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Some Fractional Hermite–Hadamard Type Inequalities for Interval-Valued Functions

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Abstract: In this paper, firstly we prove the relationship between interval h-convex functions and interval harmonically h-convex functions. Secondly, several new Hermite–Hadamard type inequalities for interval h-convex functions via interval Riemann–Liouville type fractional integrals are established. Finally, we obtain some new fractional Hadamard–Hermite type inequalities for interval harmonically h-convex functions by using the above relationship. Also we discuss the importance of our results and some special cases. Our results extend and improve some previously known results.

Keywords: Hermite–Hadamard type inequalities; interval-valued functions; fractional integrals

1. Introduction

Hermite–Hadamard inequality was firstly discovered by Hermite and Hadamard for convex functions are considerable significant in the literature. Since Hermite–Hadamard inequality has been regarded as one of the most useful inequalities in mathematical analysis and optimization, many papers have provided generalizations, refinements and extensions, see [1–4]. Due to the fractional integral has played a irreplaceable part in various scientific fields and importance of Hermite–Hadamard type inequalities, Sarikaya et al. [5] presented Hermite–Hadamard type inequalities via fractional integrals. Moreover, many papers relating to fractional integral inequalities have been obtained for different classes of functions, see [6–8].

On the other hand, interval analysis was initially developed as an attempt to deal with interval uncertainty that appears in computer graphics [9], automatic error analysis [10], and many others. Recently, several authors have extended their research by combining integral inequalities with interval-valued functions (IVFs), one can see Chalco-Cano et al. [11], Román-Flores et al. [12], Flores-Franulič et al. [13], Zhao et al. [14,15], An et al. [16]. As a further extension, more and more Hermite–Hadamard type inequalities involving interval Riemann–Liouville type fractional integral have been obtained for different classes of IVFs, see for interval convex functions [17], for interval harmonically convex functions [18] and the references therein.

Motivated by the ongoing research, We proved the relationship between interval h-convex functions and interval harmonically h-convex functions, then we establish some new Hermite–Hadamard type inequalities for interval h-convex functions and interval harmonically h-convex functions via interval Riemann–Liouville type fractional integrals. Our results extend and improve some known results. Also we discuss the importance of our results and some special cases. In addition, results obtained in this paper may be extended for other classes of convex functions

including interval (h_1, h_2)-convex functions and interval Log-h-convex functions and used as a tool to investigate the research of optimization and probability, among others.

2. Preliminaries and Result

Let us denote by $\mathbb{R}_{\mathcal{I}}$ the collection of all nonempty closed intervals of the real line $\mathbb{R}$. We call $[z] = [\underline{z}, \overline{z}]$ positive if $\underline{z} > 0$. We denote by $\mathbb{R}_{\mathcal{I}}^+$ and $\mathbb{R}^+$ the set of all positive intervals and the set of all positive numbers of $\mathbb{R}$, respectively. For $[z] = [\underline{z}, \overline{z}], [s] = [\underline{s}, \overline{s}] \in \mathbb{R}_{\mathcal{I}}$, the inclusion "$\subseteq$" is defined by

$$[\underline{z}, \overline{z}] \subseteq [\underline{s}, \overline{s}] \Leftrightarrow \underline{s} \leq \underline{z}, \ \overline{z} \leq \overline{s}.$$

For $\lambda \in \mathbb{R}$, the Minkowski addition and scalar multiplication are defined by

$$[z] + [s] = [\underline{z}, \overline{z}] + [\underline{s}, \overline{s}] = [\underline{z} + \underline{s}, \overline{z} + \overline{s}];$$

$$\lambda[z] = \lambda[\underline{z}, \overline{z}] = \begin{cases} [\lambda\underline{z}, \lambda\overline{z}], & \lambda > 0, \\ \{0\}, & \lambda = 0, \\ [\lambda\overline{z}, \lambda\underline{z}], & \lambda < 0 \end{cases}$$

respectively. The conception of Riemann integral for interval-valued function is introduced in [19]. Moreover, we have

Definition 1. *[19] Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $f = [\underline{f}, \overline{f}]$. Then the f is Riemann integrable on $[a, b]$ iff $\underline{f}$ and $\overline{f}$ are Riemann integrable on $[a, b]$ and*

$$\int_a^b f(t)dt = \left[\int_a^b \underline{f}(t)dt, \int_a^b \overline{f}(t)dt \right].$$

The set of all Riemann integrable IVFs on $[a, b]$ will be denoted by $\mathcal{IR}_{([a,b])}$. For more basic notations with interval analysis, see [19,20]. Furthermore, we recall the following results in [17].

Definition 2. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function and $f \in \mathcal{IR}_{([a,b])}$. Then the interval Riemann–Liouville type fractional integrals of f are defined by*

$$\mathcal{J}_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - v)^{\alpha-1} f(v)dv, \quad t > a.$$

and

$$\mathcal{J}_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (v - t)^{\alpha-1} f(v)dv, \quad t < b.$$

where $\alpha > 0$ and Γ is the Gamma function.

Definition 3. *[14] Let $h : [0, 1] \to \mathbb{R}^+$ be a non-negative function. We say that $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ is interval h-convex function or that $f \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, if for all $x, y \in [a, b]$ and $v \in [0, 1]$, we have*

$$f(vx + (1 - v)y) \supseteq h(v)f(x) + h(1 - v)f(y).$$

Definition 4. *[15] Let $h : [0, 1] \to \mathbb{R}^+$ be a non-negative function. We say that $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ is interval harmonically h-convex function or that $f \in SHX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, if for all $x, y \in [a, b]$ and $v \in [0, 1]$, we have*

$$f\left(\frac{xy}{vx + (1 - v)y} \right) \supseteq h(1 - v)f(x) + h(v)f(y).$$

Next, we will present the relationship between interval h-convex functions and interval harmonically h-convex functions which will be used in Section 4.

Theorem 1. $f(x) \in SHX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$ iff $f\left(\frac{1}{x}\right) \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$.

Proof. Let $\psi(x) = f\left(\frac{1}{x}\right)$. Since $f \in SHX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$,

$$f\left(\frac{xy}{vx + (1-v)y}\right) \supseteq h(1-v)f(x) + h(v)f(y). \tag{1}$$

By using $A = \frac{1}{x}$ and $B = \frac{1}{y}$ to replace x and y, respectively, applying (1)

$$f\left(\frac{\frac{1}{xy}}{v\frac{1}{x} + (1-v)\frac{1}{y}}\right) = f\left(\frac{1}{(1-v)x + vy}\right)$$

$$= \psi\left((1-v)x + vy\right)$$

$$\supseteq h(v)f\left(\frac{1}{y}\right) + h(1-v)f\left(\frac{1}{x}\right)$$

$$= h(v)\psi(y) + h(1-v)\psi(x),$$

which gives that $\psi \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$.

On the other hand, if $\psi \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, then

$$\psi\left(vx + (1-v)y\right) \supseteq h(v)\psi(x) + h(1-v)\psi(y).$$

In the same way as above, we have

$$\psi\left(v\frac{1}{x} + (1-v)\frac{1}{y}\right) = f\left(\frac{1}{v\frac{1}{x} + (1-v)\frac{1}{y}}\right) = f\left(\frac{xy}{(1-v)x + vy}\right)$$

$$\supseteq h(v)\psi\left(\frac{1}{x}\right) + h(1-v)\psi\left(\frac{1}{y}\right)$$

$$= h(v)f(x) + h(1-v)f(y),$$

which gives that $f \in SHX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$. We have completed the proof. □

Remark 1. *If $h(v) = v$ and $\underline{f} = \overline{f}$, then we get the Lemma 2.1 of [6].*

3. Fractional Hermite–Hadamard Type Inequalities of Interval h-Convex Functions

In this section, we will prove some new Hermite–Hadamard type inequalities for interval h-convex functions via interval Riemann–Liouville type integrals.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $f = [\underline{f}, \overline{f}]$ and $f \in \mathcal{IR}_{([a,b])}$, $h : [0, 1] \to \mathbb{R}^+$ be a non-negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $f \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$\frac{1}{\alpha h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[\mathcal{J}_{a^+}^\alpha f(b) + \mathcal{J}_{b^-}^\alpha f(a)\right]$$

$$\supseteq [f(a) + f(b)]\int_0^1 v^{\alpha-1}\left[h(v) + h(1-v)\right]dv \tag{2}$$

with $\alpha > 0$.

Proof. Since $f \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, we have

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{x+y}{2}\right) \supseteq f(x) + f(y).$$

Let $x = va + (1-v)b$, $y = (1-v)a + vb$, $v \in [0,1]$, then

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \supseteq f(va + (1-v)b) + f((1-v)a + vb). \tag{3}$$

Multiplying both sides (3) by $v^{\alpha-1}$ and integrating on $[0,1]$, we get

$$
\begin{aligned}
&\frac{1}{\alpha h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_0^1 v^{\alpha-1} dv \\
&\supseteq \left[\int_0^1 v^{\alpha-1} f(va + (1-v)b) dv + \int_0^1 v^{\alpha-1} f((1-v)a + vb) dv \right] \\
&= \left[\int_0^1 v^{\alpha-1} \left(\underline{f}(va + (1-v)b) + \underline{f}((1-v)a + vb) \right) dv, \right. \\
&\qquad \left. \int_0^1 v^{\alpha-1} \left(\overline{f}(va + (1-v)b) + \overline{f}((1-v)a + vb) \right) dv \right] \\
&= \left[\int_b^a (\tau(\mu))^{\alpha-1} \underline{f}(\mu) \frac{d\mu}{a-b} + \int_a^b (1-\tau(\mu))^{\alpha-1} \underline{f}(\mu) \frac{d\mu}{b-a}, \right. \\
&\qquad \left. \int_b^a (\tau(\mu))^{\alpha-1} \overline{f}(\mu) \frac{d\mu}{a-b} + \int_a^b (1-\tau(\mu))^{\alpha-1} \overline{f}(\mu) \frac{d\mu}{b-a} \right] \\
&= \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[J_{a^+}^\alpha \underline{f}(b) + J_{b^-}^\alpha \underline{f}(a), J_{a^+}^\alpha \overline{f}(b) + J_{b^-}^\alpha \overline{f}(a) \right] \\
&= \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha f(b) + \mathcal{J}_{b^-}^\alpha f(a) \right].
\end{aligned}
\tag{4}
$$

where $\tau(\mu) = \frac{b-\mu}{b-a}$.

Similarly, since $f \in SX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$,

$$f(va + (1-v)b) + f((1-v)a + vb) \supseteq [h(v) + h(1-v)][f(a) + f(b)]. \tag{5}$$

Multiplying both sides (5) by $v^{\alpha-1}$ and integrating on $[0,1]$, we have

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha f(b) + \mathcal{J}_{b^-}^\alpha f(a) \right] \supseteq [f(a) + f(b)] \int_0^1 v^{\alpha-1} [h(v) + h(1-v)] dv. \tag{6}$$

By combining (4) with (6), and the result follows. □

Example 1. *Suppose that $[a, b] = [1, 2]$. Let $h(v) = v$ for all $v \in [0, 1]$ and $\alpha = \frac{1}{2}$, $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ be defined by*

$$f(t) = \left[-\sqrt{t} + 2, \sqrt{t} + 2 \right].$$

We obtain

$$\frac{1}{\alpha h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) = \left[8 - 2\sqrt{6}, 8 + 2\sqrt{6} \right],$$

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha f(b) + \mathcal{J}_{b^-}^\alpha f(a) \right] = \left[7 - \sqrt{2} - \frac{\pi}{2} - \log(\sqrt{2} + 1), 9 + \sqrt{2} + \frac{\pi}{2} + \log(\sqrt{2} + 1) \right],$$

$$\left[f(a) + f(b)\right] \int_0^1 \nu^{\alpha-1} \left[h(\nu) + h(1-\nu)\right] d\nu = \left[6 - 2\sqrt{2}, 6 + 2\sqrt{2}\right].$$

Then we get

$$\left[8 - 2\sqrt{6}, 8 + 2\sqrt{6}\right] \supseteq \left[7 - \sqrt{2} - \frac{\pi}{2} - log(\sqrt{2}+1), 9 + \sqrt{2} + \frac{\pi}{2} + log(\sqrt{2}+1)\right] \supseteq \left[6 - 2\sqrt{2}, 6 + 2\sqrt{2}\right].$$

Consequently, Theorem 2 is verified.

Remark 2. *If $\alpha = 1$, then we get Theorem 4.1 of [14]. If $h(\nu) = \nu$, then we get Theorem 2.5 of [17]. If $\underline{f} = \overline{f}$ and $\alpha = 1$, then we get Theorem 6 of [4]. If $\underline{f} = \overline{f}$ and $h(\nu) = \nu$, then we get Theorem 2 of [5].*

Theorem 3. *Let $f, g : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ be two interval-valued functions such that $f = [\underline{f}, \overline{f}]$, $g = [\underline{g}, \overline{g}]$ and $fg \in \mathcal{IR}_{([a,b])}$, $h_1, h_2 : [0, 1] \to \mathbb{R}^+$ be non-negative functions. If $f \in SX(h_1, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, $g \in SX(h_2, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha f(b)g(b) + \mathcal{J}_{b^-}^\alpha f(a)g(a)\right]$$

$$\supseteq \mathcal{M}(a, b) \int_0^1 \nu^{\alpha-1} \left[h_1(\nu)h_2(\nu) + h_1(1-\nu)h_2(1-\nu)\right] d\nu \tag{7}$$

$$+ \mathcal{N}(a, b) \int_0^1 \nu^{\alpha-1} \left[h_1(\nu)h_2(1-\nu) + h_1(1-\nu)h_2(\nu)\right] d\nu$$

where

$$\mathcal{M}(a, b) = f(a)g(a) + f(b)g(b), \quad \mathcal{N}(a, b) = f(a)g(b) + f(b)g(a).$$

Proof. By hypothesis, one has

$$f\left(\nu a + (1-\nu)b\right) \supseteq h_1(\nu)f(a) + h_1(1-\nu)f(b),$$

$$g\left(\nu a + (1-\nu)b\right) \supseteq h_2(\nu)g(a) + h_2(1-\nu)g(b).$$

Since $f, g \in \mathbb{R}_{\mathcal{I}}^+$, we obtain

$$f\left(\nu a + (1-\nu)b\right)g\left(\nu a + (1-\nu)b\right)$$
$$\supseteq h_1(\nu)h_2(\nu)f(a)g(a) + h_1(1-\nu)h_2(1-\nu)f(b)g(b) \tag{8}$$
$$+ h_1(\nu)h_2(1-\nu)f(a)g(b) + h_1(1-\nu)h_2(\nu)f(b)g(a).$$

In the same way as above, we have

$$f\left((1-\nu)a + \nu b\right)g\left((1-\nu)a + \nu b\right)$$
$$\supseteq h_1(1-\nu)h_2(1-\nu)f(a)g(a) + h_1(\nu)h_2(\nu)f(b)g(b) \tag{9}$$
$$+ h_1(1-\nu)h_2(\nu)f(a)g(b) + h_1(\nu)h_2(1-\nu)f(b)g(a).$$

By adding (8) and (9), we obtain

$$f\left(\nu a + (1-\nu)b\right)g\left(\nu a + (1-\nu)b\right) + f\left((1-\nu)a + \nu b\right)g\left((1-\nu)a + \nu b\right)$$
$$\supseteq \left[h_1(\nu)f(a) + h_1(1-\nu)f(b)\right]\left[h_2(\nu)g(a) + h_2(1-\nu)g(b)\right]$$
$$+ \left[h_1(1-\nu)f(a) + h_1(\nu)f(b)\right]\left[h_2(1-\nu)g(a) + h_2(\nu)g(b)\right] \tag{10}$$
$$= \mathcal{M}(a, b)\left[h_1(\nu)h_2(\nu) + h_1(1-\nu)h_2(1-\nu)\right]$$
$$+ \mathcal{N}(a, b)\left[h_1(1-\nu)h_2(\nu) + h_1(\nu)h_2(1-\nu)\right].$$

Multiplying both sides (10) by $v^{\alpha-1}$ and integrating on $[0,1]$, we have

$$
\begin{aligned}
&\int_0^1 v^{\alpha-1} f(va + (1-v)b) g(va + (1-v)b) dv \\
&+ \int_0^1 v^{\alpha-1} f((1-v)a + vb) g((1-v)a + vb) dv \\
&\supseteq \mathcal{M}(a,b) \int_0^1 v^{\alpha-1} \big[h_1(v)h_2(v) + h_1(1-v)h_2(1-v) \big] dv \\
&+ \mathcal{N}(a,b) \int_0^1 v^{\alpha-1} \big[h_1(1-v)h_2(v) + h_1(v)h_2(1-v) \big] dv.
\end{aligned}
\tag{11}
$$

By Definition 2, we obtain

$$
\int_0^1 v^{\alpha-1} f(va + (1-v)b) g(va + (1-v)b) dv = \frac{\Gamma(\alpha)}{(b-a)^\alpha} \mathcal{J}_{a+}^\alpha f(b)g(b),
\tag{12}
$$

$$
\int_0^1 v^{\alpha-1} f((1-v)a + vb) g((1-v)a + vb) dv = \frac{\Gamma(\alpha)}{(b-a)^\alpha} \mathcal{J}_{b-}^\alpha f(a)g(a).
\tag{13}
$$

By substituting the equalities (12) and (13) in (11), then we have inequality (7). $\square$

Remark 3. *If $h(v) = v$, then we get Theorem 3.5 of [17]. If $\alpha = 1$, then we get Theorem 4.5 of [14]. If $\underline{f} = \overline{f}$ and $\alpha = 1$, then we get Theorem 7 of [4].*

Theorem 4. *Let $f, g : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ be two interval-valued functions such that $f = [\underline{f}, \overline{f}]$, $g = [\underline{g}, \overline{g}]$ and $fg \in \mathcal{IR}_{([a,b])}$, $h_1, h_2 : [0,1] \to \mathbb{R}^+$ be non-negative functions and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. If $f \in SX(h_1, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, $g \in SX(h_2, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$
\begin{aligned}
&\frac{1}{\alpha h_1(\frac{1}{2})h_2(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\
&\supseteq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \big[\mathcal{J}_{a+}^\alpha f(b)g(b) + \mathcal{J}_{b-}^\alpha f(a)g(a) \big] \\
&+ \mathcal{M}(a,b) \int_0^1 \big[v^{\alpha-1} + (1-v)^{\alpha-1} \big] h_1(v)h_2(1-v) dv \\
&+ \mathcal{N}(a,b) \int_0^1 \big[v^{\alpha-1} + (1-v)^{\alpha-1} \big] h_1(1-v)h_2(1-v) dv.
\end{aligned}
\tag{14}
$$

Proof. Since $f \in SX(h_1, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, $g \in SX(h_2, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, we get

$$
\begin{aligned}
&f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\
&= f\Big(\frac{va + (1-v)b}{2} + \frac{(1-v)a + vb}{2}\Big) g\Big(\frac{va + (1-v)b}{2} + \frac{(1-v)a + vb}{2}\Big) \\
&\supseteq h_1(\tfrac{1}{2})h_2(\tfrac{1}{2}) \Big[f(va + (1-v)b) + f((1-v)a + vb) \Big] \Big[q(va + (1-v)b) + q((1-v)a + vb) \Big] \\
&\supseteq h_1(\tfrac{1}{2})h_2(\tfrac{1}{2}) \Big[f(va + (1-v)b)g(va + (1-v)b) + f((1-v)a + vb)g((1-v)a + vb) \Big] \\
&+ h_1(\tfrac{1}{2})h_2(\tfrac{1}{2}) \Big[\big(h_1(v)h_2(1-v) + h_1(1-v)h_2(v) \big) \mathcal{M}(a,b) \\
&+ \big(h_1(v)h_2(v) + h_1(1-v)h_2(1-v) \big) \mathcal{N}(a,b) \Big].
\end{aligned}
\tag{15}
$$

Multiplying both sides (15) by $v^{\alpha-1}$ and integrating on $[0,1]$, we have inequality (14). $\square$

Remark 4. *If $h(\nu) = \nu$, then we get Theorem 3.6 of [17]. If $\alpha = 1$, then we get Theorem 4.6 of [14]. If $\underline{f} = \overline{f}$ and $\alpha = 1$, then we get Theorem 8 of [4].*

4. Fractional Hermite–Hadamard Type Inequalities of Interval Harmonically h-Convex Functions

In this section, we will use the above results to get some Hermite–Hadamard type inequalities for interval harmonically h-convex functions via interval Riemann–Liouville type integrals and some special cases are also discussed.

Theorem 5. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $f = [\underline{f}, \overline{f}]$ and $f \in \mathcal{IR}_{([a,b])}$, $h : [0, 1] \to \mathbb{R}^+$ be a non-negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $f \in SHX(h, [a, b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$
\frac{1}{\alpha h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right)
$$
$$
\supseteq \Gamma(\alpha) \left(\frac{ab}{b-a}\right)^\alpha \left[\mathcal{J}_{\left(\frac{1}{a}\right)^-}^\alpha (f \circ \xi)\left(\frac{1}{b}\right) + \mathcal{J}_{\left(\frac{1}{b}\right)^+}^\alpha (f \circ \xi)\left(\frac{1}{a}\right)\right] \tag{16}
$$
$$
\supseteq [f(a) + f(b)] \int_0^1 \nu^{\alpha-1} \left[h(\nu) + h(1-\nu)\right] d\nu
$$

where $\tilde{\xi}(x) = \frac{1}{x}$.

Proof. Let $\psi(x) = f\left(\frac{1}{x}\right)$. By Theorem 1, we have $\psi \in SX\left(h, [a, b], \mathbb{R}_{\mathcal{I}}^+\right)$ and

$$
\frac{1}{h\left(\frac{1}{2}\right)} \psi\left(\frac{x+y}{2xy}\right) \supseteq \psi\left(\frac{1}{x}\right) + \psi\left(\frac{1}{y}\right).
$$

Let $x = \frac{ab}{\nu a + (1-\nu)b}$, $y = \frac{ab}{(1-\nu)a + \nu b}$, $\nu \in [0, 1]$. Then

$$
\frac{1}{h\left(\frac{1}{2}\right)} \psi\left(\frac{a+b}{2ab}\right) \supseteq \psi\left(\frac{\nu a + (1-\nu)b}{ab}\right) + \psi\left(\frac{(1-\nu)a + \nu b}{ab}\right), \tag{17}
$$

Multiplying both sides (17) by $\nu^{\alpha-1}$ and integrating on $[0, 1]$, we have

$$
\frac{1}{\alpha h\left(\frac{1}{2}\right)} \psi\left(\frac{a+b}{2ab}\right)
$$
$$
\supseteq \left[\int_0^1 \nu^{\alpha-1} \psi\left(\frac{\nu a + (1-\nu)b}{ab}\right) d\nu + \int_0^1 \nu^{\alpha-1} \psi\left(\frac{(1-\nu)a + \nu b}{ab}\right) d\nu\right] \tag{18}
$$
$$
= \Gamma(\alpha) \left(\frac{ab}{b-a}\right)^\alpha \left[\mathcal{J}_{\left(\frac{1}{a}\right)^-}^\alpha \psi\left(\frac{1}{b}\right) + \mathcal{J}_{\left(\frac{1}{b}\right)^+}^\alpha \psi\left(\frac{1}{a}\right)\right]
$$

Similarly, we have

$$
\psi\left(\frac{\nu a + (1-\nu)b}{ab}\right) \supseteq h(\nu)\psi\left(\frac{1}{b}\right) + h(1-\nu)\psi\left(\frac{1}{a}\right),
$$
$$
\psi\left(\frac{(1-\nu)a + \nu b}{ab}\right) \supseteq h(\nu)\psi\left(\frac{1}{a}\right) + h(1-\nu)\psi\left(\frac{1}{b}\right).
$$

Then,

$$\psi\left(\frac{\nu a + (1-\nu)b}{ab}\right) + \psi\left(\frac{(1-\nu)a+\nu b}{ab}\right) \supseteq [h(\nu)+h(1-\nu)]\left[\psi\left(\frac{1}{a}\right)+\psi\left(\frac{1}{b}\right)\right]. \tag{19}$$

Multiplying both sides (19) by $\nu^{\alpha-1}$ and integrating on $[0,1]$, we have

$$\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{a}\right)^-}\psi\left(\frac{1}{b}\right)+\mathcal{J}^\alpha_{\left(\frac{1}{b}\right)^+}\psi\left(\frac{1}{a}\right)\right]$$
$$\supseteq \left[\psi\left(\frac{1}{a}\right)+\psi\left(\frac{1}{b}\right)\right]\int_0^1 \nu^{\alpha-1}[h(\nu)+h(1-\nu)]\,dt. \tag{20}$$

By (18) and (20), we have inequality (16). □

Remark 5. *If $h(\nu)=\nu$, then we get Theorem 3.6 of [18]. If $\alpha=1$, then we get Theorem 2 of [15]. If $\underline{f} = \overline{f}$ and $\alpha=1$, then we get Theorem 3.2 of [3]. If $\underline{f}=\overline{f}$ and $h(\nu)=\nu$, then we get Theorem 4 of [7].*

Theorem 6. *Let $f,g : [a,b] \to \mathbb{R}^+_\mathcal{I}$ be two interval-valued functions such that $f = [\underline{f},\overline{f}]$, $g = [\underline{g},\overline{g}]$ and $fg \in \mathcal{IR}_{([a,b])}$, $h_1,h_2 : [0,1] \to \mathbb{R}^+$ be non-negative functions. If $f \in SHX(h_1,[a,b],\mathbb{R}^+_\mathcal{I})$, $g \in SHX(h_2,[a,b],\mathbb{R}^+_\mathcal{I})$, then*

$$\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{b}\right)^+}(f\circ\xi)\left(\frac{1}{a}\right)(g\circ\xi)\left(\frac{1}{a}\right)+\mathcal{J}^\alpha_{\left(\frac{1}{a}\right)^-}(f\circ\xi)\left(\frac{1}{b}\right)(g\circ\xi)\left(\frac{1}{b}\right)\right]$$
$$\supseteq \mathcal{M}(a,b)\int_0^1\left[\nu^{\alpha-1}+(1-\nu)^{\alpha-1}\right]h_1(\nu)h_2(\nu)d\nu \tag{21}$$
$$+\mathcal{N}(a,b)\int_0^1\left[\nu^{\alpha-1}+(1-\nu)^{\alpha-1}\right]h_1(\nu)h_2(1-\nu)d\nu$$

where $\xi(x) = \frac{1}{x}$.

Proof. The proof is completed by combining Theorems 1, 3 and 5. □

Remark 6. *If $h_1(\nu)=h_2(\nu)=\nu$, then*

$$\Gamma(\alpha)\left(\frac{ab}{b-a}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{b}\right)^+}(f\circ\xi)\left(\frac{1}{a}\right)(g\circ\xi)\left(\frac{1}{a}\right)+\mathcal{J}^\alpha_{\left(\frac{1}{a}\right)^-}(f\circ\xi)\left(\frac{1}{b}\right)(g\circ\xi)\left(\frac{1}{b}\right)\right]$$
$$\supseteq \mathcal{M}(a,b)\int_0^1 \nu^2\left[\nu^{\alpha-1}+(1-\nu)^{\alpha-1}\right]d\nu + \mathcal{N}(a,b)\int_0^1 \nu(1-\nu)\left[\nu^{\alpha-1}+(1-\nu)^{\alpha-1}\right]d\nu.$$

If $\alpha=1$, then we get Theorem 4 of [15]. If $\underline{f}=\overline{f}$ and $\alpha=1$, then we get Theorem 3.6 of [3].

Theorem 7. *Let $f,g : [a,b] \to \mathbb{R}^+_\mathcal{I}$ be two interval-valued functions such that $f = [\underline{f},\overline{f}]$, $g = [\underline{g},\overline{g}]$ and $fg \in \mathcal{IR}_{([a,b])}$, $h_1,h_2 : [0,1] \to \mathbb{R}^+$ be non-negative functions and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. If $f \in SHX(h_1,[a,b],\mathbb{R}^+_\mathcal{I})$, $g \in SHX(h_2,[a,b],\mathbb{R}^+_\mathcal{I})$, then*

$$\frac{1}{\alpha h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)$$
$$\supseteq \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{b}\right)^+}(f\circ\xi)\left(\frac{1}{a}\right)(g\circ\xi)\left(\frac{1}{a}\right)+\mathcal{J}^\alpha_{\left(\frac{1}{a}\right)^-}(f\circ\xi)\left(\frac{1}{b}\right)(g\circ\xi)\left(\frac{1}{b}\right)\right]$$
$$+\int_0^1\left[\nu^{\alpha-1}+(1-\nu)^{\alpha-1}\right]\left[h_1(\nu)h_2(1-\nu)\mathcal{M}(a,b)+h_1(\nu)h_2(\nu)\mathcal{N}(a,b)\right]d\nu$$

where $\xi(x) = \frac{1}{x}$.

Proof. The proof is completed by combining Theorem 1, 4 and 5. □

Remark 7. *If $h_1(\nu) = h_2(\nu) = \nu$, then*

$$4f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)$$

$$\supseteq \Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{b}\right)^{+}}(f \circ \xi)\left(\frac{1}{a}\right)(g \circ \xi)\left(\frac{1}{a}\right) + \mathcal{J}^{\alpha}_{\left(\frac{1}{a}\right)^{-}}(f \circ \xi)\left(\frac{1}{b}\right)(g \circ \xi)\left(\frac{1}{b}\right)\right]$$

$$+ \int_0^1 \left[\nu^{\alpha-1} + (1-\nu)^{\alpha-1}\right]\left[\nu(1-\nu)\mathcal{M}(a,b) + \nu^2\mathcal{N}(a,b)\right]d\nu.$$

If $\alpha = 1$, then we get Theorem 4 of [15].

5. Conclusions

This paper proved the relationship between interval *h*-convex functions and interval harmonically *h*-convex functions. Further, we obtained some Hermite–Hadamard type inequalities for IVFs via interval Riemann–Liouville type fractional integrals. The results obtained in this article are the generalizations and refinements of the earlier works. Moreover, these results may be extended for other kinds of convex functions including interval (h_1, h_2)-convex functions and interval Log-*h*-convex functions and used as a method to establish the Hermite–Hadamard type inequalities for other types of interval harmonically convex functions. As a future research direction, we intend to investigate Hermite–Hadamard type inequalities for IVFs on time scales and some applications in interval optimization, probability, among others.

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