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On the Solvability of Fourth-Order Two-Point Boundary Value Problems

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Abstract: In this paper, we study the solvability of various two-point boundary value problems for $x^{(4)} = f(t, x, x', x'', x''')$, $t \in (0, 1)$, where f may be defined and continuous on a suitable bounded subset of its domain. Imposing barrier strips type conditions, we give results guaranteeing not only positive solutions, but also monotonic ones and such with suitable curvature.

Keywords: fourth-order differential equation; two-point boundary value problems; existence; positive or non-negative, monotone, convex or concave solutions; sign conditions

MSC: 34B15; 35B18

1. Introduction

This paper is devoted to the solvability of boundary value problems (BVPs) for the equation

$$x^{(4)} = f(t, x, x', x'', x'''),\ t \in (0, 1), \tag{1}$$

with boundary conditions (BCs) either

$$x(0) = A, x'(0) = B, x''(0) = C, x'''(1) = D \tag{2}$$

$$x(1) = A, x'(0) = B, x''(0) = C, x'''(1) = D, \tag{3}$$

$$x(0) = A, x'(1) = B, x''(0) = C, x'''(1) = D, \tag{4}$$

$$x(1) = A, x'(1) = B, x''(0) = C, x'''(1) = D, \tag{5}$$

$$x(0) = A, x'(0) = B, x''(1) = C, x'''(1) = D, \tag{6}$$

$$x(1) = A, x'(0) = B, x''(1) = C, x'''(1) = D, \tag{7}$$

$$x(0) = A, x'(1) = B, x''(1) = C, x'''(1) = D, \tag{8}$$

$$x(1) = A, x'(1) = B, x''(1) = C, x'''(1) = D, \tag{9}$$

$$x(0) = A, x(1) = B, x''(0) = C, x'''(1) = D, \tag{10}$$

$$x(0) = A, x(1) = B, x''(1) = C, x'''(1) = D, \tag{11}$$

$$x(0) = A, x'(0) = B, x'(1) = C, x'''(1) = D \tag{12}$$

or

$$x(1) = A, x'(0) = B, x'(1) = C, x'''(1) = D, \tag{13}$$

where $f : [0,1] \times D_x \times D_p \times D_q \times D_r \to \mathbf{R}$, and $D_x, D_p, D_q, D_r \subseteq \mathbf{R}$.

It is well known that fourth-order BVPs appear in beam analysis. Among them, the solvability of two-point ones has received much attention in the literature. Such problems for equations of the form

$$x^{(4)} = f(t,x),\ t \in (0,1),$$

were studied by A. Cabada et al. [1], J. Caballero et al. [2], J. Cid et al. [3], G. Han and Z. Xu [4], J. Harjani et al. [5], G. Infante and P. Pietramala [6], J. Li [7] (here $f(t,x)$ may be singular at $t = 0,1$ and $x = 0$), B. Yang [8] and C. Zhai and C. Jiang [9]. In [1,3–5] the BCs are

$$x(0) = x(1) = x''(0) = x''(1) = 0, \tag{14}$$

in [2,7] they are

$$x(0) = x(1) = x'(0) = x'(1) = 0, \tag{15}$$

homogeneous (12) in [8], and those in [6,9] include homogeneous ones (6).

J. Liu and W. Xu [10], D. O'Regan [11] and Q. Yao [12] studied BVPs for

$$x^{(4)} = f(t,x,x'),$$

with BCs (14) in [10], with homogeneous (6) in [12], and with either (6) (for $A, B \geq 0, C, D = 0$), (12) (for $A \geq 0, B, C, D = 0$),

$$x(0) = x(1) = x''(0) = x'''(0) = 0 \text{ or}$$
$$x(0) = A \geq 0, x'(1) = B \geq 0, x''(0) = x''(1) = 0 \tag{16}$$

in [11] (here $f(t,x,p)$ may be singular at $t = 0,1,\ x = 0$ and/or $p = 0$).

Many researchers studied BVPs for equations of the form

$$x^{(4)} = f(t,x,x''),\ t \in (0,1),$$

see Z. Bai et al. [13], D. Brumley et al. [14], M. Del Pino and R. Manasevich [15], A. El-Haffaf [16], P. Habets and M. Ramalho [17], R. Ma [18] and D. O'Regan [19] ($f(t,x,q)$ may be singular at $t = 0,1,\ x = 0$ and/or $q = 0$). The BCs in [14] are

$$\alpha_1 x(0) - \gamma_1 x(1) = \beta_1 x'(0) - \delta_1 x'(1) = -A,$$

$$\alpha_2 x''(0) - \gamma_2 x''(1) = \beta_2 x'''(0) - \delta_2 x'''(1) = B,$$

where $A, B, \alpha_i, \beta_i, \gamma_i, \delta_i > 0, i = 1,2$, in [15] they are inhomogeneous ones of the form (14),

$$x(0) = x'(0) = x''(1) = x'''(0) = 0 \tag{17}$$

in [20], periodic ones in [17], and in [13,18] they are (14). Except (16), BCs (4), (6) and (12), all with $A \geq 0, B \geq 0, C \leq 0$ and $D = 0$, are considered in [19].

The existence of solutions for more general equations of the form

$$x^{(4)} = f(t,x,x',x''),\ t \in (0,1),$$

was studied by M. Elgindi and Z. Guan [20], A. El-Haffaf [16], T. Ma [21] and Q. Yao [22] (here $f(t,x,p,q)$ may be singular at $t = 0,1,\ x = 0, p = 0$ and $q = 0$). Homogeneous BCs (4), (6), (12), (14) and (15) are considered in [20], in [16] they are again (17), in [21] are (14), and (4) in [22].

BVPs for equations of the form (1) were considered by R. Agarwal [23], Z. Bai [24], C. De Coster et al. [25], J. Ehme et al. [26], D. Franco et al. [27], A. Granas et al. [28], Y. Li and

Q. Liang [29], Y. Liu and W. Ge [30], R. Ma [31], F. Minhós et al. [32], B. Rynne [33], F. Sadyrbaev [34] and Q. Yao [35]. The BCs in those works are as follows:

$$x(0) = A, x'(0) = B, x(1) = C, x''(1) = D$$

in [23,34], homogeneous (16) in [24], homogeneous (4) in [24,31], (14) in [29,32,33], periodic in [25], (15) in [28,33],

$$x'(0) = x'(1) = x'''(0) = x'''(1) = 0$$

in [30], homogeneous (6), (12), (14), (15) in [31], general nonlinear in [34], of the form

$$g_1(\bar{x}) = 0, g_2(\bar{x}) = 0, h_1(\tilde{x}) = 0, h_2(\tilde{x}) = 0$$

in [26,27], where $g_i, h_i, i = 1,2$, are continuous, $\bar{x} = (x(0), x(1), x'(0), x'(1), x''(0), x''(1))$ in both papers, $\tilde{x} = \bar{x}$ in [26], and $\tilde{x} = (x(0), x(1), x'(0), x'(1), x''(0), x''(1), x'''(0), x'''(1))$ in [27].

In the works mentioned above, the main nonlinearity is a Carathéodory function on unbounded set, see [6,31,35], or is defined and continuous on a set such that each dependent variable changes in a left- and/or right-unbounded set, see [1–5,7,8,10,11,13–30,32–34,36,37]; an exception is [9] where it is supposed that $f(t,x)$ is continuous on a set of the form $(0,1) \times [a,b]$. Several results guaranteeing a unique solution, at least one or multiple solutions are obtained by using various techniques, conditions and tools as: the classical lower and upper solutions technique [1,7,13,16,17,24,26,27,32,34], requirements to the main nonlinearity of the equation to be positive or non-negative [2,3,5–8,10,14,18,19,21,22], Nagumo-type growth conditions [24,26,27,32,34], nonresonance conditions [15,25], monotone conditions [7,24], maximum principles [1,17], various applications of Greens function [2,3,6,8,10,14,16,29,36–38]. Because of the nature of processes leading to fourth-order BVPs, a variety of authors study the existence of positive solutions, see [2,3,5–10,12,14,18,21,22].

We do not use the above tools. Throughout the paper we assume that:

Hypothesis 1 **($\mathbf{H}_1$).** *There are constants $F_i, L_i, i = 1,2$, and a sufficiently small $\sigma > 0$ such that*

$$F_2 + \sigma \le F_1 \le D \le L_1 \le L_2 - \sigma, \ [F_2, L_2] \subseteq D_r,$$

$$f(t,x,p,q,r) \ge 0 \text{ for } (t,x,p,q,r) \in [0,1] \times D_x \times D_p \times D_q \times [L_1, L_2], \tag{18}$$

$$f(t,x,p,q,r) \le 0 \text{ for } (t,x,p,q,r) \in [0,1] \times D_x \times D_p \times D_q \times [F_2, F_1]. \tag{19}$$

Besides, we will say that some of the BVPs (1),(k), $k = 2,3,...,13$($k = \overline{2,13}$ for short) satisfies ($\mathbf{H}_2$) for constants $m_i, M_i, i = \overline{0,3}$, if:

Hypothesis 2 **($\mathbf{H}_2$).** *$m_i \le M_i, i = \overline{0,3}$, $[m_0 - \sigma, M_0 + \sigma] \subseteq D_x$, $[m_1 - \sigma, M_1 + \sigma] \subseteq D_p$, $[m_2 - \sigma, M_2 + \sigma] \subseteq D_q$,$[m_3 - \sigma, M_3 + \sigma] \subseteq D_r$, where σ is as in ($\mathbf{H}_1$), and $f(t,x,p,q,r)$ is continuous on $[0,1] \times J$, where $J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times [m_2 - \sigma, M_2 + \sigma] \times [m_3 - \sigma, M_3 + \sigma]$.*

The condition ($\mathbf{H}_1$) is a variant of the barrier strips type conditions used in [39–41], for example, for studying the solvability of first-order initial and second-order boundary value problems; we do not know another application of conditions of this type for studying the solvability of fourth-order problems. ($\mathbf{H}_1$) allows the sets D_x, D_p, D_q and D_r to be bounded and f to be continuous only on a bounded subset of its domain. Also, together with ($\mathbf{H}_2$), it allows studying not only the existence of positive solutions, but also of monotone ones as well as of solutions with suitable curvature.

The paper is organized as follows. In Section 2, we give our basic existence theorem and auxiliary results guaranteeing a priori bounds for $x'''(t), x''(t), x'(t)$ and $x(t)$, in this order, for each eventual solution $x(t) \in C^4[0,1]$ to the families of BVPs for

$$x^{(4)} = \lambda f(t, x, x', x'', x'''),\ \lambda \in [0,1],\ t \in (0,1), \tag{1$_\lambda$}$$

with one of the boundary conditions (k), $k = \overline{2,13}$. Section 3 is devoted to the solvability of BVPs (1), (2), ..., (1), (8) and (1), (9). In Section 4 we study (1), (10) and (1), (11), and in Section 5 give existence results for (1), (12) and (1), (13).

2. Basic Existence Results, Auxiliary Results

For the convenience of the reader we include basic notions and facts needed to formulate the topological transversality theorem. Moreover, we follow A. Granas et al. [28].

Let Y be a convex subset of a Banach space E and let $U \subset Y$ be open in Y. The compact map $F : \overline{U} \to Y$ is called admissible if it is fixed point free on the boundary, ∂U, of U. We denote the set of all such maps by $L_{\partial U}(\overline{U}, Y)$.

A map $F \in L_{\partial U}(\overline{U}, Y)$ is inessential if there is a compact fixed point free map $G : \overline{U} \to Y$ such that the restrictions of G and F to ∂U coincide. A map $F \in L_{\partial U}(\overline{U}, Y)$ which is not inessential is called essential.

Two maps F and G in $L_{\partial U}(\overline{U}, Y)$ are called homotopic if there is a compact homotopy $H_\lambda : \overline{U} \to Y$ for which $F = H_0$, $G = H_1$, and H_λ is admissible for each $\lambda \in [0,1]$.

Theorem 1 (Chapter I, Theorem 2.2, [28])**.** *Let $p \in U$ be fixed and $F \in L_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then F is essential.*

Lemma 1 (Chapter I, Lemma 2.4, [28])**.** *A map $F \in L_{\partial U}(\overline{U}, Y)$ is inessential if and only if it is homotopic to a fixed point free map.*

The next result is a consequence of this lemma.

Theorem 2 (Topological transversality theorem, Chapter I, Theorem 2.5, [28])**.** *Let F and G in $L_{\partial U}(\overline{U}, Y)$ be homotopic maps. Then one of these maps is essential if and only if the other is.*

In fact, the transversality theorem is used in the following equivalent form.

Theorem 3 (Chapter I, Theorem 2.6, [28])**.** *Suppose:*

(i) $F, G : \overline{U} \to Y$ *are compact maps.*
(ii) $G \in L_{\partial U}(\overline{U}, Y)$ *is essential.*
(iii) $H(x, \lambda), \lambda \in [0,1]$, *is a compact homotopy joining G and F, i.e.,*

$$H(x,0) = G(x) \text{ and } H(x,1) = F(x).$$

(iv) $H(x, \lambda), \lambda \in [0,1]$, *is fixed point free on ∂U.*

Then $H(x, \lambda), \lambda \in [0,1]$, has at least one fixed point in U and in particular there is a $x_0 \in U$ such that $x_0 = F(x_0)$.

Consider the BVP

$$x^{(4)} + \sum_{k=0}^{3} s_k(t) x^{(k)} = f(t, x, x', x'', x'''),\ t \in (0,1),\quad V_i(x) = r_i, i = \overline{1,4}, \tag{20}$$

where $s_k(t), k = \overline{0,3}$, are continuous on $[0,1]$, $f : [0,1] \times D_x \times D_p \times D_q \times D_r \to \mathbf{R}$,

$$V_i(x) \equiv \sum_{j=0}^{3}[a_{ij}x^{(j)}(0) + b_{ij}x^{(j)}(1)],\ i = \overline{1,4},$$

with constants a_{ij} and b_{ij} for which $\sum_{j=0}^{3}(a_{ij}^2 + b_{ij}^2) > 0, i = \overline{1,4}$, and $r_i \in \mathbf{R}$.

For $\lambda \in [0,1]$ consider also the family of BVPs

$$x^{(4)} + \sum_{k=0}^{3} s_k(t)x^{(k)} = g(t, x, x', x'', x''', \lambda), t \in (0,1),\ \ V_i(x) = r_i, i = \overline{1,4}, \tag{21}$$

where $g : [0,1] \times D_x \times D_p \times D_q \times D_r \times [0,1] \to \mathbf{R}$, and $s_k(t), k = \overline{0,3}, V_i, r_i, i = \overline{1,4}$, are as above.

Finally, let BC be the set of functions satisfying the boundary conditions $V_i(x) = r_i$, $i = \overline{1,4}$, let $C^4_{BC}[0,1] = C^4[0,1] \cap BC$, BC_0 be the set of functions satisfying the homogeneous boundary conditions $V_i(x) = 0, i = \overline{1,4}$, and $C^4_{BC_0}[0,1] = C^4[0,1] \cap BC_0$.

In this setting, we will prove the following basic existence result which is a variant of (Chapter I, Theorem 5.1 and Chapter V, Theorem 1.2, [28]).

Theorem 4. *Assume that:*

(i) *For $\lambda = 0$ problem (21) has a unique solution $x_0 \in C^4[0,1]$.*
(ii) *Problems (20) and (21) are equivalent when $\lambda = 1$.*
(iii) *The map $\Lambda_h : C^4_{BC_0}[0,1] \to C[0,1]$, defined by*

$$\Lambda_h x = x''' + s_3(t)x''' + s_2(t)x'' + s_1(t)x' + s_0 x,$$

is one-to-one.

(iv) *Each solution $x \in C^4[0,1]$ to family (21) satisfies the bounds*

$$m_i \le x^{(i)} \le M_i \text{ for } t \in [0,1], i = \overline{0,4},$$

where the constants $-\infty < m_i, M_i < \infty, i = \overline{0,4}$, are independent of λ and x.

(v) *There is a sufficiently small $\sigma > 0$ such that $[m_0 - \sigma, M_0 + \sigma] \subseteq D_x$, $[m_1 - \sigma, M_1 + \sigma] \subseteq D_p$, $[m_2 - \sigma, M_2 + \sigma] \subseteq D_q$, $[m_3 - \sigma, M_3 + \sigma] \subseteq D_r$, and $g(t, x, p, q, r, \lambda)$ is continuous for $(t, x, p, q, r, \lambda) \in [0,1] \times J \times [0,1]$; $m_i, M_i, i = \overline{0,4}$, are as in (iv).*

Then BVP (20) has at least one solution in $C^4[0,1]$.

Proof. Define the set

$$\overline{U} = \{x \in C^4_{BC}[0,1] : m_i - \sigma \le x^{(i)} \le M_i + \sigma \text{ on } [0,1],\ i = \overline{0,4}\}$$

and the maps

$$j : C^4_{BC}[0,1] \to C^3[0,1] \ \text{ by }\ jx = x,$$

$$\Lambda : C^4_{BC}[0,1] \to C[0,1] \ \text{ by }\ \Lambda x = x^{(4)} + s_3(t)x''' + s_2(t)x'' + s_1(t)x' + s_0(t)x,$$

and for $\lambda \in [0,1]$

$$\Phi_\lambda : C^3[0,1] \to C[0,1] \ \text{ by }\ \Phi_\lambda x = g(t, x, x', x'', x''', \lambda), x \in j(\overline{U}).$$

Our next step is to show that $\Lambda^{-1} : C[0,1] \to C^4_{BC}[0,1]$ exists and is continuous. Observe firstly that according to (iii) for each $y \in C[0,1]$ the BVP

$$x^{(4)} + s_3(t)x''' + s_2(t)x'' + s_1(t)x' + s_0(t)x = y(t),$$

$$V_i(x) = 0, i = \overline{1,4},$$

has a unique $C^4[0,1]$-solution of the form

$$x(t) = C_1^* x_1(t) + C_2^* x_2(t) + C_3^* x_3(t) + +C_4^* x_4(t) + \eta(t),$$

where $x_i(t), i = \overline{1,4}$, are linearly independend solutions to the homogeneous equation

$$x^{(4)} + s_3(t)x''' + s_2(t)x'' + s_1(t)x' + s_0(t)x = 0, \tag{22}$$

and $\eta(t)$ is a solution to the inhomogeneous equation. Since $(C_1^*, C_2^*, C_3^*, C_4^*)$ in $x(t)$ is the unique solution to the system

$$C_1 V_i(x_1) + C_2 V_i(x_2) + C_3 V_i(x_3) + C_4 V_i(x_4) = -V_i(\eta), i = \overline{1,4},$$

we must have $\det[V_i(x_j)] \neq 0$. The last means that the system

$$C_1 V_i(x_1) + C_2 V_i(x_2) + C_3 V_i(x_3) + C_4 V_i(x_4) = r_i, i = \overline{1,4},$$

also has a unique solution $(\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4)$ and so

$$l(t) = \overline{C}_1 x_1(t) + \overline{C}_2 x_2(t) + \overline{C}_3 x_3(t) + \overline{C}_4 x_4(t)$$

is the unique $C^4[0,1]$-solution to the homogeneous Equation (22) satisfying the inhomogeneous boundary conditions

$$V_i(x) = r_i, i = \overline{1,4}.$$

Now, it is not hard to check that $\Lambda^{-1}y = \Lambda_h^{-1}y + l$. On the other hand, for $x \in C^4_{BC_0}[0,1]$ we have

$$\begin{aligned}
||\Lambda_h x||_{C[0,1]} &\leq ||x^{(4)}||_{C[0,1]} + S_3||x'''||_{C[0,1]} + S_2||x''||_{C[0,1]} + S_1||x'||_{C[0,1]} + S_0||x||_{C[0,1]} \\
&\leq ||x||_{C^4[0,1]} + S_3||x||_{C^4[0,1]} + S_2||x||_{C^4[0,1]} + S_1||x||_{C^4[0,1]} + S_0||x||_{C^4[0,1]} \\
&\leq (1 + S_3 + S_2 + S_1 + S_0)||x||_{C^4[0,1]},
\end{aligned}$$

where $S_k = \max_{[0,1]} |s_k(t)|, i = \overline{0,3}$. This means that the linear map Λ_h is bounded and so it is continuous. Then, Λ_h^{-1} is continuous and so Λ^{-1} is also continuous.

Furthermore, let the homotopy $H_\lambda : \overline{U} \times [0,1] \to C^4_{BC}[0,1]$ be defined by $H_\lambda = \Lambda^{-1}\Phi_\lambda j$. Since j is a completely continuous embedding and U is bounded, the set $j(\overline{U})$ is compact. Because of the continuity of g on the set $[0,1] \times J \times [0,1]$, the map Φ_λ is continuous on $j(\overline{U})$ for each $\lambda \in [0,1]$. These facts together with the proved above continuity of Λ^{-1} imply that the homotopy is compact. For its fixed points we have

$$x = \Lambda^{-1}\Phi_\lambda j x$$

and

$$\Lambda x = \Phi_\lambda j x,$$

which means that the fixed points of H_λ are precisely the solutions of family (21). Consequently, by (iv), the homotopy is fixed point free on the boundary of U. Finally, using (i), we see that $H_0 = x_0$. Since $x_0 \in U$, H_0 is essential by Theorem 1. Then, H_1 is also essential by Theorem 3 which means that H_1 has a fixed point, i.e., (21) has a solution in $C^4[0,1]$ when $\lambda = 1$, and, by (ii), problem (20) has a solution in $C^4[0,1]$. $\square$

The following auxiliary results ensure the a priori bounds which (iv) of Theorem 4 requires.

Lemma 2. *Let* ($\mathbf{H}_1$) *hold. Then each solution* $x \in C^4[0,1]$ *to a BVP for* (1)$_\lambda$ *with one of the boundary conditions* (*k*), $k = \overline{2,13}$, *satisfies the bounds*

$$F_1 \le x'''(t) \le L_1 \ \text{ on } [0,1].$$

Proof. Assume on the contrary that the set

$$S_+ = \{t \in [0,1] : L_1 < x'''(t) \le L_2\}$$

is not empty. Then $x'''(1) \le L_1$ and $x''' \in C[0,1]$ imply that there is a $\gamma \in S_+$ with the property

$$x^{(4)}(\gamma) < 0.$$

On the other hand, from $(\gamma, x(\gamma), x'(\gamma), x''(\gamma), x'''(\gamma)) \in S_+ \times D_x \times D_p \times D_q \times (L_1, L_2]$ and (18) it follows $f(\gamma, x(\gamma), x'(\gamma), x''(\gamma), x'''(\gamma)) \ge 0$, which means

$$x^{(4)}(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma), x''(\gamma), x'''(\gamma)) \ge 0,$$

a contradiction. Thus,

$$x'''(t) \le L_1 \text{ for } t \in [0,1].$$

Furthermore, by essentially the same reasoning as above, assuming that the set

$$S_- = \{t \in [0,1] : F_2 \le x'''(t) < F_1\}$$

is not empty and using (19), we reach a contradiction which yields

$$F_1 \le x'''(t) \text{ for } t \in [0,1].$$

□

Lemma 3. *Let* ($\mathbf{H}_1$) *hold. Then each solution* $x(t) \in C^4[0,1]$ *to a BVP for* (1)$_\lambda$ *with one of the boundary conditions* (*k*), $k = \overline{2,11}$, *satisfies the bound*

$$|x''(t)| \le |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,1].$$

Proof. If $x''(0) = C$, by the mean value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x''(t) = x'''(\xi)t + x''(0).$$

Now, keeping in mind that according to Lemma 2

$$|x'''(t)| \le \max\{|F_1|, |L_1|\}, \ t \in [0,1],$$

derive the bound for $|x''(t)|$. If $x''(1) = C$, we obtain similarly that for each $t \in [0,1)$ there is an $\eta \in (t,1)$ with the property

$$x''(1) - x''(t) = x'''(\eta)(1-t),$$

from where the assertion follows as above. □

Lemma 4. *Let* ($\mathbf{H}_1$) *hold. Then each solution* $x(t) \in C^4[0,1]$ *to* (1)$_\lambda$, (12) *or* (1)$_\lambda$, (13) *satisfies the bound*

$$|x''(t)| \le |C - B| + \max\{|F_1|, |L_1|\}, \ t \in [0,1].$$

Proof. Clearly, there is a $\mu \in (0,1)$ for which $x''(\mu) = C - B$. Then, for each $t \in [0, \mu)$ there is a $\xi \in (t, \mu)$ such that

$$x''(\mu) - x''(t) = x'''(\xi)(\mu - t),$$

from where, using Lemma 2, we get

$$|x''(t)| \le |C - B| + \max\{|F_1|, |L_1|\}, \ t \in [0, \mu].$$

We can see similarly that the same bound is valid for $t \in [\mu, 1]$. □

Lemma 5. *Let* ($\mathbf{H}_1$) *hold. Then each solution* $x(t) \in C^4[0,1]$ *to a BVP for* (1)$_\lambda$ *with one of the boundary conditions (k),* $k = \overline{2,9}$, *satisfies the bounds*

$$|x(t)| \le |A| + |B| + |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,1],$$

$$|x'(t)| \le |B| + |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,1]. \tag{23}$$

Proof. Let $x(t) \in C^4[0,1]$ be a solution of (1)$_\lambda$,(4); the assertion follows similarly for all the rest families of BVPs. By the mean value theorem, for each $t \in [0,1)$ there is a $\xi \in (t,1)$ such that

$$x'(1) - x'(t) = x''(\xi)(1 - t),$$

from where, using Lemma 3, we get (23). Using again the mean value theorem, we obtain that for each $t \in (0,1]$ and some $\eta \in (0,t)$ we have

$$x(t) - x(0) = x'(\eta)t.$$

This together with (23) gives the bound for $|x(t)|$. □

Lemma 6. *Let* $A, B, C, D \ge 0$ *and* ($\mathbf{H}_1$) *hold with* $F_1 \ge 0$. *Then each solution* $x \in C^4[0,1]$ *to* (1)$_\lambda$, (2) *satisfies the bounds*

$$A \le x(t) \le A + B + C + L_1, \ t \in [0,1],$$

$$B \le x'(t) \le B + C + L_1, \ t \in [0,1], \tag{24}$$

$$C \le x''(t) \le C + L_1, \ t \in [0,1]. \tag{25}$$

Proof. By Lemma 2, we know that

$$0 \le F_1 \le x'''(t) \le L_1 \ \text{ on } [0,1].$$

Then, for $t \in (0,1]$ we have

$$\int_0^t F_1 ds \le \int_0^t x'''(s) ds \le \int_0^t L_1 ds,$$

which yields consequtively $F_1 t \le x''(t) - C \le L_1 t, t \in [0,1]$, and $0 \le x''(t) - C \le L_1$, $t \in [0,1]$, from where (25) follows. Next, by integration of (25) from 0 to $t \in (0,1]$ we get (24), and a new integration from 0 to $t \in (0,1]$ gives the bound for $x(t)$. □

Lemma 7. *Let* $A \ge 0, B, C, D \le 0$ *and* ($\mathbf{H}_1$) *hold with* $L_1 \le 0$. *Then each solution* $x \in C^4[0,1]$ *to* (1)$_\lambda$, (3) *satisfies the bounds*

$$A \le x(t) \le A - B - C - F_1, \ t \in [0,1],$$

$$B + C + F_1 \le x'(t) \le B, \ t \in [0,1], \tag{26}$$

$$C + F_1 \le x''(t) \le C, \ t \in [0,1]. \tag{27}$$

Proof. Following the proof of Lemma 6, we check that (26) and (27) are valid, and integrating (26) from $t \in [0,1)$ to 1 gives the bound for $x(t)$. □

Lemma 8. *Let* ($\mathbf{H}_1$) *hold. Then each solution* $x \in C^4[0,1]$ *to* $(1)_\lambda$, *(10) or* $(1)_\lambda$, *(11) satisfies the bounds*

$$|x(t)| \le |A| + |B-A| + |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,1],$$

$$|x'(t)| \le |B-A| + |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,1].$$

Proof. By the mean value theorem, there is a $\mu \in (0,1)$ with the property $x'(\mu) = B - A$. Then, for each $t \in [0,\mu)$ there is a $\xi \in (t,\mu)$ such that

$$x'(\mu) - x'(t) = x''(\xi)(\mu - t),$$

from where, using Lemma 3, we get

$$|x'(t)| \le |B-A| + |C| + \max\{|F_1|, |L_1|\}, \ t \in [0,\mu].$$

Arguing similarly, establish that the same bound is valid on the interval $t \in [\mu, 1]$. Next, for each $t \in (0,1]$ there is an $\eta \in (0,t)$ such that

$$x(t) - x(0) = x'(\eta)t,$$

from where using the obtained bound for $|x'(t)|$, we reach the bound for $|x(t)|$. □

Lemma 9. *Let* $A, B \ge 0, C, D \le 0$ *and* ($\mathbf{H}_1$) *hold with* $L_1 \le 0$. *Then each solution* $x \in C^4[0,1]$ *to* $(1)_\lambda$, *(10) satisfies the bounds*

$$\min\{A, B\} \le x(t) \le A + |B-A| + |C| + |F_1|, \ t \in [0,1],$$
$$B - A + C + F_1 \le x'(t) \le B - A - C - F_1, \ t \in [0,1],$$
$$C + F_1 \le x''(t) \le C, \ t \in [0,1]. \tag{28}$$

Proof. Using again Lemma 2, we get

$$\int_0^t F_1 ds \le \int_0^t x'''(s)ds \le \int_0^t L_1 ds, \ t \in (0,1],$$
$$F_1 t \le x''(t) - C \le L_1 t, \ t \in (0,1],$$

and so (28) holds. Next, use that there is a $\mu \in (0,1)$ such that $x'(\mu) = B - A$. Then,

$$\int_t^\mu (C + F_1)ds \le \int_t^\mu x''(s)ds \le \int_t^\mu C ds, t \in [0,\mu),$$

gives consecutively

$$(C + F_1)(\mu - t) \le x'(\mu) - x'(t) \le C(\mu - t), \ t \in [0,\mu),$$

$$C + F_1 \le x'(\mu) - x'(t) \le 0, \ t \in [0,\mu),$$
$$B - A \le x'(t) \le B - A - C - F_1, \ t \in [0,\mu).$$

Integrating (28) from μ to $t \in (\mu, 1]$, we establish similarly

$$B - A + C + F_1 \le x'(t) \le B - A,\ t \in (\mu, 1].$$

Hence,

$$B - A + C + F_1 \le x'(t) \le B - A - C - F_1 \text{ for } t \in [0, 1]$$

and so

$$|x'(t)| \le |B - A| + |C| + |F_1| \text{ for } t \in [0, 1].$$

By the mean value theorem, for each $t \in (0, 1]$ there is a $\xi \in (0, t)$ such that

$$x(t) - x(0) = x'(\xi)t,$$

which yields

$$|x(t)| \le A + |B - A| + |C| + |F_1| \text{ for } t \in [0, 1].$$

Since $C \le 0$, in view of (28), $x(t)$ is concave on $[0, 1]$. This together with $A, B \ge 0$ gives

$$x(t) \ge \min\{A, B\} \ge 0 \text{ for } t \in [0, 1],$$

from where the bounds for $x(t)$ follow. □

Lemma 10. *Let $A, B, D \ge 0, C \le 0$ and ($\mathbf{H}_1$) hold with $F_1 \ge 0$. Then each solution $x \in C^4[0, 1]$ to (1)$_\lambda$, (11) satisfies the bounds*

$$\begin{gathered} \min\{A, B\} \le x(t) \le A + |B - A| + |C| + L_1,\ t \in [0, 1], \\ B - A + C - L_1 \le x'(t) \le B - A - C + L_1,\ t \in [0, 1], \\ C - L_1 \le x''(t) \le C,\ t \in [0, 1]. \end{gathered} \tag{29}$$

Proof. Using Lemma 2, we get

$$\int_t^1 F_1 ds \le \int_t^1 x'''(s) ds \le \int_t^1 L_1 ds,\ t \in [0, 1),$$

and $F_1(1 - t) \le x''(1) - x''(t) \le L_1(1 - t),\ t \in [0, 1]$, from where (29) follows. Furthermore, similar arguments to those in the proof of Lemma 9 give the bounds for $x'(t)$ and $x(t)$. □

Lemma 11. *Let ($\mathbf{H}_1$) hold. Then each solution $x \in C^4[0, 1]$ to (1)$_\lambda$, (12) or (1)$_\lambda$, (13) satisfies the bounds*

$$|x(t)| \le |A| + |B| + |C - B| + \max\{|F_1|, |L_1|\},\ t \in [0, 1],$$

$$|x'(t)| \le |B| + |C - B| + \max\{|F_1|, |L_1|\},\ t \in [0, 1].$$

Proof. Let $x(t) \in C^4[0, 1]$ be a solution to (1)$_\lambda$, (13); the assertion follows similarly for problem (1)$_\lambda$, (12). For each $t \in (0, 1]$ there is a $\xi \in (0, t)$ such that

$$x'(t) - x'(0) = x''(\xi)t.$$

This and $|x''(t)| \le |C - B| + \max\{|F_1|, |L_1|\},\ t \in [0, 1]$, which we have, thanks to Lemma 4, give the bound for $|x'(t)|$. Next, again by the mean value theorem, for each $t \in [0, 1)$ there is an $\eta \in (t, 1)$ such that

$$x(1) - x(t) = x'(\eta)(1 - t),$$

from where, using the obtained already bound for $|x'(t)|$, obtain one for $|x(t)|$. □

Lemma 12. *Let $A, B, C \geq 0, D \leq 0$ and ($\mathbf{H}_1$) hold with $L_1 \leq 0$. Then each solution $x \in C^4[0,1]$ to (1)$_\lambda$, (12) satisfies the bounds*

$$A \leq x(t) \leq A + B + |C - B| + |F_1|,\ t \in [0,1],$$

$$\min\{B, C\} \leq x'(t) \leq B + |C - B| + |F_1|,\ t \in [0,1],$$

$$C - B + F_1 \leq x''(t) \leq C - B - F_1,\ t \in [0,1].$$

Proof. Observe firstly that there is a $\mu \in (0,1)$ such that $x''(\mu) = C - B$. Next, let us recall, by Lemma 2 we have $F_1 \leq x'''(t) \leq L_1$ on $[0,1]$. Integrating this inequality from $t \in [0,\mu)$ to μ, we get

$$C - B \leq x''(t) \leq C - B - F_1, t \in [0,\mu],$$

and integrating it from μ to $t \in (\mu, 1]$, we get

$$C - B + F_1 \leq x''(t) \leq C - B, t \in [\mu, 1].$$

Thus,

$$C - B + F_1 \leq x''(t) \leq C - B - F_1,\ t \in [0,1].$$

Furthermore, by the mean value theorem, for each $t \in (0,1]$ there exists a $\xi \in (0,t)$ such that

$$x'(t) - x'(0) = x''(\xi)t,$$

which yields $|x'(t)| \leq B + |C - B| + |F_1|$ for $t \in [0,1]$. However, $x'(t)$ is concave on $[0,1]$ because $x'''(t) \leq L_1 \leq 0$ for $t \in [0,1]$. This fact together with $B, C \geq 0$ means that $x'(t) \geq \min\{B, C\}$ on $[0,1]$. Thus,

$$0 \leq \min\{B, C\} \leq x'(t) \leq B + |C - B| + |F_1| \text{ for } t \in [0,1].$$

Integrating from 0 to $t \in (0,1]$ yields

$$0 \leq x(t) - x(0) \leq (B + |C - B| + |F_1|)t,$$

from where the bound for $x(t)$ follows. □

Lemma 13. *Let $A, D \geq 0, B, C \leq 0$ and ($\mathbf{H}_1$) hold with $F_1 \geq 0$. Then each solution $x \in C^4[0,1]$ to (1)$_\lambda$, (13) satisfies the bounds*

$$A \leq x(t) \leq A + |B| + |C - B| + L_1,\ t \in [0,1],$$

$$-(|B| + |C - B| + L_1) \leq x'(t) \leq \max\{B, C\},\ t \in [0,1],$$

$$C - B - L_1 \leq x''(t) \leq C - B + L_1,\ t \in [0,1].$$

Proof. Following the proof of Lemma 12, we obtain the bounds for $x''(t)$. Next, applying again the mean value theorem, we get

$$|x'(t)| \leq |B| + |C - B| + L_1 \text{ for } t \in [0,1].$$

Besides, from Lemma 2 we have $0 \leq F_1 \leq x'''(t) \leq L_1$ on $[0,1]$, which means that $x'(t)$ is convex on $[0,1]$. Therefore, $x'(t) \leq \max\{B, C\}$ on $[0,1]$, i.e.

$$-(|B| + |C - B| + L_1) \leq x'(t) \leq \max\{B, C\} \leq 0 \text{ for } t \in [0,1].$$

Finally, integrating from $t \in [0,1)$ to 1 yields

$$-(|B| + |C - B| + L_1)(1 - t) \leq x(1) - x(t) \leq 0 \text{ for } t \in [0,1],$$

from where the bound for $x(t)$ follows. □

3. Problems (1),(2),...,(1),(8) and (1),(9)

Theorem 5. *Let* ($\mathbf{H}_1$) *hold and let* ($\mathbf{H}_2$) *hold for*

$$M_0 = |A| + |B| + |C| + \max\{|F_1|, |L_1|\},\ m_0 = -M_0,$$

$$M_1 = |B| + |C| + \max\{|F_1|, |L_1|\},\ m_1 = -M_1,$$

$$M_2 = |C| + \max\{|F_1|, |L_1|\},\ m_2 = -M_2,\ m_3 = F_1,\ M_3 = L_1.$$

Then each BVP for Equation (1) with one of the boundary conditions (k), $k = \overline{2,9}$, *has at least one solution in* $C^4[0,1]$.

Proof. We will show that each BVP for $(1)_\lambda$, $\lambda \in [0,1]$, with one of the boundary conditions (*k*), $k = \overline{2,9}$, satisfies all hypotheses of Theorem 4. It is not hard to check that (i) holds for each BVP for $(1)_0$ with one of the boundary conditions (*k*), $k = \overline{2,9}$. Obviously, each pair of BVPs for (1) and $(1)_1$ with one and the same boundary conditions are equivalent, i.e., (ii) is satisfied. It is easy to check also that for an arbitrary $y(t) \in C[0,1]$ each BVP for the equation $x^{(4)} = y(t)$ with one of the homogeneous boundary conditions (*k*), $k = \overline{2,9}$, has a unique solution in $C^4[0,1]$, that is, the map $\Lambda_h : C^4_{BC_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x^{(4)}$, is one-to-one. Thus, (iii) holds. Furthermore, for each solution $x(t) \in C^4[0,1]$ to a BVP for $(1)_\lambda$, $\lambda \in [0,1]$, with one of the boundary conditions (*k*), $k = \overline{2,9}$, we have

$$m_0 \le x(t) \le M_0, t \in [0,1], \text{by Lemma 5},$$

$$m_1 \le x'(t) \le M_1, t \in [0,1], \text{by Lemma 5},$$

$$m_2 \le x''(t) \le M_2, t \in [0,1], \text{ by Lemma 3},$$

$$m_3 \le x''(t) \le M_3, t \in [0,1], \text{ by Lemma 2}.$$

Because of the continuity of f on $[0,1] \times J$, there are constants m_4 and M_4 such that

$$m_4 \le \lambda f(t,x,p,q,r) \le M_4 \text{ for } \lambda \in [0,1] \text{ and } (t,x,p,q,r) \in [0,1] \times J.$$

Since for $t \in [0,1]$ we have $(x(t), x'(t), x''(t), x'''(t)) \in J$, the equation $(1)_\lambda$ implies

$$m_4 \le x^{(4)}(t) \le M_4 \text{ for } t \in [0,1].$$

Hence, (iv) also holds. Finally, (v) follows again from the continuity of f on the set J. Therefore, we can apply Theorem 4 to conclude that assertion is true. □

Under a suitable combination of the signs of A, B, C and D, ($\mathbf{H}_1$) and ($\mathbf{H}_2$) guarantee solutions with important properties.

Theorem 6. *Let* $A, B > 0 (A, B = 0), C, D \ge 0$. *Suppose* ($\mathbf{H}_1$) *holds with* $F_1 \ge 0$ *and* ($\mathbf{H}_2$) *holds for*

$$m_0 = A,\ M_0 = A + B + C + L_1,\ m_1 = B,\ M_1 = B + C + L_1,$$

$$m_2 = C,\ M_2 = C + L_1,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(2) has at least one positive, increasing (non-negative, non-decreasing), convex solution in $C^4[0,1]$.

Proof. Folowing the proof of Theorem 5, we establish that there is a solution $x(t) \in C^4[0,1]$. Now, the bounds $m_i \le x^{(i)}(t) \le M_i, t \in [0,1], i = 0,1,2,$ follow from Lemma 6. In fact, Lemma 6 implies in particular $x(t) \ge A > 0, x'(t) \ge B > 0$ $(x(t) \ge 0,\ x'(t) \ge 0)$ and $x''(t) \ge C \ge 0$ for $t \in [0,1]$, which yields the assertion. □

Theorem 7. *Let $A > 0, B < 0 (A, B = 0), C, D \le 0$. Suppose ($\mathbf{H}_1$) holds with $L_1 \le 0$ and ($\mathbf{H}_2$) holds for*

$$m_0 = A,\ M_0 = A - B - C - F_1,\ m_1 = B + C + F_1,\ M_1 = B,$$

$$m_2 = C + F_1,\ M_2 = C,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(3) has at least one positive, decreasing (non-negative, non-increasing), concave solution in $C^4[0,1]$.

Proof. Using Lemma 7, as in the proof of Theorem 5 we establish that (1),(3) has a solution $x(t) \in C^4[0,1]$. Since Lemma 7 implies $x(t) \ge A > 0, x'(t) \le B < 0$ $(x(t) \ge 0,\ x'(t) \le 0)$ and $x''(t) \le C \le 0$ for $t \in [0,1],\ x(t)$ has the desired properties. □

We provide the reader to formulate variants of Theorems 6 and 7 for the rest BVPs (1),(k), $k = \overline{4,9}$.

Example 1. *Consider BVPs for the equation*

$$x^{(4)} = P_n(x'''), t \in (0,1), \tag{30}$$

with one of the boundary conditions (k), $k = \overline{2,9}$, where the polynomial $P_n(r)$, $n \ge 2$, has simple zeros r_1 and r_2 such that $r_1 > D > r_2$.

Fix some $\theta > 0$ with the properties $r_1 - \theta \ge D \ge r_2 + \theta$ and

$$P_n(r) \ne 0 \text{ on } r \in \cup_{i=1}^{2}(r_i - \theta, r_i + \theta) \setminus r_i, i = 1,2,$$

and consider the case

$$P_n(r) > 0 \text{ for } r \in (r_1, r_1 + \theta] \text{ and } P_n(r) < 0 \text{ for } r \in [r_2 - \theta, r_2);$$

the other cases for the sign of $P_n(r)$ around the zeros can be studied by analogy. It is easy to check in this case that if we choose, for example, $F_2 = r_2 - \theta$, $F_1 = r_2$, $L_1 = r_1$, $L_2 = r_1 + \theta$ and $\sigma = \theta/2$, ($\mathbf{H}_1$) and ($\mathbf{H}_2$) hold and so each BVP for (30) with one of the boundary conditions (k), $k = \overline{2,9}$, has a solution in $C^4[0,1]$ by Theorem 5.

Example 2. *Consider the BVP*

$$x^{(4)} = \frac{t(x''' + 2)\sqrt{100 - x^2}\sqrt{400 - x'^2}}{\sqrt{225 - x''^2}\sqrt{900 - x'''^2}}, t \in (0,1),$$

$$x(1) = 2, x'(0) = -1, x''(0) = -2, x'''(1) = -3.$$

It is not hard to see that this problem has a positive, decreasing, concave solution in $C^4[0,1]$ by Theorem 7; moreover, $F_2 = -5, F_1 = -4, L_1 = -2, L_2 = -1$ and $\sigma = 0.1$, for example. Notice also, the right side of the equation is defined on a bounded set.

4. Problems (1),(10) and (1),(11)

Theorem 8. *Let ($\mathbf{H}_1$) hold and let ($\mathbf{H}_2$) hold for*

$$M_0 = |A| + |B - A| + |C| + \max\{|F_1|, |L_1|\},\ m_0 = -M_0,$$

$$M_1 = |B - A| + |C| + \max\{|F_1|, |L_1|\},\ m_1 = -M_1,$$

$$M_2 = |C| + \max\{|F_1|, |L_1|\},\ m_2 = -M_2, m_3 = F_1,\ M_3 = L_1.$$

Then BVPs (1),(10) and (1),(11) have at least one solution in $C^4[0,1]$.

Proof. Following the proof of Theorem 5, we check that all hypotheses of Theorem 4 are fulfilled for family $(1)_\lambda$,(10) and BVP (1),(10) as well as for $(1)_\lambda$,(11) and BVP (1),(11) and so the assertion is true. Moreover,

$$m_0 \le x(t) \le M_0 \text{ on } [0,1], \text{by Lemma 8},$$

$$m_1 \le x'(t) \le M_1 \text{ on } [0,1], \text{by Lemma 8},$$

$$m_2 \le x''(t) \le M_2 \text{ on } [0,1],\ \text{by Lemma 3}.$$

$$m_3 \le x'''(t) \le M_3 \text{ on } [0,1],\ \text{by Lemma 2}.$$

□

The remaining result of this section provide solutions with various properties.

Theorem 9. *Let* $A, B > 0(A.B = 0$ *and* $A + B \ge 0)$ *and* $C, D \le 0$. *Suppose* ($\mathbf{H}_1$) *holds with* $L_1 \le 0$, *and* ($\mathbf{H}_2$) *holds for*

$$m_0 = \min\{A, B\},\ M_0 = A + |B - A| + |C| + |F_1|,$$

$$m_1 = B - A + C + F_1,\ M_1 = B - A - C - F_1,$$

$$m_2 = C + F_1,\ M_2 = C,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(10) has at least one positive (non-negative), concave solution in $C^4[0,1]$.

Proof. Following the proof of Theorem 5 and using Lemmas 9 and 2, we establish that there is a solution $x(t) \in C^4[0,1]$. In fact, from Lemma 9 we know that $x(t) \ge \min\{A, B\} > 0$ $(x(t) \ge 0)$ and $x''(t) \le C \le 0$ for $t \in [0,1]$, which completes the proof. □

Corollary 1. *Let* $B > A > 0, A - B < C \le 0$ *and* $D \le 0$. *Suppose* ($\mathbf{H}_1$) *holds with* $L_1 \le 0$ *and* $A - B - C < F_1 \le 0\,(A - B - C \le F_1 < 0)$, *and* ($\mathbf{H}_2$) *holds for* $m_i, M_i, i = \overline{0,3}$, *as in Theorem 9. Then BVP (1),(10) has at least one positive, increasing (non-decreasing), concave solution in* $C^4[0,1]$.

Proof. According to Theorem 9 problem (1),(10) has at least one positive, concave solution $x(t) \in C^4[0,1]$. From Lemma 9 we know that

$$x'(t) \ge B - A + C + F_1 > 0\ (x'(t) \ge B - A + C + F_1 \ge 0),\ t \in [0,1],$$

and as a result $x(t)$ is increasing (non-decreasing); let us note, since $A - B - C < 0$ the inequality $A - B - C < F_1 \le 0$ $(A - B - C \le F_1 < 0)$ is possible and so $B - A + C + F_1 > 0$ $(B - A + C + F_1 \ge 0)$. □

Corollary 2. *Let* $A = B > 0(A, B = 0)$ *and* $C, D \le 0$. *Suppose* ($\mathbf{H}_1$) *holds with* $L_1 \le 0$, *and* ($\mathbf{H}_2$) *holds for* $m_i, M_i, i = \overline{0,3}$, *as in Theorem 9. Then BVP (1),(10) has at least one positive (non-negative), concave solution* $x(t) \in C^4[0,1]$ *for which there is a* $\mu \in (0,1)$ *with the property* $x(\mu) = \max_{[0,1]} x(t)$.

Proof. A positive (non-negative), concave solution $x(t) \in C^4[0,1]$ exists by Theorem 9. By the mean value theorem there is a $\mu \in (0,1)$ such that $x'(\mu) = B - A = 0$, which yields the assertion. □

Theorem 10. *Let $A, B > 0(A.B = 0$ and $A + B \geq 0)$, $C \leq 0$ and $D \geq 0$. Suppose $(\mathbf{H}_1)$ holds with $F_1 \geq 0$, and $(\mathbf{H}_2)$ holds for*

$$m_0 = \min\{A, B\},\ M_0 = A + |B - A| + |C| + L_1,$$

$$m_1 = B - A + C - L_1,\ M_1 = B - A - C + L_1,$$

$$m_2 = C - L_1,\ M_2 = C,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(11) has at least one positive (non-negative), convex solution in $C^4[0, 1]$.

Proof. Following the proof of Theorem 5 and using Lemmas 10 and 2, we establish that there is a solution $x(t) \in C^4[0, 1]$. In fact, from Lemmas 10 we know that $x(t) \geq \min\{A, B\} > 0$ $(x(t) \geq 0)$ and $x''(t) \leq C \leq 0$ for $t \in [0, 1]$, which completes the proof. □

Corollary 3. *Let $B > A > 0, A - B < C \leq 0$ and $D \geq 0$. Suppose $(\mathbf{H}_1)$ holds with $F_1 \geq 0$ and $0 \leq L_1 < B - A + C(0 < L_1 \leq B - A + C)$, and $(\mathbf{H}_2)$ holds for $m_i, M_i, i = \overline{0, 3}$, as in Theorem 10. Then BVP (1),(11) has at least one positive, increasing (non-decreasing), convex solution in $C^4[0, 1]$.*

Proof. According to Theorem 10 problem (1),(11) has at least one positive, convex solution $x(t) \in C^4[0, 1]$. From Lemma 10 we know that

$$x'(t) \geq B - A + C - L_1 > 0\ (x'(t) \geq B - A + C - L_1 \geq 0),\ t \in [0, 1],$$

and as a result $x(t)$ is increasing (non-decreasing); the inequality $B - A + C - L_1 > 0$ $(B - A + C - L_1 \geq 0)$ is possible since $B - A + C > 0$. □

Corollary 4. *Let $A = B > 0(A, B = 0)$, $C \leq 0$ and $D \geq 0$. Suppose $(\mathbf{H}_1)$ holds with $F_1 \geq 0$, and $(\mathbf{H}_2)$ holds for $m_i, M_i, i = \overline{0, 3}$, as in Theorem 10. Then BVP (1),(11) has at least one positive (non-negative), convex solution $x(t) \in C^4[0, 1]$ for which there is a $\mu \in (0, 1)$ with the property $x(\mu) = \max_{[0,1]} x(t)$.*

Proof. A positive (non-negative), convex solution $x(t) \in C^4[0, 1]$ exists by Theorem 10. By the mean value theorem there is a $\mu \in (0, 1)$ such that $x'(\mu) = B - A = 0$, which yields the assertion. □

Example 3. *Consider the BVP*

$$x^{(4)} = \frac{(x''' - 3)\sqrt{400 - x'^2}}{t(x''^2 + 1)\sqrt{225 - x^2}}, t \in (0, 1), \tag{31}$$

$$x(0) = 1, x(1) = 0, x''(0) = 2, x'''(1) = 3.$$

The assumptions of Theorem 8 hold for $F_2 = 1, F_1 = 2, L_1 = 4, L_2 = 5$ and $\sigma = 0.1$, for example. Hence, the considered problem has a solution in $C^4[0, 1]$.

Example 4. *Consider the BVP for (31) with boundary conditions*

$$x(0) = 1, x(1) = 6, x''(0) = -2, x'''(1) = -2.$$

This problem has a positive, increasing, concave solution in $C^4[0, 1]$ by Corollary 1. Now, $F_2 = -3$, $F_1 = -2, L_1 = -1, L_2 = 0$ and $\sigma = 0.1$, for example.

Example 5. *Consider the BVP for (31) with boundary conditions*

$$x(0) = 2, x(1) = 2, x''(1) = -1, x'''(1) = 2.$$

The assumptions of Corollary 4 are satisfied for $F_2 = 0, F_1 = 1, L_1 = 3, L_2 = 4$ and $\sigma = 0.1$, for example. Thus, the considered problem has a positive, convex solution $x(t) \in C^4[0,1]$ which has a maximum on $(0,1)$.

5. Problems (1),(12) and (1),(13)

Theorem 11. *Let* ($\mathbf{H}_1$) *hold and let* ($\mathbf{H}_2$) *hold for*

$$M_0 = |A| + |B| + |C - B| + \max\{|F_1|, |L_1|\},\ m_0 = -M_0,$$

$$M_1 = |B| + |C - B| + \max\{|F_1|, |L_1|\},\ m_1 = -M_1,$$

$$M_2 = |C - B| + \max\{|F_1|, |L_1|\},\ m_2 = -M_2, m_3 = F_1,\ M_3 = L_1.$$

Then BVPs (1),(12) and (1),(13) have at least one solution in $C^4[0,1]$.

Proof. As in the proof of Theorem 5, we check that family $(1)_\lambda$, (12) and BVP (1),(12) as well as family $(1)_\lambda$,(13) and BVP (1),(13) satisfy all hypotheses of Theorem 4 and so the assertion is true. Moreover, now

$$m_0 \le x(t) \le M_0 \text{ on } [0,1], \text{by Lemma 11},$$

$$m_1 \le x'(t) \le M_1 \text{ on } [0,1], \text{by Lemma 11},$$

$$m_2 \le x''(t) \le M_2 \text{ on } [0,1], \text{ by Lemma 4},$$

$$m_3 \le x'''(t) \le M_3 \text{ on } [0,1], \text{ by Lemma 2}.$$

□

Theorem 12. *Let $A, B, C > 0 (A, B, C = 0)$ and $D \le 0$. Suppose* ($\mathbf{H}_1$) *holds with $L_1 \le 0$, and* ($\mathbf{H}_2$) *holds for*

$$m_0 = A,\ M_0 = A + B + |C - B| + |F_1|,$$

$$m_1 = \min\{B, C\},\ M_1 = B + |C - B| + |F_1|,$$

$$m_2 = C - B + F_1,\ M_2 = C - B - F_1,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(12) has at least one positive, increasing (non-negative, non-decreasing), concave solution in $C^4[0,1]$.

Proof. Following the proof of Theorem 5 and using Lemma 12 and Lemma 2, we establish that there is a solution $x(t) \in C^4[0,1]$. However, from Lemma 12 we know $x(t) \ge A > 0$ ($x(t) \ge 0$) and $x'(t) \ge \min\{B, C\} > 0$ ($x'(t) \ge 0$), $t \in [0,1]$, and by Lemma 2, $x''(t) \le L_1 \le 0,\ t \in [0,1]$, which completes the proof. □

Theorem 13. *Let $A > 0, B, C < 0 (A, B, C = 0)$ and $D \ge 0$. Suppose* ($\mathbf{H}_1$) *holds with $F_1 \ge 0$, and* ($\mathbf{H}_2$) *holds for*

$$m_0 = A,\ M_0 = A + |B| + |C - B| + L_1,$$

$$m_1 = -(|B| + |C - B| + L_1),\ M_1 = \max\{B, C\},$$

$$m_2 = C - B - L_1,\ M_2 = C - B + L_1,\ m_3 = F_1,\ M_3 = L_1.$$

Then BVP (1),(13) has at least one positive, decreasing (non-negative, non-increasing), convex solution in $C^4[0,1]$.

Proof. Using Lemma 13 and following the proof of Theorem 5, we establish that there is a solution $x(t) \in C^4[0,1]$. However, from Lemma 13 we know that $x(t) \ge A > 0$ ($x(t) \ge 0$) and

$x'(t) \leq \max\{B, C\} < 0$ $(x'(t) \leq 0)$ for $t \in [0,1]$, and by Lemma 2, $x''(t) \geq F_1 \geq 0,\ t \in [0,1]$, which completes the proof. □

Example 6. *Consider the BVP*

$$x^{(4)} = x'''^3(x'^2 + x' + 1) + ln(x''' + 8) - 1,\ t \in (0,1),$$

$$x(0) = 1, x'(0) = 0, x'(1) = -1, x'''(1) = 2.$$

The assumptions of Theorem 11 are satisfied for $F_2 = -7.5, F_1 = -7$, $L_1 = 2$, $L_2 = 3$ *and* $\sigma = 0.1$. *Thus, the problem has a solution in* $C^4[0,1]$.

Example 7. *Consider the BVP*

$$x^{(4)} = (x''^2 + 5)(\exp(x''' + 3) - 1) + (t - 1)\sin(x + 2) - 2,\ t \in (0,1),$$

$$x(0) = 2, x'(0) = 1, x'(1) = 5, x'''(1) = -3.$$

It is not hard to check that we can apply Theorem 12 if $F_2 = -5, F_1 = -4$, $L_1 = -2$, $L_2 = -1$ *and* $\sigma = 0.1$. *Thus, the considered problem has a positive, increasing, concave solution in* $C^4[0,1]$.

Example 8. *Consider the BVP*

$$x^{(4)} = t\sqrt{x' + 15}\sin(x''' - 2),\ t \in (0,1),$$

$$x(1) = 1, x'(0) = -4, x'(1) = -1, x'''(1) = 2.$$

This problem has a positive, decreasing, convex solution in $C^4[0,1]$ *by Theorem 13. Now,* $F_2 = 0, F_1 = 1$, $L_1 = 2.5$, $L_2 = 3$ *and* $\sigma = 0.1$.

6. Conclusions

This paper introduces the reader to the possibility of the barrier strips technique (based here on condition $(\mathbf{H}_1)$) for investigating not only the solvability of various two-point BVPs for fourth-order nonlinear differential equations, but also for investigating the existence of solutions with important properties; we cannot cite known results that guarantee similar conclusions for Examples 6–8. The number of similar BVPs on which this technique is applicable can be substantially increased by using another suitable condition of barrier strips type, alone or in combination with $(\mathbf{H}_1)$.

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