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A Note on NIEP for Leslie and Doubly Leslie Matrices

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Abstract: The nonnegative inverse eigenvalue problem (NIEP) consists of finding necessary and sufficient conditions for the existence of a nonnegative matrix with a given list of complex numbers as its spectrum. If the matrix is required to be Leslie (doubly Leslie), the problem is called the Leslie (doubly Leslie) nonnegative eigenvalue inverse problem. In this paper, necessary and/or sufficient conditions for the existence and construction of Leslie and doubly Leslie matrices with a given spectrum are considered.

Keywords: Leslie matrices; doubly Leslie matrices; nonnegative inverse eigenvalue problem

MSC: 15A18; 15A29

1. Introduction

A matrix $A = (a_{ij})_{i,j}^n$ of order n is called nonnegative if all its entries a_{ij} are nonnegative and is denoted by $A \geq 0$. The nonnegative inverse eigenvalue problem (from now on, the NIEP) consists to find necessary and sufficient conditions for a list $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers to be the spectrum of a nonnegative matrix of order n. A list $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n complex numbers is said to be realizable if there exists some $n \times n$ nonnegative matrix A with spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and that A is the realizing matrix. This problem was firstly considered by Suleĭmanova [1] in 1949. The NIEP remains open for $n \geq 5$. In Reference [2] this problem was solved for $n = 3$, and for $n = 4$ the problem was solved in References [3,4]. It has been studied by several researchers in, for example, References [2,5–11]. T. Laffey and H. Šmigoc proposed studying the NIEP for lists of complex numbers and a certain class of structured nonnegative matrices, such as symmetric, stochastic, circulant, normal matrices, among others. In this work, we deal with Leslie and doubly Leslie matrices. When the realizing nonnegative matrix is required to be Leslie (doubly Leslie) matrix we call the Leslie (doubly Leslie) nonnegative inverse eigenvalue problem (hereafter, LNIEP and DLNIEP, respectively).

The analysis of a mathematical model that considers internal/external variables, and derives in spectral information (eigenvalues) that allow the behavior of the phenomenon shown in the model to be induced, is called the direct eigenvalues problem. On the contrary, the inverse problem of eigenvalues is to estimate the variables of the system from the behavior of the system(eigenvalues). The nonnegative inverse eigenvalues problem or inverse eigenvalues problem for nonnegative matrices arise from and is applied in different areas such as dynamic systems, pole assignment problem, applied mechanics, inverse Sturm-Liouville problem, applied physics, numerical analysis, signal and data processing, geology, demographic growth, among others [12].

It is well known that a Leslie matrix has a single positive real eigenvalue of modulus greater than or equal to the modulus of the other eigenvalues (see Reference [13]). Therefore, we consider studying the LNIEP (DLNIEP) for a given list of complex numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 > 0$ and

$\lambda_1 \geq |\lambda_i|, i = 2, \ldots, n$. This study will allow the estimation of the variables of a Leslie (doubly Leslie) model from the list Λ by reconstructing the matrix that represents it.

In References [14,15], the construction of Leslie stochastic matrices are considered; in the first, the construction of Leslie and doubly Leslie stochastic matrices with zero traces from the coefficients of their characteristic polynomial, and in the second, the construction of Leslie stochastic matrices from a list of nonzero complex numbers, which is a subset of its spectrum. Reference [16] presents the construction of Leslie and doubly Leslie matrices, and companion and doubly companion matrices from particular spectral data. These constructions are independent.

In this paper, we present a different construction of the Leslie matrices from a list closed under complex conjugation and less restive conditions.

On the other hand, in the inverse eigenvalue problem for nonnegative matrices of order n, there exist four necessary conditions. If $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix, then:

(i) The Perron root $\lambda_1 = \max\{|\lambda_i| : \lambda_i \in \Lambda\}$ belongs to Λ.
(ii) $\overline{\Lambda} = \Lambda$, i.e., Λ must be closed under complex conjugation.
(iii) $s_k \geq 0, k = 1, 2, \ldots$, where s_k is the kth power sum of the Λ defined by

$$s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k,$$

since if $A \geq 0$ realized Λ, then $s_k = \text{trace}(A^k) \geq 0$
(iv) $s_m^k \leq n^{k-1} s_{km}$, for all $k, m = 1, 2, \ldots$. This necessary condition is due to Loewy and London [2] and Johnson [6]. For $n = 4$, these necessary conditions are not sufficient.

In Reference [17], one of the most important results of the NIEP was established by T. Laffey and H. Šmigoc in the following:

Theorem 1. *[17] Let $(\lambda_2, \ldots, \lambda_n)$ be complex numbers with real parts less than or equal to zero and let ρ be a positive real number. Then the list $\sigma = (\rho, \lambda_2, \ldots, \lambda_n)$ is the spectrum of a nonnegative matrix if and only if the following conditions are satisfied:*

(1) The list $(\rho, \lambda_2, \ldots, \lambda_n)$ is closed under complex conjugation.
(2) $s_1 = \rho + \sum_{i=2}^{n} \lambda_i \geq 0$.
(3) $s_2 = \rho^2 + \sum_{i=2}^{n} \lambda_i^2 \geq 0$.
(4) $s_1^2 \leq n s_2$.

The Leslie matrix, introduced by P. H. Leslie (see References [13,18]), is an $n \times n$ nonnegative matrix of the form:

$$\mathcal{L} = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_{n-1} & 0 & \ldots & \ldots & 0 \\ 0 & b_{n-2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & b_1 & 0 \end{pmatrix}$$

with $0 < b_j \leq 1, j = 1, 2, \ldots, n-1$, and $a_k \geq 0, k = 1, 2, \ldots, n$, where not all are zeros. The Leslie matrix is used to study the model of population growth. This model is based on studying the rate of fertility a_k and mortality b_j in subsets of the initial population associated with ages.

Given P the matrix with ones along the secondary diagonal and zeros elsewhere. The matrix

$$L = P\mathcal{L}P^{-1} = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-2} & 0 \\ 0 & \dots & \dots & 0 & b_{n-1} \\ a_n & a_{n-1} & \dots & a_2 & a_1 \end{pmatrix} \tag{1}$$

is similar to $\mathcal{L}$. If $b_j = 1, j = 1, 2, \dots, n-1$, L is called Frobenius companion matrix or simply companion matrix and denoted by C. It is clear that given $b_1, b_2, \dots, b_{n-1}$, L is similar to the nonnegative companion matrix C. Indeed, we have $C = DLD^{-1}$ with $D = \mathrm{diag}(1, b_1, b_1 b_2, \dots, b_1 b_2 \cdots b_n)$.

Remark 1. *It is easy to check that the characteristic polynomial of L is*

$$p(x) = x^n - a_1 x^{n-1} - a_2 b_{n-1} x^{n-2} - \dots - a_n b_{n-1} b_{n-2} \cdots b_1. \tag{2}$$

Without loss of generality throughout the paper, we shall consider Leslie matrices of the form (1) with $a_n \neq 0$.

On the other hand, as L is such that $b_j \neq 0, j = 1, 2, \dots, n-1$, and $a_n \neq 0$ its associated directed graph is strongly connected, then L is irreducible. Furthermore, if v is an eigenvector of L associated with the Perron eigenvalue λ, then it is true that

$$Lv = \lambda v.$$

Developing this system and resolving for $v_n = 1$, then it is obtained that

$$v = \left(\frac{b_1 b_2 \cdots b_{n-2} b_{n-1}}{\lambda^{n-1}}, \frac{b_2 b_3 \cdots b_{n-2} b_{n-1}}{\lambda^{n-2}}, \dots, \frac{b_{n-2} b_{n-1}}{\lambda^2}, \frac{b_{n-1}}{\lambda}, 1 \right).$$

Definition 1. *[19] An $n \times n$ matrix A is said to be nonderogatory if every eigenvalue of A have geometric multiplicity 1.*

It is well known that a companion matrix of the form (1), with $b_j = 1, j = 1, 2, \dots, n-1$, is nonderogatory with characteristic polynomial

$$q(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \dots - a_n.$$

In Reference [19] the following results were proved.

Theorem 2. *[19] Every monic polynomial is both the minimal polynomial and the characteristic polynomial of its companion matrix.*

Theorem 3. *[19] An $n \times n$ matrix A is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.*

The doubly Leslie matrices were defined in Reference [20] as a generalization of a doubly companion matrix introduced in Reference [21]. We define a doubly Leslie matrix as follows.

Definition 2. *A* doubly Leslie *matrix is an $n \times n$ nonnegative matrix of the form*

$$\mathcal{B} = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n + c_n \\ b_{n-1} & 0 & \dots & 0 & c_{n-1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & b_2 & 0 & c_2 \\ 0 & \dots & 0 & b_1 & c_1 \end{pmatrix}$$

with $a_k, c_k \geq 0, k = 1, \dots, n$ and $0 < b_j \leq 1$, $j = 1, \dots, n-1$.

The matrix

$$B = P\mathcal{B}P^{-1} = \begin{pmatrix} c_1 & b_1 & 0 & \dots & 0 \\ c_2 & 0 & b_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_{n-1} & 0 & \dots & 0 & b_{n-1} \\ c_n + a_n & a_{n-1} & \dots & a_2 & a_1 \end{pmatrix} \tag{3}$$

is similar to $\mathcal{B}$. In particular, if $b_j = 1, j = 1, 2, \dots, n-1$, B is called doubly companion matrix which was introduced in Reference [21] by Butcher and Chartier. It is clear that if $c_j = 0$, (or $a_j = 0$) $j = 1, \dots, n$, then B is a Leslie matrix.

The paper is organized as follows—in Section 2 we obtain a sufficient condition for the LNIEP and lists of numbers in the complex plane. Then, we completely solve the LNIEP for lists $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1 > 0, \operatorname{Re}\lambda_i \leq 0, |\operatorname{Re}(\lambda_i)| \geq |\operatorname{Im}\lambda_i|$, $i = 2, \dots, n$ (Laffey-Šmigoc type lists) and for lists of real numbers $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1 > 0 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ (Suleĭmanova type lists). Our results allow calculation of the matrix solution. We also show some examples to illustrate the results. In Section 3 we derive a sufficient condition for the DLNIEP and we completely solve the DLNIEP for Laffey-Šmigoc and Suleĭmanova type lists where the solution matrix is of the form (3) with some additional zeros. Finally, at Section 4 conclusions are presented.

The next result will be very useful later.

Lemma 1. *[17] Let t be a nonnegative real number and let $\lambda_2, \lambda_3 \dots, \lambda_n$ be complex numbers with real parts less than or equal to zero, such that the list $(\lambda_2, \lambda_3, \dots, \lambda_n)$ is closed under complex conjugation. Let*

$$\lambda_1 = 2t - \lambda_2 - \dots - \lambda_n$$

and

$$f(x) = (x - \lambda_1)\prod_{j=2}^{n}(x - \lambda_j) = x^n - 2tx^{n-1} + \alpha_2 x^{n-2} + \dots + \alpha_n.$$

Then $\alpha_2 \leq 0$ implies $\alpha_j \leq 0$ for $j = 3, 4, \dots, n$.

2. Leslie Matrices with Prescribed Spectrum

In this section, we derive a sufficient condition for the LNIEP and lists of complex numbers $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Also, we give necessary and/or sufficient conditions for the existence and construction of Leslie matrices for Laffey-Šmigoc and Suleĭmanova type lists. We start with the following definition.

Definition 3. *[19] The kth* elementary symmetric function *of the n numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $k \leq n$, is*

$$S_k(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}$$

the sum of all $\binom{n}{k}$ k-fold products of distinct item from $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Theorem 4. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be the spectrum of an $n \times n$ Leslie matrix, then*

$$(-1)^{k+1} S_k(\Lambda) \geq 0,\ k = 1, 2, \ldots, n. \tag{4}$$

Proof. Let L be a Leslie matrix with spectrum $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and characteristic polynomial as (2).

It is easy to see that

$$\begin{aligned} f(x) = &\ (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \\ = &\ x^n - S_1(\Lambda)x^{n-1} + S_2(\Lambda)x^{n-2} - \cdots + (-1)^{n+1} S_n(\Lambda). \end{aligned} \tag{5}$$

It follows from (2) and (5) that

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &\geq 0 \\ -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n) &\geq 0 \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n &\geq 0 \\ &\vdots \\ (-1)^{n+1}\lambda_1\lambda_2\cdots\lambda_n &\geq 0, \end{aligned}$$

and the inequalities in (4) hold. □

Theorem 5. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of complex numbers such that $\overline{\Lambda} = \Lambda, \lambda_1 \geq |\lambda_i|, i = 1, \ldots, n$, and satisfies the inequalities given in (4). Then Λ is realizable by an $n \times n$ Leslie matrix.*

Proof. Suppose that (4) holds. Let $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_{p-1}$ be real numbers and let

$$\lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+l}, \overline{\lambda}_{p+l}, \ldots, \overline{\lambda}_p = \lambda_n$$

be complex nonreal numbers. We define

$$b_j = \frac{|\lambda_{n+1-j}|}{M},\ j = 1, \ldots, n-1, \tag{6}$$

with $M = \max\{1, \max\{|\lambda_{n+1-j}|, j = 2, \ldots, n\}\}$, and

$$a_1 = S_1(\Lambda),\ \ a_k = \frac{(-1)^{k+1} S_k(\Lambda)}{\prod\limits_{\ell=2}^{k} b_{n+1-\ell}},\ k = 2, \ldots, n. \tag{7}$$

Notice that $0 < b_j \leq 1$, $j = 1, \ldots, n-1$ and $a_k \geq 0$, $k = 1, \ldots, n$.

Then, with b_j, a_k, defined in (6) and (7), we obtain a matrix L of the form (1). From Remark 1, the obtained matrix L has characteristic polynomial as in (2), which is equal to (5). Therefore, L is the desired Leslie matrix with spectrum Λ. □

Example 1. *We want to construct a Leslie matrix with spectrum* $\Lambda = \{5, \frac{1}{10} + i, \frac{1}{10} - i, -1 + i, -1 - i\}$. *From Theorem 5 we obtain the Leslie matrix*

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{101}}{10\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{101}}{10\sqrt{2}} \\ 20 & \frac{1216}{101} & \frac{1143\sqrt{2}}{10\sqrt{101}} & \frac{639\sqrt{2}}{10\sqrt{101}} & \frac{16}{5} \end{pmatrix}$$

that has polynomial characteristic

$$p(x) = x^5 - \frac{16}{5}x^4 - \frac{639}{100}x^3 - \frac{1143}{100}x^2 - \frac{152}{25}x - \frac{101}{10}.$$

and spectrum Λ.

Remark 2. *Notice that in Example 1, some complex numbers of list* Λ *have the positive real part, therefore we could not apply Theorem 3 in Reference [17]. The conditions form Theorem 5 allow us to solve the LNIEP, in consequence the NIEP, for lists that lie outside the left half plane.*

Now, we shall show that the 2nd elementary symmetric function of a Suleĭmanova type list is nonnegative.

Lemma 2. *Let* $\Lambda = (\lambda_1, \ldots, \lambda_n)$ *be a list of real numbers with* $\lambda_1 > 0 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ *and* $\sum_{i=1}^{n} \lambda_i \geq 0$. *Then* $S_2(\Lambda) \leq 0$.

Proof. Since $\sum_{i=1}^{n} \lambda_i \geq 0$, it follows that

$$\sum_{k=1}^{j} \lambda_k \geq \sum_{k=1}^{j+1} \lambda_k \geq 0, \; j = 1, \ldots, n-1.$$

Next,

$$\begin{array}{rcl} \lambda_n(\lambda_1 + \cdots + \lambda_{n-1}) & \leq & 0 \\ \lambda_{n-1}(\lambda_1 + \cdots + \lambda_{n-2}) & \leq & 0 \\ & \vdots & \\ \lambda_3(\lambda_1 + \lambda_2) & \leq & 0 \\ \lambda_1\lambda_2 & \leq & 0. \end{array}$$

Then,

$$\lambda_1(\lambda_1 + \cdots + \lambda_n) + \lambda_2(\lambda_3 + \cdots + \lambda_n) + \lambda_{n-1}\lambda_n \leq 0.$$

Therefore,

$$\begin{aligned} S_2(\Lambda) &= \lambda_1\lambda_2 + \lambda_3(\lambda_1 + \lambda_2) + \cdots + \lambda_n(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}) \\ &= \lambda_1(\lambda_1 + \cdots + \lambda_n) + \lambda_2(\lambda_3 + \cdots + \lambda_n) + \lambda_{n-1}\lambda_n \\ &\leq 0. \end{aligned}$$

□

Theorem 6. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of complex numbers such that $\overline{\Lambda} = \Lambda$, $\lambda_1 \geq |\lambda_i|$, $\mathrm{Re}(\lambda_i) \leq 0$ and $|\,\mathrm{Re}(\lambda_i)| \geq |\,\mathrm{Im}(\lambda_i)|$ for $i = 2, \ldots, n$. Then Λ is realizable by an $n \times n$ Leslie matrix L if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$.*

Proof. It is clear that the condition is necessary. Let $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_{p-1}$ be real numbers and let

$$\lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+\ell}, \overline{\lambda_{p+\ell}}, \ldots, \overline{\lambda_p} = \lambda_n$$

be complex nonreal numbers. Since $\sum_{i=1}^{n} \lambda_i \geq 0$, we obtain that $\sum_{i=2}^{\ell} \mathrm{Re}(\lambda_i) < 0$ and

$$\lambda_1 + \sum_{i=2}^{\ell} \mathrm{Re}(\lambda_i) \geq \lambda_1 + 2\sum_{i=2}^{\ell} \mathrm{Re}(\lambda_i) \geq 0.$$

By the Lemma 2, we have $\sum_{i<j}^{p-1} \lambda_i \lambda_j \leq 0$.

On the other hand, notice that

$$\sum_{j=1}^{p-1} \lambda_j + \sum_{k=p}^{p+\ell} \mathrm{Re}(\lambda_k) \geq 0.$$

Next,

$$\begin{aligned}
S_2(\Lambda) &= \lambda_1(\lambda_2 + \ldots + \lambda_{p-1} + \lambda_p + \lambda_{p+1} + \cdots + \lambda_{p+\ell} + \overline{\lambda}_{p+\ell} + \ldots + \overline{\lambda}_p) \\
&\quad + \lambda_2(\lambda_3 + \ldots + \lambda_{p-1} + \lambda_p + \lambda_{p+1} + \cdots + \lambda_{p+\ell} + \overline{\lambda}_{p+\ell} + \cdots + \overline{\lambda}_p) \\
&\quad \vdots \\
&\quad + \lambda_{p-1}(\lambda_p + \lambda_{p+1} + \cdots + \lambda_{p+\ell}, \overline{\lambda}_{p+\ell} + \cdots + \overline{\lambda}_p) \\
&\quad \vdots \\
&\quad + \overline{\lambda}_{p+1}\overline{\lambda}_p \\
&= \sum_{k=p}^{p+\ell} 2\,\mathrm{Re}(\lambda_k) \sum_{j=1}^{p-1} \lambda_j + \sum_{1 \leq i < j \leq p-1} \lambda_i \lambda_j + \sum_{k=p}^{p+\ell} \|\lambda_k\|^2 \\
&\qquad + 2\left(2 \sum_{p \leq s < t \leq p+\ell} \mathrm{Re}(\lambda_s)\,\mathrm{Re}(\lambda_t)\right) \\
&\leq 2\sum_{k=p}^{p+\ell} \mathrm{Re}(\lambda_k) \sum_{j=1}^{p-1} \lambda_j + \sum_{1 \leq i < j \leq p-1} \lambda_i \lambda_j + 2\sum_{k=p}^{p+\ell} \mathrm{Re}(\lambda_k)^2 \\
&\qquad + 2\left(2 \sum_{p \leq s < t \leq p+\ell} \mathrm{Re}(\lambda_s)\,\mathrm{Re}(\lambda_t)\right) \\
&= 2\sum_{k=p}^{p+\ell} \mathrm{Re}(\lambda_k) \left(\sum_{j=1}^{p-1} \lambda_j + \sum_{k=p}^{p+\ell} \mathrm{Re}(\lambda_k)\right) + \sum_{1 \leq i < j \leq p-1} \lambda_i \lambda_j \\
&\leq 0.
\end{aligned}$$

Then, from Lemma 1 we have $(-1)^{k+1} S_k(\Lambda) \leq 0$, $k = 3, \ldots, n$. Thus, Λ is realizable by an $n \times n$ Leslie matrix. $\square$

Remark 3. *Notice that the condition of Theorem 6 is simpler than the conditions of Theorem 1. In addition, unlike References [14,17] here the trace is not necessarily greater than or equal to zero*

Corollary 1. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, and $\lambda_1 \geq |\lambda_i|$, $i = 2, \ldots, n$. Then Λ is realizable by an $n \times n$ Leslie matrix if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$.*

Proof. It is immediate from Theorem 6. □

Example 2. *Let $\Lambda = (9, -2+i, -2+2i, -2-i, -2-2i)$. Then from Theorem 6 the matrix*

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{2\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2\sqrt{5}} \\ 576 & \frac{3424}{5} & \frac{418\sqrt{2}}{\sqrt{5}} & \frac{86\sqrt{2}}{\sqrt{5}} & 1 \end{pmatrix} \tag{8}$$

is a Leslie matrix with spectrum Λ.

3. Doubly Leslie Matrices with Prescribed Spectrum

In this section, we present a sufficient condition for the DLNIEP and necessary and sufficient conditions for the existence and construction of doubly Leslie matrices for Laffey-Šmigoc and Suleĭmanova type lists are derived.

From now on, to simplify the calculations, we shall consider the doubly Leslie matrices of the form:

$$B = \begin{pmatrix} -c_1 & b_1 & 0 & \ldots & 0 \\ -c_2 & 0 & b_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -c_{n-1} & 0 & \ldots & 0 & b_{n-1} \\ -c_n - a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \end{pmatrix} \tag{9}$$

with $0 < b_j \leq 1, j = 1, 2, \ldots, n-1$ and $a_i, c_i \leq 0, i = 1, 2, \ldots, n$.

The following theorem establishes that a doubly Leslie matrix is similar to a companion matrix.

Theorem 7. *The doubly Leslie matrix B defined in (9) is nonderogatory and its characteristic polynomial is*

$$\begin{aligned} p(x) &= x^n + (d_1 + f_1)x^{n-1} + \left(\sum_{i+j=2} d_i f_j + d_2 + f_2 \right) x^{n-2} + \cdots \\ &+ \left(\sum_{i+j=n-1} d_i f_j + d_{n-1} + f_{n-1} \right) x + \left(\sum_{i+j=n} d_i f_j + d_n + f_n \right), \end{aligned} \tag{10}$$

where $d_1 = a_1, f_1 = c_1$, and $d_i = a_i \prod_{k=n+1-i}^{n-1} b_k$, $f_j = c_j \prod_{k=1}^{j-1} b_k$, for $i, j = 2, 3, \ldots, n$.

Proof. Consider the lower triangular matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ \frac{c_1}{b_1} & \frac{1}{b_1} & 0 & \ddots & \ddots & 0 \\ \frac{c_2}{b_2} & \frac{c_1}{b_1 b_2} & \frac{1}{b_1 b_2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{c_{n-2}}{b_{n-2}} & \frac{c_{n-3}}{b_{n-3}b_{n-2}} & \frac{c_{n-4}}{b_{n-4}b_{n-3}b_{n-2}} & \dots & \frac{1}{b_1 b_2 \cdots b_{n-2}} & 0 \\ \frac{c_{n-1}}{b_{n-1}} & \frac{c_{n-2}}{b_{n-2}b_{n-1}} & \frac{c_{n-3}}{b_{n-3}b_{n-2}b_{n-1}} & \dots & \frac{c_1}{b_1 b_2 \cdots b_{n-1}} & \frac{1}{b_1 b_2 \cdots b_{n-1}} \end{pmatrix}$$

where

$$T^{-1} = \begin{pmatrix} e_1^T \\ e_1^T B \\ e_1^T B^2 \\ \vdots \\ e_1^T B^{n-1} \end{pmatrix}, \quad \text{and} \quad e_1^T = (1 \; 0 \; \dots \; 0) \,.$$

Then, we obtain a companion matrix

$$C = T^{-1}BT = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_n & -\alpha_{n-1} & \dots & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

with

$$\begin{aligned} \alpha_1 &= d_1 + f_1 \\ \alpha_2 &= \sum_{i+j=2} d_i f_j + d_2 + f_2 \\ &\vdots \\ \alpha_{n-1} &= \sum_{i+j=n-1} d_i f_j + d_{n-1} + f_{n-1} \\ \alpha_n &= \sum_{i+j=n} d_i f_j + d_n + f_n \end{aligned}$$

where $d_1 = a_1, f_1 = c_1$ and $d_i = a_i \prod_{k=n+1-j}^{n-1} b_k;\; f_j = c_j \prod_{k=1}^{j-1} b_k$, for $i, j = 2, 3, \dots, n$. Thus, C is a companion matrix with characteristic polynomial as in (10). Therefore, from Theorem 3 the doubly Leslie matrix B is a nonderogatory. □

In the following Theorem, we shall consider a doubly Leslie matrix of the form

$$B = \begin{pmatrix} 0 & b_1 & 0 & \dots & \dots & 0 \\ -c_2 & 0 & b_2 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & b_3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & b_{n-1} \\ -b_{n,1} & \dots & 0 & -a_3 & 0 & -a_1 \end{pmatrix} \tag{11}$$

with $b_{n,1} = c_n$ if n is odd and $b_{n,1} = a_n$ if n is even.

Theorem 8. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of complex numbers such that $\overline{\Lambda} = \Lambda, \lambda_1 \geq |\lambda_i|, i = 1, \ldots, n$ and satisfies the inequalities given in (4). Then Λ is realizable by an $n \times n$ doubly Leslie matrix.*

Proof. Consider $b_j, j = 1, \ldots, n-1$ as in Theorem 5 and define

$$-a_1 = S_1(\Lambda), \; -a_{2k} = 0,$$

$$-a_{2k-1} = \left(S_{2k-1}(\Lambda) - \sum_{j=i}^{k-1} d_{2j-1} f_{2(k-j)} \right) \left(\prod_{\ell=n-2(k-1)}^{n-1} b_\ell \right)^{-1}$$

for $k = 2, \ldots, \lceil \frac{n}{2} \rceil$ and

$$-c_{2k} = \frac{S_{2k}(\Lambda)}{\prod\limits_{\ell=1}^{2k-1} b_\ell}, \; -c_{2k-1} = 0,$$

for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greater integer least than or equal to x and the least integer greater or equal to x, respectively. Notice that $0 < b_j \leq 1, j = 1, \ldots, n-1$ and $a_{2k-1}, c_{2k} \leq 0$. Then, from Theorem 7 the matrix B, with b_j, a_{2k-1} and c_{2k} newly defined, has characteristic polynomial

$$p(x) = x^n + a_1 x^{n-1} + c_2 b_1 x^{n-2} + \cdots + c_{n-1} b_1 \cdots b_{n-2} x + \left(\sum_{i+j=n} d_i f_j + a_n b_1 \cdots b_{n-1} \right),$$

if n is odd, and

$$p(x) = x^n + a_1 x^{n-1} + c_2 b_1 x^{n-2} + \cdots + \left(\sum_{i+j=n-1} d_i f_j + a_n b_1 \cdots b_{n-1} \right) x + c_n b_1 \cdots b_{n-1}$$

if n is even.

Therefore, B is the desired doubly Leslie matrix of the form (11) with spectrum Λ. □

Corollary 2. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of complex numbers such that $\overline{\Lambda} = \Lambda$, $\lambda_1 \geq |\lambda_i|$, $\mathrm{Re}(\lambda_i) \leq 0$ and $|\mathrm{Re}(\lambda_i)| \geq |\mathrm{Im}(\lambda_i)|, i = 2, \ldots, n$. Then Λ is realizable by a doubly Leslie matrix if and only if $\sum\limits_{i=1}^{n} \lambda_i \geq 0$.*

Proof. It is immediate from Theorem 6 and Theorem 8. □

Corollary 3. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, and $\lambda_1 \geq |\lambda_i|, i = 2, \ldots, n$. Then Λ is realizable by an $n \times n$ doubly Leslie matrix if and only if $\sum\limits_{i=1}^{n} \lambda_i \geq 0$.*

Proof. It is a consequence of Corollary 1 and Theorem 8. □

Example 3. *Let $\Lambda = (7, -1+2i, -2+2i, -1-2i, -2-2i)$. Then, from Theorem 8 we have that*

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 21 & 0 & \frac{5}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{424\sqrt{2}}{\sqrt{5}} & 0 & 0 & 0 & \frac{5}{2\sqrt{2}} \\ \frac{26112}{5} & 0 & \frac{264}{2\sqrt{2}} & 0 & 1 \end{pmatrix}$$

is a doubly Leslie matrix with spectrum Λ.

4. Conclusions

In this paper, we give necessary and sufficient conditions for the LNIEP and DLNIEP for Laffey-Šmigoc type lists and as a consequence for Suleĭmanova type lists. Our results provide algorithms for the reconstruction of the matrix from spectral data provided. Such algorithms are applied in the examples presented.

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