



Article On the Kernel of a Polynomial of Scalar Derivations

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Abstract: In this paper, by using a vector variable, the procedure of characteristic systems allows us to describe the kernel of a polynomial of scalar derivations by solving Cauchy Problems for the corresponding system of ODEs. Moreover, a gradient representation for the associated Cauchy Problem solution is derived.

Keywords: scalar derivation; characteristic system; Lie algebra; gradient system; kernel

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1. Introduction and Problem Formulation

The gradient-type representations for some solutions, Lie algebras, gradient systems in a Lie algebra, algebraic representation of gradient systems and their integral manifolds, have been studied for a long time, with remarkable results, by Vârsan [1] and Barbu et al. [2]. Apart from the linear higher order PDEs, the characteristic system method is intensively used for solving linear or nonlinear SPDEs and in this respect, we mention Iftimie et al. [3]. For other different but related viewpoints on this subject, the reader is directed to Friedman [4], Sussmann [5], Crandall and Souganidis [6], Sontag [7], Bressan and Shen [8], Evans [9], Brezis [10], Parveen and Akram [11], Treanță and Vârsan [12], Treanță [13].

In this paper, by using a vector variable, the procedure of characteristic systems allows us to describe the kernel of a polynomial of scalar derivations by solving Cauchy Problems for the corresponding system of ODEs. Moreover, a gradient representation for the associated Cauchy Problem solution is derived. As the main motivation of this study, the mathematical framework developed in this work can be extended for the study of some higher-order hyperbolic, parabolic or Hamilton–Jacobi equations involving a finite set of derivations. For instance, a simple *m*-th order Hamilton–Jacobi equation has the following expression

$$(\vec{Z})^{m}(\varphi)(t,x) = \sum_{k=0}^{m-1} a_{k}(t)(\vec{Z})^{k}(\varphi)(t,x) + f(t), \ (t,x) \in [0,T] \times \mathbb{R}^{n},$$
(1)

where $\vec{Z} : C^1([0, T] \times R^n; R) \to C([0, T] \times R^n; R)$ is a linear application defined by

$$\vec{Z}(\varphi)(t,x) = \partial_t \varphi(t,x) + \langle \partial_x \varphi(t,x), X(t,x) \rangle, \ t \in [0,T], \ x \in \mathbb{R}^n$$

(see \vec{Z} as being generated by the vector field Z(z) = col(1, X(z)), z = (t, x), where $X \in (C_b^1 \cap C^m)([0, T] \times R^n; R^n), m \ge 2$, and the index *b* of C_b^1 is for *bounded*) and $\{a_k, f : 0 \le k \le m-1\} \subseteq C([0, T]; R)$. Using standard notation, $\varphi = y_0, (\vec{Z})^k(\varphi) = y_k, 0 \le k \le m-1$, rewrite (1) as a system of Hamilton–Jacobi equations

$$\vec{Z}(y_0)(t,x) = y_1(t,x), \quad \cdots \quad , \vec{Z}(y_{m-2})(t,x) = y_{m-1}(t,x),$$
(2)

$$\vec{Z}(y_{m-1})(t,x) = \sum_{k=0}^{m-1} a_k(t) y_k(t,x) + f(t), \ (t,x) \in [0,T] \times \mathbb{R}^n.$$

A classical solution $y(t, x) = (y_0(t, x), ..., y_{m-1}(t, x)) \in \mathbb{R}^m$, $(t, x) \in [0, T] \times \mathbb{R}^n$, associated with (2), means a first order continuously differentiable mapping $y \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ satisfying (2) for any $(t, x) \in [0, T] \times \mathbb{R}^n$. The first component of a solution verifying (2) stands for a classical solution of the higher order Hamilton–Jacobi equation (1). It is well known that the characteristic system method is associated with the classical solutions of the PDEs (at least continuous functions). In the previous context, a solution of the corresponding higher order PDEs involves a characteristic system containing a bounded variation component as solution for some ODEs.

Throughout this paper, let $0 \in I \subseteq R$ be an open interval. Consider a polynomial of the scalar derivation $\frac{d}{dt}$,

$$P_m\left(t;\frac{d}{dt}\right) = a_1(t) + a_2(t)\left(\frac{d}{dt}\right) + \dots + a_m(t)\left(\frac{d}{dt}\right)^{m-1} - \left(\frac{d}{dt}\right)^m,\tag{3}$$

where $m \ge 1$, $a_j \in L_{\infty}(I)$, $j \in \{1, 2, ..., m\}$. Define

$$H_{\infty}^{m}(I) = \left\{ h \in C^{m-1}(I) \colon \left(\frac{d}{dt}\right)^{m}(h) \in L_{\infty}(I) \right\}$$
(4)

and consider $Ker(P_m) \subseteq H_{\infty}^m(I)$, where

$$Ker(P_m) = \left\{ h \in H^m_{\infty}(I) : P_m\left(t; \frac{d}{dt}\right)(h)(t) = 0, \ a.e. \ t \in I \right\}.$$
(5)

The procedure of characteristic systems (see Friedman [4], Vârsan [1]) allows us to describe $Ker(P_m)$ by solving *Cauchy Problems* for the corresponding system of ODEs using a vector variable

$$y = \operatorname{col}(y_1, y_2, ..., y_m), \ \frac{dy}{dt} = Ay + \sum_{i=1}^m a_i(t)B_iy, \ y(0) = y_0 \in \mathbb{R}^m.$$
(6)

Here, the $(m \times m)$ constant matrices *A* and B_i , $i = \overline{1, m}$, are defined by

$$A = [0 \ e_1 \ \cdots \ e_{m-1}], \ B_1 = [e_1 \ 0 \ \cdots \ 0], \ \cdots , \ B_m = [0 \ 0 \ \cdots \ e_m],$$
(7)

where $\{e_1, ..., e_m\}$ is the canonical basis and $0 \in R^m$ is the origin. By definition

$$[B_i, B_j] := B_j B_i - B_i B_j, \ i, j \in \{1, 2, ..., m\}, \quad (Lie \ bracket),$$
(8)

and making a direct computation, we get

$$O = [B_i, B_j], \ i, j \in \{1, 2, ..., m\},$$
(9)

with O - null matrix, and

$$A^m = O, \quad (A \text{ is a nilpotent matrix}). \tag{10}$$

The Cauchy Problem solution for (4) is represented by

$$y(t;y_0) = [\exp At] \,\hat{y}(t;y_0), \ t \in I, \tag{11}$$

where $\{\hat{y}(t; y_0) : t \in I\}$ fulfils the following linear system (initial value problem)

$$\frac{dy}{dt} = \sum_{i=1}^{m} a_i(t) A_i(t) y, \ t \in I, \ y(0) = y_0 \in \mathbb{R}^m.$$
(12)

By using the linear mapping $ad_A : M_{m \times m} \to M_{m \times m}$, $ad_A(B) := BA - AB$ (see [A, B]), write the $(m \times m)$ matrices

$$A_i(t) := [\exp(-tA)] B_i [\exp tA], \ i \in \{1, 2, ..., m\},$$
(13)

as follows

$$A_{i}(t) = B_{i} + \frac{t}{1!} \operatorname{ad}_{A}(B_{i}) + \dots + \frac{t^{m-1}}{(m-1)!} \operatorname{ad}_{A}^{m-1}(B_{i}), \ i \in \{1, ..., m\}.$$
(14)

In addition, taking into account (9), (10) and (14), we get

$$A_i(t) = [\exp \ t \ \mathrm{ad}_A] (B_i), \ i \in \{1, 2, ..., m\}, \ t \in I.$$
(15)

Denote $N = m^2$ and define N matrices $\{C_1, C_2, ..., C_N\} \subseteq M_{m \times m}$, as follows

$$\{C_1, C_2, ..., C_N\} = \left\{ \operatorname{ad}_A^k(B_1) : k \in \{0, 1, 2, ..., m-1\} \right\} \cup \dots \cup$$

$$\left\{ \operatorname{ad}_A^k(B_m) : k \in \{0, 1, 2, ..., m-1\} \right\}.$$
(16)

Moreover, let { $\alpha_1(t), \alpha_2(t), ..., \alpha_N(t) : t \in I$ } be given by

$$\{\alpha_1(t), ..., \alpha_m(t)\} = a_1(t) \left[1, \frac{t}{1!}, ..., \frac{t^{m-1}}{(m-1)!}\right],$$
(17)

:
$$\{\alpha_{N-m+1}(t),...,\alpha_N(t)\} = a_m(t) \left[1,\frac{t}{1!},...,\frac{t^{m-1}}{(m-1)!}\right]$$

With these notations, we write ODE (12) as follows

$$\frac{dy}{dt} = \sum_{j=1}^{N} \alpha_j(t) Y_j(y), \ t \in I, \ y(0) = y_0,$$
(18)

where $Y_j(y) := C_j y, \ j \in \{1, 2, ..., N\}.$

2. Main Results

In this section, the main results of the present paper are formulated and proved.

Theorem 1. Consider $\{C_1, C_2, ..., C_N\}$ defined in (16), with $N = m^2$. Then, $\{C_1, C_2, ..., C_N\}$ is a basis for $M_{m \times m}$ and

$$\{Y_1(y) = C_1 y, Y_2(y) = C_2 y, ..., Y_N(y) = C_N y\}$$

is a system of generators for the Lie algebra $L(Y_1, ..., Y_N) \subseteq C^{\infty}(\mathbb{R}^m; \mathbb{R}^m)$ *generated by* $\{Y_1, ..., Y_N\}$ *.*

Proof. By direct computation, we rewrite the matrices $\{C_1, C_2, ..., C_N\}$ as follows:

$$\mathrm{ad}_{A}^{k}(B_{1}) = \begin{bmatrix} (-1)^{k} e_{m-k} & c_{1,1}^{k} e_{m-k+1} \dots c_{1,k}^{k} e_{m} & 0 \dots 0 \end{bmatrix},$$
(19)

for some constants $c_{1,i}^k$, $0 \le k \le m - 1$. For any $1 \le i \le m$, we get

$$\mathrm{ad}_{A}^{k}(B_{i}) = \begin{bmatrix} \underbrace{0 \dots 0}_{(i-1) \ times} \ (-1)^{k} e_{m-k} & c_{i,1}^{k} e_{m-k+1} \dots c_{i,k(i)}^{k} e_{m-k+k(i)} & 0 \dots 0 \end{bmatrix},$$
(20)

where $k(i) := min \{k, m - i\}$ and $c_{i,i}^k$ are some constants.

The particular structure given in (20) leads us directly to the conclusion that $\{C_1, C_2, ..., C_N\} \subseteq M_{m \times m}$ are linearly independent. Therefore, $\{C_1, C_2, ..., C_N\}$ is a basis for $L(C_1, C_2, ..., C_N) = M_{m \times m}$ and using

$$L(Y_1, Y_2, ..., Y_N) = \{Cy \colon C \in L(C_1, C_2, ..., C_N)\}$$
(21)

we get that $\{Y_1, Y_2, ..., Y_N\}$ is a system of generators for $L(Y_1, Y_2, ..., Y_N)$. The proof is complete. \Box

The next remark contains several mathematical tools (some of these, introduced in Vârsan [1]) and their hints, which are necessary for proving Theorem 2.

Remark 1. Consider the linear vector fields $Y_j(y) := C_j y$, $j \in \{1, ..., N\}$, $y \in \mathbb{R}^m$, where $\{C_1, C_2, ..., C_N\} \subseteq M_{m \times m}$ is a basis (see Theorem 1). Let $L(Y_1, ..., Y_N)$ be the finite dimensional Lie algebra generated by $\{Y_1, ..., Y_N\}$. Then, the following statements are valid:

$$\dim L(Y_1, ..., Y_N)(y_0) = \dim [\operatorname{span} \{Y_1(y_0), ..., Y_N(y_0)\}] = m,$$
(22)

for any $y_0 \neq 0 \in \mathbb{R}^m$;

$$y(p;\lambda) = G(p)\lambda := [\exp t_1 C_1] \cdots [\exp t_N C_N] \lambda, \qquad (23)$$

with $p = (t_1, ..., t_N) \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^m$, satisfies a gradient system (GS) in $L(Y_1, Y_2, ..., Y_N)$ (see Vârsan [1] for more details; $G(p) \in M_{m \times m}$, $p \in \mathbb{R}^N$, is solution for a (GS) in $M_{m \times m}$)

$$\partial_p y(p;\lambda) = \{Y_1, ..., Y_N\} (y(p;\lambda)) A(p), \ p \in \mathbb{R}^N,$$
(24)

where A(p) is an $(N \times N)$ analytic matrix fulfilling $A(0) = I_N$ and det $A(p) \neq 0$ for $p \in B(0, \rho) \subseteq \mathbb{R}^N$; if

$$\varphi(y(p;y_0)) = const. (= \varphi(y_0)), \ \forall p \in B(0,\rho) \subseteq \mathbb{R}^N,$$
(25)

for some $y_0 \neq 0 \in \mathbb{R}^m$ and $\varphi \in C^1$ $(D \supseteq B(y_0, a))$, where $B(y_0, a) \subseteq \{y(p; y_0) \neq 0 \colon p \in B(0, \rho) \subseteq \mathbb{R}^N\}$, then $\partial_y \varphi(y) = 0 \in \mathbb{R}^m$, $\forall y \in B(y_0, a)$.

The conclusion (22) relies on Theorem 1 and we get

$$\dim L(Y_1, ..., Y_N)(y_0) = \dim [\operatorname{span} \{Y_1(y_0), ..., Y_N(y_0)\}]$$

$$= \dim [\text{span} \{C_1 y_0, ..., C_N y_0\}] = \dim \{C y_0 \colon C \in M_{m \times m}\} = m, \quad \forall y_0 \neq 0 \in \mathbb{R}^m.$$

Using (23) and (24), we compute the following Lie derivatives

$$0 = \langle \partial_{p} [\varphi (y(p; y_{0}))], A^{-1}(p)e_{j} \rangle = \langle \partial_{y} \varphi (y(p; y_{0})), \partial_{p} y(p; y_{0}) A^{-1}(p)e_{j} \rangle$$

$$= \langle \partial_{y} \varphi (y(p; y_{0})), Y_{j} (y(p; y_{0})) \rangle, \quad j \in \{1, 2, ..., N\},$$
(26)

where $\{y(p;y_0) \neq 0: p \in B(0,\rho)\} \supseteq B(y_0,a)$ and $\{e_1,e_2,...,e_N\} \subseteq \mathbb{R}^N$ is the canonical basis. Taking into account (22) and (26), we obtain $\partial_y \varphi(y(p;y_0)) = 0$, $p \in B(0,\rho)$, and $\partial_y \varphi(y) = 0$ for any $y \in B(y_0,a)$. In other words, as far as

dim [span { $Y_1(y_0), ..., Y_N(y_0)$ }] = m,

for any $y_0 \neq 0$ (see (22)), from (26), we obtain both $\partial_y \varphi(y(p; y_0)) = 0 \in \mathbb{R}^m$ for any $p \in B(0, \rho) \subseteq \mathbb{R}^N$ and $\partial_y \varphi(y) = 0 \in \mathbb{R}^m$, for any $y \in B(y_0, a) \subseteq \mathbb{R}^m$.

Theorem 2. Assume $a_j(t) \equiv 0, j \in \{1, 2, ..., i\}$, for some $1 \leq i \leq m$. Define a subspace $S_i =$ span $\{e_1, ..., e_i\} \subseteq R^m$ and its orthogonal complement $S_i^{\perp} \subseteq R^m$, where $\{e_1, ..., e_m\} \subseteq R^m$ is the canonical basis. Then

$$\forall y_0 \in S_i \text{ is a stationary point for ODE (12).}$$
 (27)

In addition, for each $y_0 \in S_i^{\perp}$, $y_0 \neq 0$, the following statements are valid:

$$\dim L\left(Y_{j(i)},...,Y_{N}\right)(y_{0}) = \dim \left[\operatorname{span}\left\{Y_{j(i)}(y_{0}),...,Y_{N}(y_{0})\right\}\right] = m - i,$$

$$\left\{C_{j(i)},...,C_{N}\right\} \text{ is a basis for } M^{i}_{m \times m} := L\left(C_{j(i)},...,C_{N}\right), \ j(i) := mi + 1$$

$$\hat{y}(t,y_{0}) \in S^{\perp}_{i}, \ t \in I, \ for \ any \ solution \ of \ ODE \ (12).$$

$$(28)$$

Proof. As far as any $y_0 \in S_i$ is a stationary point for each $Y_j(y) = C_j y$, j = j(i), ..., N, j(i) = mi + 1, and ODE (12) is written using only $\{Y_{j(i)}(y), ..., Y_N(y)\}$, we get the conclusion (27) is satisfied. By definition, (see Theorem 1) the matrices $\{C_{j(i)}, ..., C_N\} \subseteq M_{m \times m}^i$ determine a basis in the space $M_{m \times m}^i$ consisting of all $(m \times m)$ matrices $C = \left[\underbrace{0 \dots 0}_{i-times} c_{i+1} \dots c_m\right]$, with $c_j \in R^m$, j = i+1, ..., m. Using $\dim L\left(Y_{j(i)}, ..., Y_N\right)(y_0) = \dim \left[\operatorname{span}\left\{Y_{j(i)}(y_0), ..., Y_N(y_0)\right\}\right]$ $= \dim \left[\operatorname{span}\left\{C_{j(i)}y_0, ..., C_Ny_0\right\}\right]$ $= \dim \left\{C^i y_0 \colon C^i \in M_{m \times m}^i\right\} = m - i$,

we get the conclusion (28). Notice that $\frac{d}{dt} < \hat{y}(t; y_0), e_j >= 0$, $t \in I$, and $< \hat{y}(t; y_0), e_j >= < y_0, e_j >= 0$, $t \in I$, for any $j = \overline{1, i}$, which stands for the conclusion (29). The proof is complete. \Box

3. Conclusions

In this paper, we investigated (by using the characteristic system method) the kernel of a polynomial of scalar derivations by solving Cauchy Problems for the corresponding system of ODEs. In addition, a gradient representation for the associated Cauchy Problem solution has been formulated. Moreover, as further research directions, some applications were highlighted in this study of some higher-order hyperbolic, parabolic or Hamilton-Jacobi equations involving a finite set of derivations.

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