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Nonlocal Integro-Differential Equations of the Second Order with Degeneration

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Abstract: We study the solvability for boundary value problems to some nonlocal second-order integro–differential equations that degenerate by a selected variable. The possibility of degeneration in the equations under consideration means that the statements of the corresponding boundary value problems have to change depending on the nature of the degeneration, while the nonlocality in the equations implies that the boundary conditions will also have a nonlocal form. For the problems under study, the paper provides conditions that ensure their well-posedness.

Keywords: integro-differential equation; degeneration; boundary value problem; non-local conditions; well-posedness

MSC: 35M10; 35M12; 35M13

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1. Introduction

Let Ω be a bounded domain in the space $\mathbb{R}^n$ with a smooth (infinitely differentiable for the simplicity) boundary Γ, Q is the cylinder $\Omega \times (0,T)$, $0 < T < +\infty$, $a_k(x,t)$, $k = 0,1,2$, while $b_0(x)$ and $f(x,t)$ are given functions that defined at $x \in \overline{\Omega}$, $t \in [0,T]$, B is an integral functional on the space $L_2(\Omega)$, defined by the equality

$$B\varphi = \int\limits_{\Omega} b_0(x)\varphi(x)\,dx, \quad \varphi(x) \in L_2(\Omega).$$

The purpose of the work is the study of the solvability of boundary value problems for the integro-differential equations

$$\Delta u + \sum_{k=0}^{2} a_k(x,t)\frac{\partial^{2-k}}{\partial t^{2-k}}(Bu) = f(x,t), \tag{1}$$

where $u = u(x,t)$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}$ is the Laplace operator with respect to the variables $x_1, x_2, \ldots, x_n$. The features of the above equations are, firstly, that they are nonlocal, or loaded [1,2], and secondly, that these equations allow degeneration by the selected variable t (degeneration occurs if the function $a_0(x,t)$ somehow turns to zero, and with a possible change of sign, for certain values of the variable t).

As in general for degenerate or mixed-type equations, the boundary conditions for (1) may differ depending on the nature of the degeneration, see [3–9]. Note that in many cases, for degenerate equations, for mixed-type equations, only the existence of generalized solutions to certain boundary value problems is proved. In our work, the main goal will be to prove the existence and uniqueness of regular solutions, i.e., solutions that have all generalized in the sense of S.L. Sobolev derivatives included in the equation.

Below, we show that for Equation (1) one, two, or three conditions can be defined for t, or even conditions may not set at all, and for all such problems in the work conditions will be given that ensure their correctness.

The nonlocal character of the Equation (1) leads to the nonlocal character of the boundary conditions at $t = 0$ or $t = T$. We will specify exactly what conditions will be discussed below.

Let $\overline{b}_0(x)$ be a solution of the Dirichlet problem

$$\Delta\overline{b}_0(x) = b_0(x), \quad x \in \Omega,$$

$$\overline{b}_0(x)|_\Gamma = 0.$$

Define the functional $\overline{B}$:

$$\overline{B}\varphi = \int\limits_\Omega \overline{b}_0(x)\varphi(x)\, dx, \quad \varphi(x) \in L_2(\Omega).$$

Further define the nonlocal condition for $u(x,t)$, which will be used in various boundary value problems for (1):

$$\overline{B}u(x,t)|_{x\in\Omega,\, t=0} = 0; \tag{2}$$

$$\overline{B}u(x,t)|_{x\in\Omega,\, t=T} = 0; \tag{3}$$

$$\overline{B}u_t(x,t)|_{x\in\Omega,\, t=0} = 0; \tag{4}$$

$$\overline{B}u_t(x,t)|_{x\in\Omega,\, t=T} = 0. \tag{5}$$

As mentioned above, boundary value problems for (1) will use one, two, or three nonlocal conditions from conditions (2)–(5), or, in one case, all nonlocal conditions (2)–(5) will not be used.

The technique used is based on the regularization method, a priori estimates, and limit transition.

Some of the results presented below are new even for local (non-integro-differential) equations.

One last note. Some comments on the results obtained, their possible generalizations and amplifications will be discussed at the end of the article.

2. Solvability of the Boundary Value Problem without Nonlocal Conditions

Consider the boundary value problem: *Find a function $u(x,t)$ that is a solution of* (1) *in the cylinder Q satisfying*

$$u(x,t)|_{\Gamma\times(0,T)} = 0. \tag{6}$$

The solvability of this problem is established using an auxiliary result on the solvability of degenerate ordinary differential equations.

Everywhere below, L_p, W_p^l $(p \geq 1)$ denotes the usual Lebesgue spaces of summable functions, or, respectively, functions that have generalized in the sense of S.L. Sobolev derivatives of order up to and including l, whose modulus is summed over a given domain with degree p. The definition of these spaces can be found, for example, in the monograph [10].

We will denote by h', h'', h''' and so on the first, the second, the third derivative with respect to the unique variable respectively.

Let $h_0(t)$, $h_1(t)$ and $h_2(t)$ be given functions defined at $t \in [0,T]$, and let $F_1(\xi,\eta)$ and $F_2(\xi,\eta)$ be the quadratic forms

$$F_1(\xi,\eta) = \left[h_1(0) - h_0'(0)\right]\xi^2 - 2h_2'(0)\xi\eta - \left[h_1(0) + h_0'(0)\right]\eta^2,$$

$$F_2(\xi,\eta) = \left[h_0'(T) - h_1(T)\right]\xi^2 + 2h_2'(T)\xi\eta + \left[h_1(T) + h_0'(T)\right]\eta^2.$$

Theorem 1. *Let the conditions*

$$h_k(t) \in C^2([0,T]), \quad k = 0,1,2, \tag{7}$$

be satisfied;

$$h_0(t) \geq 0 \quad \text{at } t \in [0,T]; \tag{8}$$

$$h_0(0) = h_0(T) = 0, \quad F_1(\xi,\eta) \geq 0, \quad F_2(\xi,\eta) \geq 0 \quad \text{at } (\xi,\eta) \in \mathbb{R}^2; \tag{9}$$

$$h_2(t) - \frac{1}{2}h_1'(t) + \frac{1}{2}h_0''(t) \leq -\bar{h}_0 < 0 \quad \text{at } t \in [0,T]; \tag{10}$$

$$h_2(t) + \frac{1}{2}h_1'(t) \leq -\bar{h}_1 < 0 \quad \text{at } t \in [0,T]; \tag{11}$$

$$h_2(t) + \frac{3}{2}h_1'(t) + \frac{1}{2}h_0''(t) \leq -\bar{h}_2 < 0 \quad \text{at } t \in [0,T], \tag{12}$$

and let $g(t)$ be a given function, such that one of the conditions

(a) $g(t) \in W_2^2([0,T])$, $g'(0) = g'(T) = 0$;

(b) $g(t) \in L_2([0,T])$, $[h_0(t)]^{-\frac{1}{2}} g'(t) \in L_2([0,T])$

be satisfied. Then the differential equation

$$h_0(t)v'' + h_1(t)v' + h_2(t)v = g(t) \tag{13}$$

has a unique solution $v(t)$ in the space $W_2^2([0,T])$, moreover, for this solution the inclusion $[h_0(t)]^{\frac{1}{2}} v'''(t) \in L_2([0,T])$ is fulfilled.

Proof. Let ε be a positive number. Consider the boundary value problem: *Find a function $v(t)$, which is the solution on the segment $[0,T]$ of the equation*

$$-\varepsilon v'''' + h_0(t)v'' + h_1(t)v' + h_0(t)v = g(t) \tag{14}$$

and such that the conditions

$$v'(0) = v'(T) = v'''(0) = v'''(T) = 0 \tag{15}$$

is satisfied. The solvability of this problem for a fixed ε and if the function $g(t)$ belongs to the space $L_2([0,T])$ is not difficult to establish using the continuation method for parameter [11] and the first a priori estimate

$$\varepsilon \int_0^T v''^2(t)\,dt + \int_0^T h_0(t)v'^2(t)\,dt + \int_0^T v^2(t)\,dt \leq K_1 \int_0^T g^2(t)\,dt \tag{16}$$

(the constant K_1 in (16) is defined by the functions $h_k(t)$, $k = 0,1,2$, only), which is true for solutions of boundary value problem (13), (14) if conditions (7)–(10) are fulfilled. We show that for solutions $v(t)$ of this problem, the a priori estimates are valid, which will allow us to perform the limit transition procedure for the ε parameter (with the choice of a convergent subsequence) and to prove the existence of the required solution of Equation (13).

Multiply Equation (13) by the function $v''(t)$ and integrate over the segment $[0,T]$. After integration in parts we get the equality

$$\varepsilon \int_0^T v'''^2(t)\,dt + \int_0^T h_0(t)v''^2(t)\,dt - \int_0^T \left[h_2(t) + \frac{1}{2}h_1'(t)\right] v'^2(t)\,dt = \int_0^T h_2'(t)v(t)v'(t)\,dt - \int_0^T g'(t)v'(t)\,dt.$$

From this, using the conditions of the theorem, first a priori estimate (16), and Young's inequality, it is easy to deduce the second a priori estimate for solutions $v(t)$ of boundary value problem (14), (15)

$$\varepsilon \int_0^T v'''^2(t)\,dt + \int_0^T h_0(t)v''^2(t)\,dt + \int_0^T v'^2(t)\,dt \leq K_2 \tag{17}$$

with a constant K_2, defined by the functions $h_k(t)$, $k = 0, 1, 2$, and $g(t)$ only.

On the next step multiply Equation (13) by the function $-v''''(t)$ and integrate over the segment $[0, T]$. After the integration in parts we obtain the equality

$$\varepsilon \int_0^T v''''^2(t)\,dt + \int_0^T h_0(t)v'''^2(t)\,dt - \int_0^T \left[h_2(t) + \frac{3}{2}h_1'(t) + \frac{1}{2}h_0''(t)\right] v''^2\,dt +$$

$$+\frac{1}{2}F_1\left(v(0), v''(0)\right) + \frac{1}{2}F_2\left(v(T), v''(T)\right) = -\int_0^T g'(t)v'''(t)\,dt +$$

$$+\int_0^T h_1''(t)v'(t)v''(t)\,dt + 2\int_0^T h_2'(t)v'(t)v''(t)\,dt + \int_0^T h_2''(t)v(t)v''(t)\,dt -$$

$$-\frac{1}{2}\left[h_1(0) - h_0'(0)\right] v^2(0) - \frac{1}{2}\left[h_0'(T) - h_1(T)\right] v^2(T).$$

Taking into account conditions (7)–(12) of the theorem, estimates (16) and (17), applying Young's inequality and under condition a) for the function $g(t)$, additionally integrating in parts in the term with the function $g'(t)$, we can show that the result of this equality implies the third a priori estimate

$$\varepsilon \int_0^T v''''^2(t)\,dt + \int_0^T h_0(t)v'''^2(t)\,dt + \int_0^T v''^2(t)\,dt \leq K_3 \tag{18}$$

with a constant K_3, defined by the functions $h_k(t)$, $k = 0, 1, 2$, and $g(t)$ only.

It follows from estimates (16)–(18) and from the reflexivity property of a Hilbert space that there are sequences $\{\varepsilon_m\}_{m=1}^{\infty}$ of positive numbers, $\{u_m(t)\}_{m=1}^{\infty}$ of solutions to boundary value problems (14), (15) with $\varepsilon = \varepsilon_m$, and a function $v(t)$ such that at $m \to \infty$ we have the convergences

$$\varepsilon_m \to 0,$$

$$v_m(t) \to v(t) \text{ weakly in the space } W_2^2([0, T]),$$

$$\sqrt{h_0(t)}v_m'''(t) \to \sqrt{h_0(t)}v'''(t) \text{ weakly in the space } L_2([0, T]),$$

$$\varepsilon_m v_m''''(t) \to 0 \text{ weakly in the space } L_2([0, T]).$$

From these convergences, it follows that the limit function $v(t)$ will be a solution of Equation (13) belonging to the space $W_2^2([0, T])$, and this solution will satisfy the inclusion $[h_0(t)]^{\frac{1}{2}}v'''(t) \in L_2([0, T])$.

The uniqueness of a solution to (13) is easy to establish by multiplying this equation by the function $-v(t)$, integrating the resulting equality over the segment $[0, T]$ and using conditions (7)–(10). □

We will denote by the subscript the partial derivative with respect to the corresponding variable, for example, $f_t = \frac{\partial f}{\partial t}$.

Let's return to the Equation (1). Everywhere else (here and in what follows), we assume that $h_k(t)$, $k = 0, 1, 2$, have the form

$$h_k(t) = \int\limits_{\Omega} a_k(x,t)\bar{b}_0(x)\,dx, \quad k = 0,1, \quad h_2(t) = \int\limits_{\Omega} a_2(x,t)\bar{b}_0(x)\,dx + 1.$$

Theorem 2. *Let the conditions*

$$\frac{\partial^i a_k(x,t)}{\partial t^i} \in C(\overline{Q}), \quad i,k = 0,1,2, \quad b_0(x) \in C(\overline{\Omega}), \tag{19}$$

be fulfilled and functions $h_k(t)$, $k = 0,1,2$, satisfy conditions (8)–(12), and

$$|a_0(x,t)| \leq K\sqrt{h_0(t)}, \quad K \geq 0, \quad (x,t) \in \overline{Q}. \tag{20}$$

Then for every function $f(x,t)$, which satisfies one of the conditions

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f_t(x,0) = f_t(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$

boundary value problem (1), (6) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0,T])$, and it is unique.

Proof. Consider Equation (13) with a function $g(t)$ of the form

$$g(t) = \int\limits_{\Omega} f(x,t)\bar{b}_0(x)\,dx.$$

Theorem 1 implies the existence of a unique solution $v(t)$ of this equation, such that

$$v(t) \in W_2^2([0,T]), \quad [h_0(t)]^{\frac{1}{2}} v'''(t) \in L_2([0,T]). \tag{21}$$

Consider a boundary value problem: *Find a function $u(x,t)$ which is a solution of the equation*

$$\Delta u = f(x,t) - a_0(x,t)v''(t) - a_1(x,t)v'(t) - a_2(x,t)v \tag{22}$$

in the cylinder Q, such that condition (6) is fulfilled. The right-hand side of this equation belongs to the space $L_\infty(0,T;L_2(\Omega))$ (due to condition (20) and inclusions (21)). According to the general theory of elliptic equations, boundary value problem (22), (6) has a solution $u(x,t)$ from the space $L_2(0,T;W_2^2([0,T]))$, see, e. g., [12,13]. Multiply equation (22) by the function $\bar{b}_0(x)$ and integrate over the domain Ω. After simple transformations, we obtain the equality

$$\int\limits_{\Omega} b_0(x)u(x,t)\,dx = v(t).$$

It follows from this equality that a solution $u(x,t)$ of boundary value problem (22), (6) will be a solution to boundary value problem (1), (2) simultaneously. It is obvious that the function $u(x,t)$ belongs to the required class.

Now let us establish the uniqueness of a solution. Let $f(x,t) \equiv 0$. Then the function $v(t)$ will be identical to the zero function on the interval $(0,T)$. But then the function $u(x,t)$ as a solution to problem (22), (6) with the zero right-hand side will be zero function in the cylinder Q. This means that a solution of boundary value problem (1), (6) is unique in the required class. □

3. The Boundary Value Problem with One Nonlocal Condition

In this section the solvability of the boundary value problem for Equation (1) with boundary condition (6) and one nonlocal condition at $t = 0$, condition (2), or (4), will be investigated (boundary value problem with one nonlocal condition at $t = T$, condition (3), or condition (5), by the obvious way with replacing $t' = T - t$ can be reduced to the problem with condition (2) or condition (4)). We will not highlight the auxiliary results on the solvability of the boundary value problem for the degenerate ordinary differential equation.

Theorem 3. *Let condition (19) be fulfilled, functions $h_k(t)$, $k = 0, 1, 2$, satisfy conditions (8), (10)–(12), (20), and*

$$h_0(0) > 0, \quad h_0(T) = h_1(0) = 0, \quad F_2(\xi, \eta) \geq 0 \text{ at } (\xi, \eta) \in \mathbb{R}^2.$$

Then for every function $f(x, t)$, which satisfies one of the conditions

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f(x,0) = f_t(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$, $f(x,0) = 0$ *at* $x \in \Omega$

boundary value problem (1), (2), (6) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0, T; W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0, T])$, and it is unique.

Proof. Let $g(t)$ has the form

$$g(t) = \int\limits_{\Omega} f(x,t)\bar{b}_0(x)\, dx$$

again. Consider a boundary value problem: *Find a function $v(t)$ which is a solution of Equation (13) on the interval $(0, T)$ and satisfies the condition*

$$v(0) = 0. \tag{23}$$

Again applying the regularization of Equation (13) by Equation (14), using also the boundary conditions

$$v(0) = v''(0) = v'(T) = v'''(T) = 0,$$

it is easy to establish, that boundary value problem (13), (23) has in the space $W_2^2([0, T])$ exactly one solution $v(t)$, for whom the inclusion $[h_0(t)]^{\frac{1}{2}} v'''(t) \in L_2([0, T])$ is fulfilled. Next, we define the function $u(x,t)$ as the solution of problem (22), (6). This function will be the required solution to problem (1), (2), (6).

Uniqueness of problem (1), (2), (6) solution follows from the uniqueness of a solution to boundary value problem (13), (23) once more. □

Theorem 4. *Let condition (19) be fulfilled, functions $h_k(t)$, $k = 0, 1, 2$, satisfy conditions (8), (10)–(12), (20), and*

$$h_0(0) > 0, \quad h_0(T) = 0, \quad F_1(\xi, \eta) \geq 0, \quad F_2(\xi, \eta) \geq 0 \quad \text{at } (\xi, \eta) \in \mathbb{R}^2.$$

Then for every function $f(x, t)$, which satisfies one of the conditions

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f_t(x,0) = f_t(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$

boundary value problem (1), (4), (6) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0, T; W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0, T])$, and it is unique.

The proof of this theorem is carried out quite similarly to the proof of Theorem 3. The necessary auxiliary result on the unique solvability in the space $W_2^2([0,T])$ of the boundary value problem for Equation (13) with the condition

$$v'(0) = 0 \tag{24}$$

is also proved analogously to the proof of Theorem 1.

The proof of Theorem 1 on the solvability of degenerate ordinary differential Equation (13), as well as the proof of the corresponding results on the solvability of boundary value problems (13), (23) and (13), (24) (used in the proof of Theorem 3 and Theorem 4), are based on the "elliptic-parabolic" [3–5] approach. But for problem (13), (23), you can use another, "hyperbolic-parabolic" approach [5,6,9]. This approach will give different conditions for the solvability of boundary value problem (1), (2), (6) compared to Theorem 3.

Theorem 5. *Let condition* (19) *be fulfilled, functions* $h_k(t)$, $k = 0,1,2$, *satisfies the conditions*

$$h_0(0) \le 0, \quad h_0(T) \ge 0; \tag{25}$$

$$h_1(t) - \frac{1}{2}h_0'(t) \ge \overline{h}_0 > 0, \quad h_1(t) + \frac{3}{2}h_0'(t) \ge \overline{h}_1 > 0 \quad \text{at } t \in [0,T]; \tag{26}$$

$$h_2(t) + \frac{1}{2}h_1'(t) \le -\overline{h}_1 < 0 \quad \text{at } t \in [0,T]; \tag{27}$$

$$h_2(T) \ge 0, \quad h_2'(t) \le 0 \quad \text{at } t \in [0,T]. \tag{28}$$

Then for every function $f(x,t)$ *such that*

$$f(x,t) \in L_2(Q),\ f_t(x,t) \in L_2(Q),\ f_{tt}(x,t) \in L_2(Q),$$

boundary value problem (1), (2), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^3([0,T])$.

Proof. Let ε be a positive number. Consider a boundary value problem: *Find a function* $v(t)$ *which is a solution of the equation*

$$\varepsilon v^{(5)}(t) + h_0(t)v''(t) + h_1(t)v'(t) + h_2(t)v(t) = g(t) \tag{29}$$

on the interval $(0,T)$ *and satisfies the conditions*

$$v(0) = v'''(0) = v''''(0) = 0, \quad v'''(T) = v''''(T) = 0. \tag{30}$$

For a solution of problem (29), (30) under conditions (25)–(28) there are a priori estimates

$$\varepsilon \int_0^T v'''^2(t)\,dt + \int_0^T v'^2(t)\,dt \le N_1 \int_0^T g^2(t)\,dt, \tag{31}$$

$$\varepsilon \int_0^T [v^{(5)}(t)]^2\,dt + \int_0^T v'''^2(t)\,dt \le N_2 \int_0^T [g^2(t) + g'^2(t) + g''^2(t)]\,dt \tag{32}$$

with constants N_1 and N_2, defined only by the functions $h_k(t)$, $k = 0,1,2$, and by the number T (we prove these estimates by analyzing the equalities obtained after multiplying equation (29) sequentially on the functions $v'(t)$ and $v^{(5)}(t)$ and integrating over the segment $[0,T]$). Estimates (31) and (32) imply, first, that for a fixed ε and if the function $g(t)$ belongs to the space $L_2([0,T])$, boundary value problem (29), (30) is uniquely solvable in the space $W_2^3([0,T])$, and, secondly, that in this problem, if the function $g(t)$ belongs to the space $W_2^2([0,T])$, the limit transition procedure can be performed (with the selection of appropriate numerical and functional sequences). As a result, we get that when conditions (25)–(28) are met and the function $g(t)$ belongs to the space $W_2^2([0,T])$, boundary value problem (13), (23) has a solution $v(t)$ belonging to the space $W_2^3([0,T])$, and there is exactly one solution.

Let now $g(t)$ be

$$g(t) = \int\limits_{\Omega} f(x,t)\bar{b}_0(x)\,dx,$$

$v(t)$ be a solution of problem (13), (23) with such $g(t)$. Let us define the function $u(x,t)$ in a standard way (within the framework of this work). This function will be the required solution to boundary value problem (1), (2), (6).

The uniqueness of the solutions is obvious. □

Note that the conditions of Theorem 5 have significant differences from the conditions of Theorems 1–4, since they do not require nonnegativity of the function $h_0(t)$.

4. The Case of Two Nonlocal Conditions

The study of the solvability of the boundary value problem for Equation (1) with condition (6) and two nonlocal conditions is quite similar to the study of the solvability of the problem with condition (6) and with one nonlocal condition.

Theorem 6. *Let condition (19) be fulfilled, for function $h_k(t)$, $k = 0,1,2$, conditions (8), (10)–(12), (20) be satisfied, and*

$$h_0(0) > 0, \quad h_0(T) > 0, \quad h_1(0) = h_1(T) = 0.$$

Then for every function $f(x,t)$ which satisfies one of the conditions

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f(x,0) = f(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$, $f(x,0) = f(x,T) = 0$ *at* $x \in \Omega$

boundary value problem (1), (2), (3), (6) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0,T])$, and it is unique.

Proof. Using condition (14) and the boundary conditions

$$v(0) = v''(0) = v(T) = v''(T) = 0,$$

it is easy to establish the existence of an unique function $v(t)$, which is a solution of Equation (13) and such that inclusions (21) are satisfied. Using this function, the required solution of boundary value problem (1), (2), (3), (6) is constructed in a standard way.

The uniqueness of a solution is obvious. □

Theorem 7. *Let condition (19) be satisfied, functions $h_k(t)$, $k = 0,1,2$, satisfy conditions (8), (10)–(12), (20), and*

$$h_0(0) > 0, \quad h_0(T) > 0, \quad h_1(0) = 0, \quad F_2(\xi,\eta) \geq 0 \text{ at } (\xi,\eta) \in \mathbb{R}^2.$$

Then for every function $f(x,t)$ which satisfies one of the conditions

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f(x,0) = f_t(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$, $f(x,0) = 0$ *at* $x \in \Omega$

boundary value problem (1), (2), (5), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0,T])$, *and it is unique.*

Theorem 8. *Let condition* (19) *be satisfied, functions* $h_k(t)$, $k = 0,1,2$, *satisfy conditions* (8), (10)–(12), (20), *and*

$$h_0(0) > 0, \quad h_0(T) > 0, \quad F_1(\xi,\eta) \geq 0, \quad F_2(\xi,\eta) \geq 0 \quad \text{at } (\xi,\eta) \in \mathbb{R}^2.$$

Then for every function $f(x,t)$ *which satisfies one of the conditions*

(a) $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f_{tt}(x,t) \in L_2(Q)$;
(b) $f_t(x,0) = f_t(x,T) = 0$ *at* $x \in \Omega$;
(c) $f(x,t) \in L_2(Q)$, $[h_0(t)]^{-\frac{1}{2}} f_t(x,t) \in L_2(Q)$

boundary value problem (1), (4), (5), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0,T])$, *and it is unique.*

The proof of these theorems is carried out in the same way as the proof of Theorem 2, we only specify that Equation (14) is supplemented with the conditions

$$v(0) = v''(0) = 0, \quad v'(T) = v'''(T) = 0,$$

when proving Theorem 7, and by the conditions

$$v'(0) = v'''(0) = 0, \quad v'(T) = v'''(T) = 0,$$

for the proof of Theorem 8.

Remark 1. *It is obvious that boundary value problem* (1), (3), (4), (6) *by the change of variables* $t' = T - t$ *reduced to problem* (1), (2), (5), (6).

We give two more results on the solvability of boundary value problems (1), (2), (4), (6) and (1), (2), (5), (6), the proof of which will be based on the "hyperbolic-parabolic" approach.

Theorem 9. *Let condition* (19) *be satisfied, functions* $h_k(t)$, $k = 0,1,2$, *satisfy conditions* (26), (28), *and*

$$h_0(0) > 0, \quad h_0(T) \geq 0.$$

Then for every function $f(x,t)$ *such that*

$$f(x,t) \in L_2(Q), \; f_t(x,t) \in L_2(Q), \; f_{tt}(x,t) \in L_2(Q),$$

$$f(x,0) = f_t(x,0) = 0 \text{ at } x \in \Omega$$

boundary value problem (1), (2), (4), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^3([0,T])$.

Theorem 10. *Let condition* (19) *be satisfied, functions* $h_k(t)$, $k = 0,1,2$, *satisfy conditions* (26), (28), *and*

$$h_0(0) \leq 0, \quad h_0(T) < 0.$$

Then for every function $f(x,t)$ *such that*

$$f(x,t) \in L_2(Q), \; f_t(x,t) \in L_2(Q), \; f_{tt}(x,t) \in L_2(Q),$$

$$f(x,T) = f_t(x,T) = 0 \text{ at } x \in \Omega$$

boundary value problem (1), (2), (5), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T; W_2^2(\Omega))$, $(Bu)(t) \in W_2^3([0,T])$.

The proof of Theorem 9 and Theorem 10 is carried out analogously to the proof of Theorem 5, only the boundary conditions for Equation (29) change. When proving Theorem 9, instead of conditions (30), the conditions

$$v(0) = v'(0) = v''(0) = 0, \quad v'''(T) = v''''(T) = 0$$

are used. For the proof of Theorem 10 we use the conditions

$$v(0) = v'''(0) = v''''(0) = 0, \quad v'(T) = v''(T) = 0$$

instead of conditions (30).

5. The Boundary Value Problem with Three Nonlocal Conditions

The "hyperbolic-parabolic" approach to mixed-type equations (both for ordinary differential equations and partial differential equations) allows us to show that for second-order differential equations, the boundary value problem with three boundary conditions can also be correct.

Theorem 11. *Let condition* (19) *be satisfied, functions* $h_k(t)$, $k = 0,1,2$, *satisfy conditions* (26), (28), *and*

$$h_0(0) > 0, \quad h_0(T) < 0.$$

Then for every function $f(x,t)$ *such that*

$$f(x,t) \in L_2(Q), \; f_t(x,t) \in L_2(Q), \; f_{tt}(x,t) \in L_2(Q),$$

$$f(x,0) = f_t(x,0) = f(x,T) = f_t(x,T) = 0 \text{ at } x \in \Omega$$

boundary value problem (1), (2), (4), (5), (6) *has a solution* $u(x,t)$ *such that* $u(x,t) \in L_\infty(0,T; W_2^2(\Omega))$, $(Bu)(t) \in W_2^2([0,T])$.

The proof of this theorem is carried out quite similarly to the proof of Theorem 5, but with the replacement of conditions (30) by the conditions

$$v(0) = v'(0) = v''(0) = 0, \quad v'(T) = v''(T) = 0.$$

Remark 2. *As in other cases, replacing* $t' = T - t$ *it is not difficult to convert boundary value problem* (1), (3), (4), (5), (6) *to one considered and thus to get a theorem about its solvability.*

6. Comments and Additions

1. The integro-differential equations studied in this paper, as already mentioned in the Introduction, can be called nonlocal equations, or loaded equations. In general, a very significant number of papers have been devoted to the theory of integro-differential equations. Studies of the solvability of such equations and boundary value problems for them are important both from the point of view of the development of mathematics, and from the point of view of mathematical modeling, since nonlocal equations, integro-differential equations, and loaded equations arise in the mathematical modeling of many processes in mechanics, physics, and biology. Note also that nonlocal integro-differential equations arise naturally in fractional calculus [14], in some studies related to the theory of inverse coefficient problems [15], and in some other sections of the mathematical theory. The equations considered in this paper can also be interpreted as equations that are not resolved with respect to the derivative, namely, as an equation $lBu + \Delta u = f$ with a differential operator l acting on a time variable [16,17].

On the other hand, we note that nonlocal integro-differential equations of form (1) with degeneration have not been studied before. Here we can only note the work [18], which studied first-order with respect to t equations close to (1) (we specify that the methods of studying degenerate ordinary differential equations of the second order have a number of significant differences from the methods of studying similar first-order equations).

2. It is obvious that the Laplace operator in Equation (1) can be replaced by a general elliptic operator, including an operator of the order $2m$ (with the natural addition of the necessary boundary conditions).

3. Theorem 1 and auxiliary results on the existence of solutions to boundary value problems for degenerate ordinary differential equations of the second order can be interpreted as a refinement in relation to the one-dimensional case of some results of works [4,6,9].

4. If for the auxiliary boundary value problems for degenerate ordinary differential Equation (13) we establish the existence of solutions which are smoother than in Sections 1–4, then it is not difficult to prove that the corresponding nonlocal boundary value problem for Equation (1) will have a solution $u(x,t)$ such that $u_t(x,t)$, $u_{tt}(x,t)$ exist and belong to the space $L_\infty(0,T;W_2^2(\Omega))$. Here is just one such result.

Theorem 12. *Let functions $a_k(x,t)$, $k = 0,1,2$, have continuous in $\overline{Q}$ derivatives with respect to t up to and including the fourth order, function $b_0(x)$ be continuous in $\overline{\Omega}$. Next, let the conditions (25)–(28) be met. Then for every function $f(x,t)$ such that it and its derivatives in the variable t up to and including the fourth order belong to the space $L_2(Q)$, boundary value problem (1), (2), (6) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $u_t(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $u_{tt}(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $(Bu)(t) \in W_2^5([0,T])$.*

Proof. Consider the auxiliary value problem: *Find a function $v(t)$ which is a solution on $(0,T)$ of the equation*

$$\varepsilon v^{(9)}(t) + h_0(t)v''(t) + h_1(t)v'(t) + h_2(t)v(t) = g(t) \tag{33}$$

and such the conditions

$$v(0) = v^{(5)}(0) = v^{(6)}(0) = v^{(7)}(0) = v^{(8)}(0) = 0,$$

$$v^{(5)}(T) = v^{(6)}(T) = v^{(7)}(T) = v^{(8)}(T) = 0$$

are satisfied for it. The a priori estimates obtained after multiplying Equation (33) by the functions $v'(t)$ and $v^{(9)}(t)$ and integration will allow us to organize a limit transition and get that the boundary value problem for Equation (13) with the condition $v(0) = 0$ has a solution $v(t)$ belonging to the space $W_2^5([0,T])$. It implies that in boundary value problem (22), (6), the right-hand side and its derivatives with respect to the variable t up to and including the second order will belong to the space $L_\infty(0,T;W_2^2(\Omega))$. This means that the solution $u(x,t)$ of boundary value problem (1), (2), (6) will be such that the functions $u_t(x,t)$ and $u_{tt}(x,t)$ are correctly defined and belong to the space $L_\infty(0,T;W_2^2(\Omega))$. $\square$

7. Conclusions

A new class of integro-differential equations with degeneracy is studied. Statements of non-local boundary value problems are proposed for these equations, and theorems of existence and uniqueness of regular solutions (solutions belonging to Sobolev spaces) are proved. Let us clarify that the problem statements are essentially determined by the nature of the degeneracy in the equation itself.

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