 2: Numbers of iterations are 181, 172, 145, 78, 47 respectively.

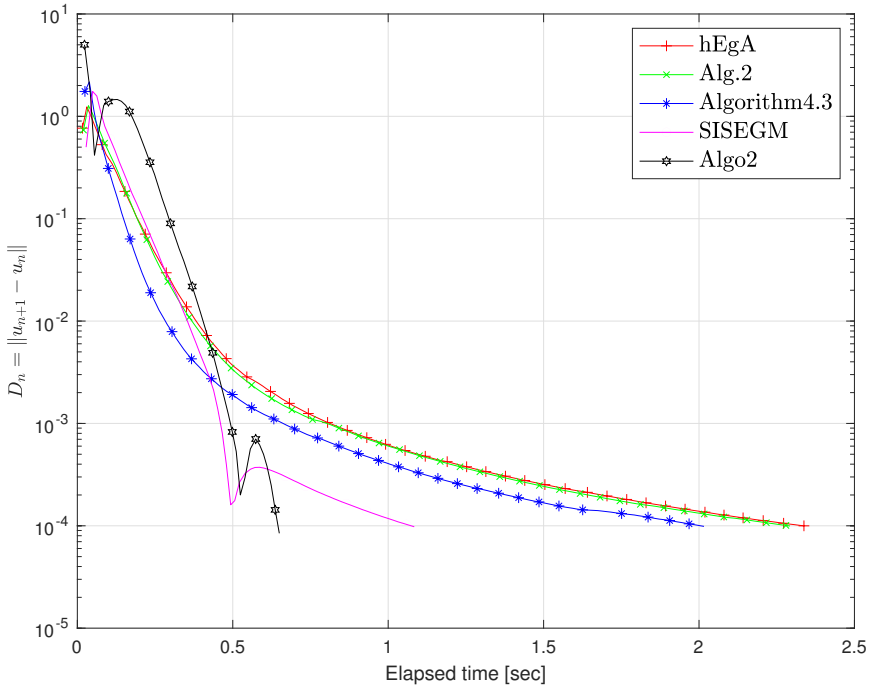

Figure 14. Example 2: Elapsed time 2.33, 2.29, 2.01, 1.08, 0.64 respectively.

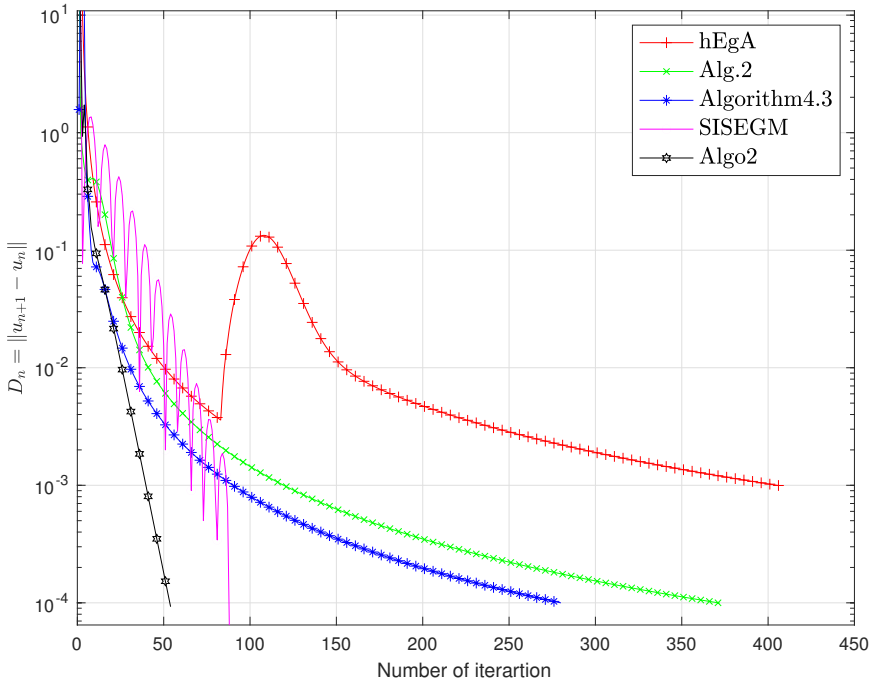

Figure 15. Example 3: Numbers of iterations are 406, 372, 280, 88, 54 respectively.

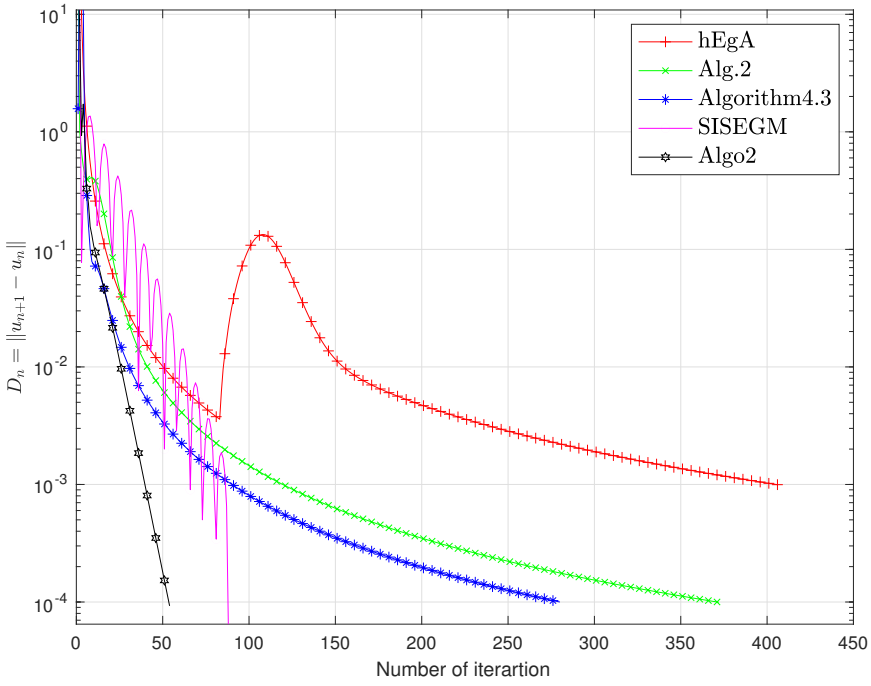

Figure 16. Example 3: Elapsed time 2.28, 2.40, 1.35, 0.58, 0.37 respectively.

For the strong convergent algorithm, it can be observed that for Example 1, 2 and 3 in Figures 9–12, Figures 13 and 14 and Figures 15 and 16 respectively, the proposed algorithm is faster, more efficient and more robust than the compared existing algorithms.

5. Conclusions

A self-adaptive subgradient extragradient algorithms with an inertial extrapolation step is presented in this work. The proposed algorithms involve a more general class of operators, the iterates generated by the first scheme converge weakly to the solution of the variational inequality problem. Furthermore, a modified version of the proposed algorithm is given by using the viscosity method to obtain a strong convergence algorithm. The main advantage of the proposed algorithms is that they do not require the prior knowledge of the Lipchitz constant of the cost operator and the iterates generated converge faster to the solution of the problem due to the inertial extrapolation step. Numerical comparison of the proposed algorithms with some of the existing state of the art algorithms shows that the proposed algorithms are fast, robust and efficient.

Author Contributions: Conceptualization, J.A. and A.H.I.; methodology, J.A.; software, H.u.R.; validation, J.A., P.K. and A.H.I.; formal analysis, J.A.; investigation, P.K.; resources, P.K.; data curation, H.u.R.; writing—original draft preparation, J.A.; writing—review and editing, J.A. and A.H.I.; visualization, H.u.R.; supervision, P.K.; project administration, P.K.; funding acquisition, P.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT. The first, third and the fourth authors were supported by "the Petchra Pra Jom Klao Ph.D. Research Scholarship" from 'King Mongkut's University of Technology Thonburi" (Grant No. 38/2018, 35/2017 and 16/2018 respectively).

Acknowledgments: The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Noor, M.A. Some developments in general variational inequalities. *Appl. Math. Comput.* **2004**, *152*, 199–277.
2. Aubin, J.; Ekeland, I. *Applied Nonlinear Analysis*; John Wiley and Sons: New York, NY, USA, 1984.
3. Marcotte, P. Application of Khobotov's algorithm to variational inequalities and network equilibrium problems. *INFOR Inf. Syst. Oper. Res.* **1991**, *29*, 258–270. [CrossRef]
4. Khobotov, E.N. Modification of the extra-gradient method for solving variational inequalities and certain optimization problems. *USSR Comput. Math. Math. Phys.* **1987**, *27*, 120–127. [CrossRef]
5. Abbas, M.; Ibrahim, Y.; Khan, A.R.; de la Sen, M. Strong Convergence of a System of Generalized Mixed Equilibrium Problem, Split Variational Inclusion Problem and Fixed Point Problem in Banach Spaces. *Symmetry* **2019**, *11*, 722. [CrossRef]
6. Kinderlehrer, D.; Stampacchia, G. *An Introduction to Variational Inequalities and Their Applications*; Siam: Philadelphia, PA, USA, 1980; Volume 31.
7. Baiocchi, C. Variational and quasivariational inequalities. *Appl. Free Bound. Probl.* **1984**.
8. Konnov, I. *Combined Relaxation Methods for Variational Inequalities*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2001; Volume 495.
9. Jaiboon, C.; Kumam, P. An extragradient approximation method for system of equilibrium problems and variational inequality problems. *Thai J. Math.* **2012**, *7*, 77–104.
10. Kumam, W.; Piri, H.; Kumam, P. Solutions of system of equilibrium and variational inequality problems on fixed points of infinite family of nonexpansive mappings. *Appl. Math. Comput.* **2014**, *248*, 441–455. [CrossRef]
11. Chamnarnpan, T.; Phiangsungnoen, S.; Kumam, P. A new hybrid extragradient algorithm for solving the equilibrium and variational inequality problems. *Afr. Mat.* **2015**, *26*, 87–98. [CrossRef]
12. Deepho, J.; Kumam, W.; Kumam, P. A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems. *J. Math. Model. Algorithms Oper. Res.* **2014**, *13*, 405–423. [CrossRef]
13. ur Rehman, H.; Kumam, P.; Cho, Y.J.; Yordsorn, P. Weak convergence of explicit extragradient algorithms for solving equilibirum problems. *J. Inequalities Appl.* **2019**, *2019*, 1–25. [CrossRef]
14. Kazmi, K.R.; Rizvi, S. An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping. *Optim. Lett.* **2014**, *8*, 1113–1124. [CrossRef]
15. Facchinei, F.; Pang, J.S. *Finite-Dimensional Variational Inequalities and Complementarity Problems*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2007.
16. Solodov, M.V.; Svaiter, B.F. A new projection method for variational inequality problems. *SIAM J. Control. Optim.* **1999**, *37*, 765–776. [CrossRef]
17. Dong, Q.; Cho, Y.; Zhong, L.; Rassias, T.M. Inertial projection and contraction algorithms for variational inequalities. *J. Glob. Optim.* **2018**, *70*, 687–704. [CrossRef]
18. Denisov, S.; Semenov, V.; Chabak, L. Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. *Cybern. Syst. Anal.* **2015**, *51*, 757–765. [CrossRef]
19. Dong, Q.L.; Lu, Y.Y.; Yang, J. The extragradient algorithm with inertial effects for solving the variational inequality. *Optimization* **2016**, *65*, 2217–2226. [CrossRef]
20. Malitsky, Y.V. Projected reflected gradient methods for variational inequalities. *SIAM J. Optim.* **2015**, *25*, 502–520. [CrossRef]
21. Abubakar, J.; Sombut, K.; Ibrahim, A.H.; Rehman, H.R. An Accelerated Subgradient Extragradient Algorithm for Strongly Pseudomonotone Variational Inequality Problems. *Thai J. Math.* **2019**, *18*, 166–187.
22. Sombut, K.; Kitkuan, D.; Padcharoen, A.; Kumam, P. Weak Convergence Theorems for a Class of Split Variational Inequality Problems. In Proceedings of the 2018 International Conference on Control, Artificial Intelligence, Robotics & Optimization (ICCAIRO), Prague, Czech Republic, 19–21 May 2018; pp. 277–282.
23. Chamnarnpan, T.; Wairojjana, N.; Kumam, P. Hierarchical fixed points of strictly pseudo contractive mappings for variational inequality problems. *SpringerPlus* **2013**, *2*, 540. [CrossRef]
24. Crombez, G. A geometrical look at iterative methods for operators with fixed points. *Numer. Funct. Anal. Optim.* **2005**, *26*, 157–175. [CrossRef]
25. Thong, D.V.; Van Hieu, D. Modified subgradient extragradient method for variational inequality problems. *Numer. Algorithms* **2018**, *79*, 597–610. [CrossRef]

26. Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **2011**, *148*, 318–335. [CrossRef] [PubMed]
27. Yang, J.; Liu, H.; Liu, Z. Modified subgradient extragradient algorithms for solving monotone variational inequalities. *Optimization* **2018**, *67*, 2247–2258. [CrossRef]
28. Van Hieu, D.; Thong, D.V. New extragradient-like algorithms for strongly pseudomonotone variational inequalities. *J. Glob. Optim.* **2018**, *70*, 385–399. [CrossRef]
29. Shehu, Y.; Dong, Q.L.; Jiang, D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. *Optimization* **2019**, *68*, 385–409. [CrossRef]
30. Khanh, P.D.; Vuong, P.T. Modified projection method for strongly pseudomonotone variational inequalities. *J. Glob. Optim.* **2014**, *58*, 341–350. [CrossRef]
31. Thong, D.V.; Van Hieu, D. Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. *J. Comput. Appl. Math.* **2018**, *341*, 80–98. [CrossRef]
32. Bauschke, H.H.; Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*; Springer: Berlin/Heidelberg, Germany, 2011; Volume 408.
33. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive*; Marcel Dekker Inc.: New York, NY, USA, 1984.
34. Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM J. Optim.* **2004**, *14*, 773–782. [CrossRef]
35. Ofoedu, E. Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space. *J. Math. Anal. Appl.* **2006**, *321*, 722–728. [CrossRef]
36. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
37. Thong, D.V.; Vuong, P.T. Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. *Optimization* **2019**, *68*, 2207–2226. [CrossRef]
38. Vuong, P.T. On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. *J. Optim. Theory Appl.* **2018**, *176*, 399–409. [CrossRef] [PubMed]
39. Sun, D. A projection and contraction method for the nonlinear complementarity problem and its extensions. *Math. Numer. Sin.* **1994**, *16*, 183–194.
40. Yang, J. Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. *Appl. Anal.* **2019**. [CrossRef]
41. Thong, D.V.; Van Hieu, D.; Rassias, T.M. Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems. *Optim. Lett.* **2020**, *14*, 115–144. [CrossRef]
42. Kraikaew, R.; Saejung, S. Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **2014**, *163*, 399–412. [CrossRef]



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).