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Tempered Fractional Integral Inequalities for Convex Functions

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Abstract: Certain new inequalities for convex functions by utilizing the tempered fractional integral are established in this paper. We also established some new results by employing the connections between the tempered fractional integral with the (R-L) fractional integral. Several special cases of the main result are also presented. The obtained results are more in a general form as it reduced certain existing results of Dahmani (2012) and Liu et al. (2009) by employing some particular values of the parameters.

Keywords: fractional integrals; tempered fractional integral; inequalities

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1. Introduction

The domain of fractional calculus (FC) as engaged in derivatives and integrals of non-integer order. This area has a long history. The basis of it can be traced back to the letter between L'Hôpital and Leibniz in 1695 (See [1]). In the last three centuries, several mathematicians and physicists have devoted to the developments of the theories of fractional calculus [2–13]. Furthermore, fractional and fractal calculus applications are found in various fields [14–18]. In practical applications, certain various types of fractional operators such as Riemann–Liouville, Caputo, Riesz [11,12] and Hilfer [19] fractional operators are introduced. Freshly, the researchers have studied certain new fractional integral and derivative operators and their possible applications in various disciplines of sciences.

Khalil et al. [20] have introduced the notion of fractional conformable derivative (FCD) operators with some shortcomings. Abdeljawad [21] investigated the properties of the fractional conformable derivative operators. In [22], Jarad et al. introduced the fractional conformable integral and derivative operators. Anderson and Unless [23] developed the idea of conformable derivative by employing local proportional derivatives. Abdeljawad and Baleanu [24] investigated certain monotonicity results for fractional difference operators with discrete exponential kernels. Abdeljawad and Baleanu [25] have established fractional derivative operators with exponential kernel and their discrete versions. In [26], Atangana and Baleanu defined a new fractional derivative operator with the non-local and non-singular kernel. Caputo and Fabrizio [27] defined fractional derivative without a singular kernel. Certain properties of fractional derivative without a singular kernel can be found in the work of Losada and Nieto [28]. In [29], Jarad et al. defined generalized fractional derivatives generated by a class of local proportional derivatives.

On the other hand, fractional integral inequalities and its applications have also an essential role in the theory of differential equations and applied mathematics. A large number of several interesting integral inequalities are established by the researchers such as weighted Grüss type inequalities [30], Inequalities via R-L integrals [31], inequalities for extended gamma and confluent hypergeometric k-function [32], Gronwall inequalities involving k-fractional integral [33], inequalities involving generalized R-L integrals [34], the generalized R-L integrals with applications [35] and Grüss-type inequalities involving the generalized R-L integrals [36].

In [37], the following inequalities are presented

$$\int_0^1 v^{\mu+1}(\theta)d\theta \geq \int_0^1 \theta^\mu v(\theta)d\theta \tag{1}$$

and

$$\int_0^1 v^{\mu+1}(\theta)d\theta \geq \int_0^1 \theta v^\sigma(\theta)d\theta, \tag{2}$$

where $\theta > 0$ and v on $[0,1]$, which is the positive continuous function, such that

$$\int_x^1 v(\theta)d\theta \geq \int_x^1 \theta d\theta, x \in [0,1].$$

In [38], the following inequalities are presented

$$\int_a^b v^{\mu+\nu}(\theta)d\theta \geq \int_a^b (\theta-a)^\mu v^\nu(\theta)d\theta, \tag{3}$$

where $\mu > 0$, $\nu > 0$ and the positive continuous v on $[a,b]$ such that

$$\int_a^b v^\omega(\theta)d\theta \geq \int_a^b (\theta-a)^\omega d\theta, \omega = \min(1,\nu), \theta \in [a,b].$$

The following theorems are presented by Liu et al. [39]:

Theorem 1. *Let the two positive functions u and v be continuous functions on $[a,b]$ such that $u(\theta) \leq v(\theta)$ for all $\theta \in [a,b]$. Assume that the function $\frac{u}{v}$ is decreasing and the function u is increasing. Suppose that Ψ is a convex function with $\Psi(0) = 0$. Then the following inequality hold*

$$\frac{\int_a^b u(\theta)d\theta}{\int_a^b v(\theta)d\theta} \geq \frac{\int_a^b \Psi\left(u(\theta)\right)d\theta}{\int_a^b \Psi\left(v(\theta)\right)d\theta}.$$

Theorem 2. *Let the functions u, w and v be positive continuous on $[a,b]$ with $u(\theta) \leq v(\theta)$ for all $\theta \in [a,b]$. Assume that the function $\frac{u}{v}$ is decreasing and u and w are increasing functions. Assume that Ψ is a convex function with $\Psi(0) = 0$. Then the following inequality hold*

$$\frac{\int_a^b u(\theta)dt}{\int_a^b v(\theta)dt} \geq \frac{\int_a^b \Psi\left(u(\theta)\right)w(\theta)dt}{\int_a^b \Psi\left(v(\theta)\right)w(\theta)dt}.$$

The applications of inequalities (1)–(3) can be found in the work of the various researchers. We refer the readers to [40–44].

Alzabut et al. [45] recently studied the Gronwall inequalities by considering generalized proportional fractional derivative operator. Rahman et al. [46] presented the Minkowski inequalities by employing proportional fractional integral. Dahmani [47] presented some classes of fractional integral inequalities by considering a family of n positive functions. Certainly, remarkable inequalities such as Hermite-Hadamard type [48], Chebyshev type [49–51], inequalities via generalized conformable integrals [52], Grüss type [53,54], fractional proportional inequalities and inequalities for convex functions [55], Hadamard proportional fractional integrals [56], bounds of proportional integrals with applications [57], inequalities for the weighted and the extended Chebyshev functionals [58], certain new inequalities for a class of $n(n \in \mathbb{N})$ positive continuous and decreasing functions [59] and certain generalized fractional inequalities [60] are recently presented by utilizing several different kinds of fractional calculus approaches.

2. Preliminaries

In this section, we give basic definitions and properties of tempered fractional integrals.

Definition 1. *The left and right sided R-L fractional integrals are respectively defined by*

$$\left({}_a\mathcal{T}^{\eta}u\right)(\theta) = \frac{1}{\Gamma(\eta)}\int_a^{\theta}(\theta-t)^{\eta-1}u(t)dt,\ \theta > a \tag{4}$$

and

$$\left(\mathcal{T}_b^{\eta}u\right)(\theta) = \frac{1}{\Gamma(\eta)}\int_{\theta}^{b}(t-\theta)^{\eta-1}u(t)dt, \theta < b \tag{5}$$

where $\eta \in \mathbb{C}$ *and* $\Re(\eta) > 0$.

The tempered fractional integral was first studied by Buschman [61], but Li et al. [62] and Meerschaert et al. [63] have described the associated tempered fractional calculus more explicitly.

Definition 2 ([62–64])**.** *Let* $[a,b]$ *be a real interval and* $\eta, \xi \in \mathbb{C}$ *with* $\Re(\eta) > 0$ *and* $\Re(\xi) \geq 0$, *then the left and right sided tempered fractional integral operators are respectively defined by*

$$\left({}_a\mathcal{T}^{\eta,\xi}u\right)(\theta) = e^{-\xi\theta}\,{}_a\mathfrak{I}_{\theta}^{\eta}\left(e^{\xi\theta}u(\theta)\right) = \frac{1}{\Gamma(\eta)}\int_a^{\theta}\exp[-\xi(\theta-t)](\theta-t)^{\eta-1}u(t)dt, a < \theta \tag{6}$$

and

$$\left(\mathcal{T}_b^{\eta,\xi}u\right)(\theta) = e^{-\xi\theta}\mathfrak{I}_b^{\eta}\left(e^{\xi\theta}u(\theta)\right) = \frac{1}{\Gamma(\eta)}\int_{\theta}^{b}\exp[-\xi(t-\theta)](t-\theta)^{\eta-1}u(t)dt, \theta < b. \tag{7}$$

Remark 1. *If we take* $\xi = 0$ *in the Equations* (6) *and* (7), *then we have the left and right R-L operators* (4) *and* (5) *respectively.*

The tempered fractional integral (6) satisfies the following semigroup property

$${}_a\mathcal{T}^{\eta,\xi}\left({}_a\mathcal{T}^{\lambda,\xi}u(t)\right) = {}_a\mathcal{T}^{\eta+\lambda,\xi}u(t), \Re(\eta), \Re(\lambda) > 0.$$

For further basic various properties, we refer the readers to see [64].

3. Main Results

Inequalities for convex functions by utilizing tempered fractional integral presented in this section.

Theorem 3. *Let the two positive functions u and v be continuous on $[a, b]$ and $u(\theta) \leq v(\theta)$ for all $\theta \in [a, b]$. If the function $\frac{u}{v}$ is decreasing and the function u is increasing on $[a, b]$. Then for any convex function Ψ with $\Psi(0) = 0$. Then the following inequality holds for the tempered integral (6)*

$$\frac{{}_a\mathcal{T}^{\eta,\xi}\left[u(\theta)\right]}{{}_a\mathcal{T}^{\eta,\xi}\left[v(\theta)\right]} \geq \frac{{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(u(\theta))\right]}{{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(v(\theta))\right]}, \tag{8}$$

where $\eta \in \mathbb{C}$ and $\Re(\eta) > 0$.

Proof. By the assumption of theorem, Ψ is convex with the property that $\Psi(0) = 0$. Then $\frac{\Psi(\theta)}{\theta}$ is increasing function. Since the function u is increasing, therefore $\frac{\Psi(u(\theta))}{u(\theta)}$ is also increasing function. Clearly, the function $\frac{u(\theta)}{v(\theta)}$ is decreasing. Thus for all $\rho, \vartheta \in [a, b]$, we have

$$\left(\frac{\Psi(u(\rho))}{u(\rho)} - \frac{\Psi(u(\vartheta))}{u(\vartheta)}\right)\left(\frac{u(\vartheta)}{v(\vartheta)} - \frac{u(\rho)}{v(\rho)}\right) \geq 0. \tag{9}$$

It follows that

$$\frac{\Psi(u(\rho))}{u(\rho)}\frac{u(\vartheta)}{v(\theta)} + \frac{\Psi(u(\vartheta))}{v(\vartheta)}\frac{u(\rho)}{v(\rho)} - \frac{\Psi(u(\vartheta))}{u(\vartheta)}\frac{u(\vartheta)}{v(\vartheta)} - \frac{\Psi(u(\rho))}{u(\rho)}\frac{u(\rho)}{v(\rho)} \geq 0. \tag{10}$$

Multiplying (10) by $v(\rho)v(\vartheta)$, we have

$$\frac{\Psi(u(\rho))}{u(\rho)}u(\vartheta)v(\rho) + \frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\rho)v(\vartheta) - \frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\vartheta)v(\rho) - \frac{\Psi(u(\rho))}{u(\rho)}u(\rho)v(\vartheta) \geq 0. \tag{11}$$

Multiplying (11) by $\dfrac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}$ and integrating (11) with respect to ρ over $[a, \theta]$, $a < \theta \leq b$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}u(\vartheta)v(\rho)d\rho \\
&+\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\rho)v(\vartheta)d\rho \\
&-\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\vartheta)v(\rho)d\rho \\
&-\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}u(\rho)v(\vartheta)d\rho \geq 0.
\end{aligned}$$

This follows that

$$\begin{aligned}
&u(\vartheta)\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right) + \left(\frac{\Psi(u(\vartheta))}{u(\vartheta)}v(\vartheta)\right)\,{}_a\mathcal{T}^{\eta,\xi}(u(\theta)) \\
&-\left(\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\vartheta)\right)\,{}_a\mathcal{T}^{\eta,\xi}(v(\theta)) - v(\vartheta)\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}u(\theta)\right) \geq 0.
\end{aligned} \tag{12}$$

Again, multiplying both sides of (12) by $\frac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\vartheta)](\theta-\vartheta)^{\eta-1}$, and integrating the resultant identity with respect to ϑ over $[a,\theta]$, $a<\theta\leq b$, we have

$$\begin{aligned}&{}_a\mathcal{T}^{\eta,\xi}(u(\theta))\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right)+{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right){}_a\mathcal{T}^{\eta,\xi}(u(\theta))\\ \geq\,&{}_a\mathcal{T}^{\eta,\xi}(v(\theta))\,{}_a\mathcal{T}^{\eta,\xi}(\Psi(u(\theta)))+{}_a\mathcal{T}^{\eta,\xi}(\Psi(u(\theta)))\,{}_a\mathcal{T}^{\eta,\xi}(v(\theta)).\end{aligned}$$

It follows that

$$\frac{{}_a\mathcal{T}^{\eta,\xi}(u(\theta))}{{}_a\mathcal{T}^{\eta,\xi}(v(\theta))}\geq\frac{{}_a\mathcal{T}^{\eta,\xi}(\Psi(u(\theta)))}{{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right)}. \tag{13}$$

Now, since $u(\theta)\leq v(\theta)$ for all $\theta\in[a,b]$ and $\frac{\Psi(\theta)}{\theta}$ is an increasing function, therefore for $\rho\in[a,\theta]$, $a<\theta\leq b$, we have

$$\frac{\Psi(u(\rho))}{u(\rho)}\leq\frac{\Psi(v(\rho))}{v(\rho)}, \tag{14}$$

multiplying both sides of (14) by $\dfrac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}v(\rho)$ and integrating the resultant identity with respect to ρ over $[a,\theta]$, $a<\theta\leq b$, we get

$$\begin{aligned}&\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}v(\rho)d\rho\\ \leq&\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(v(\rho))}{v(\rho)}v(\rho)d\rho,\end{aligned} \tag{15}$$

which in view of (6) can be written as

$${}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right)\leq{}_a\mathcal{T}^{\eta,\xi}(\Psi(v(\theta))). \tag{16}$$

Hence from (13) and (16), we get (8). □

Remark 2. *Setting $\xi=0$ in Theorem 3 will lead to Theorem 3.1 proved by [65].*

Remark 3. *Setting $\eta=1$, $\xi=0$ and $x=b$ in Theorem 3 will lead to Theorem 1.*

Theorem 4. *Let the two positive functions u and v be continuous on $[a,b]$ such that $u(\theta)\leq v(\theta)$ for all $\theta\in[a,b]$. If the function $\frac{u}{v}$ is decreasing and the function u is increasing on $[a,b]$. Then for any convex function Ψ with $\Psi(0)=0$. The following inequality holds for tempered integral (6)*

$$\frac{{}_a\mathcal{T}^{\eta,\xi}[u(\theta)]\,{}_a\mathcal{T}^{\lambda,\xi}[\Psi(v(\theta))]+{}_a\mathcal{T}^{\lambda,\xi}[u(\theta)]\,{}_a\mathcal{T}^{\eta,\xi}[\Psi(v(\theta))]}{{}_a\mathcal{T}^{\eta,\xi}[v(\theta)]\,{}_a\mathcal{T}^{\lambda,\xi}[\Psi(u(\theta))]+{}_a\mathcal{T}^{\lambda,\xi}[v(\theta)]\,{}_a\mathcal{T}^{\eta,\xi}[\Psi(u(\theta))]}\geq1, \tag{17}$$

where $\lambda\in\mathbb{C}$, $\Re(\eta)>0$ and $\Re(\lambda)>0$.

Proof. Since by assumption of theorem, Ψ is convex with $\Psi(0)=0$. Therefore, $\frac{\Psi(\theta)}{\theta}$ is increasing function. Furthermore, since u is increasing, therefore $\frac{\Psi(u(\theta))}{u(\theta)}$ is increasing. Obviously, $\frac{u(\theta)}{v(\theta)}$ is decreasing. Thus multiplying (12) by $\frac{1}{\Gamma(\lambda)}\exp[-\xi(\theta-\vartheta)](\theta-\vartheta)^{\lambda-1}$ and integrating the resultant identity with respect to ϑ over $[a,\theta]$, $a<\theta\leq b$, we get

$$\begin{aligned}&{}_a\mathcal{T}^{\lambda,\xi}(u(\theta))\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right)+{}_a\mathcal{T}^{\lambda,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)\right){}_a\mathcal{T}^{\eta,\xi}\left(u(\theta)\right)\\ &\geq {}_a\mathcal{T}^{\eta,\xi}\left(v(\theta)\right)\,{}_a\mathcal{T}^{\lambda,\xi}\left(\frac{\Psi(u(\theta)}{u(\theta)}u(\theta)\right)+{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta)}{u(\theta)}u(\theta)\right){}_a\mathcal{T}^{\lambda,\xi}\left(v(\theta)\right).\end{aligned}\tag{18}$$

Hence, from (16) and (18), we get the needful result. □

Remark 4. *Setting $\eta=\lambda$ in Theorem 4 will lead to Theorem 3.*

Remark 5. *Setting $\xi=0$ in Theorem 4 will lead to Theorem 3.3 proved by Dahmani [65].*

Theorem 5. *Let the functions u, w and v be positive continuous on $[a,b]$ such that $u(\theta)\leq v(\theta)$ for all $\theta\in[a,b]$. If $\frac{u}{v}$ is decreasing function and u and w are increasing functions on $[a,b]$. Then for convex function Ψ with $\Psi(0)=0$. Then the following inequality holds for the tempered integral (6)*

$$\frac{{}_a\mathcal{T}^{\eta,\xi}\left[u(\theta)\right]}{{}_a\mathcal{T}^{\eta,\xi}\left[v(\theta)\right]}\geq\frac{{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(u(\theta))w(\theta)\right]}{{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(v(\theta))w(\theta)\right]},\tag{19}$$

where $\eta\in\mathbb{C}$ and $\Re(\eta)>0$.

Proof. Since by assumption of theorem, Ψ is convex with the property that $\Psi(0)=0$, therefore $\frac{\Psi(\theta)}{\theta}$ is increasing. Since u is increasing, so therefore $\frac{\Psi(u(\theta))}{u(\theta)}$ is increasing. Clearly, $\frac{u(\theta)}{v(\theta)}$ is decreasing. Thus for all $\rho,\vartheta\in[a,\theta], a<\theta\leq b$, we have

$$\left(\frac{\Psi(u(\rho))}{u(\rho)}w(\rho)-\frac{\Psi(u(\vartheta))}{u(\vartheta)}w(\vartheta)\right)\left(u(\vartheta)v(\rho)-u(\rho)v(\vartheta)\right)\geq 0.\tag{20}$$

It becomes

$$\frac{\Psi(u(\rho))w(\rho)}{u(\rho)}u(\vartheta)v(\rho)+\frac{\Psi(u(\vartheta))w(\vartheta)}{u(\vartheta)}u(\rho)v(\vartheta)-\frac{\Psi(u(\vartheta))w(\vartheta)}{u(\vartheta)}u(\vartheta)v(\rho)-\frac{\Psi(u(\rho))w(\rho)}{u(\rho)}u(\rho)v(\vartheta)\geq 0.\tag{21}$$

Multiplying (21) by $\dfrac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}$ and integrating the identity with respect to ρ over $[a,\theta], a<\theta\leq b$, we get

$$\begin{aligned}&\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}u(\vartheta)v(\rho)w(\rho)d\rho\\ &+\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\rho)v(\vartheta)w(\vartheta)d\rho\\ &-\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\vartheta)w(\vartheta)v(\rho)d\rho\\ &-\frac{1}{\Gamma(\eta)}\int_a^\theta\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}u(\rho)w(\rho)v(\vartheta)d\rho\geq 0.\end{aligned}$$

This follows that

$$\begin{aligned}&u(\vartheta)\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right)+\left(\frac{\Psi(u(\vartheta))}{u(\vartheta)}v(\vartheta)w(\vartheta)\right){}_a\mathcal{T}^{\eta,\xi}(u(\theta))\\ &-\left(\frac{\Psi(u(\vartheta))}{u(\vartheta)}u(\vartheta)w(\vartheta)\right){}_a\mathcal{T}^{\eta,\xi}(v(\theta))-v(\vartheta)\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}u(\theta)w(\theta)\right)\geq 0.\end{aligned}\tag{22}$$

Again, multiplying both sides of (22) by $\frac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\theta)](\theta-\vartheta)^{\eta-1}$ and integrating the resultant identity with respect to ϑ over $[a,\theta]$, $a<\theta\leq b$, we get

$$\begin{aligned}
&{}_a\mathcal{T}^{\eta,\xi}(u(\theta))\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right)+{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right){}_a\mathcal{T}^{\eta,\xi}\left(u(\theta)\right)\\
\geq&{}_a\mathcal{T}^{\eta,\xi}\left(v(\theta)\right)\,{}_a\mathcal{T}^{\eta,\xi}\left(\Psi(u(\theta))w(\theta)\right)+{}_a\mathcal{T}^{\eta,\xi}\left(\Psi(u(\theta))w(\theta)\right)\,{}_a\mathcal{T}^{\eta,\xi}\left(v(\theta)\right).
\end{aligned}$$

It follows that

$$\frac{{}_a\mathcal{T}^{\eta,\xi}(u(\theta))}{{}_a\mathcal{T}^{\eta,\xi}\left(v(\theta)\right)}\geq\frac{{}_a\mathcal{T}^{\eta,\xi}\left(\Psi(u(\theta))w(\theta)\right)}{{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right)}. \tag{23}$$

Furthermore, since $u(\theta)\leq v(\theta)$ for all $\theta\in[a,b]$ and $\frac{\Psi(\theta)}{\theta}$ is increasing function, therefore for $\rho,\vartheta\in[a,b]$, we have

$$\frac{\Psi(u(\rho))}{u(\rho)}\leq\frac{\Psi(v(\rho))}{v(\rho)}, \tag{24}$$

multiplying both sides of (24) by $\frac{1}{\Gamma(\eta)}\exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}v(\rho)w(\rho)$ and integrating the resultant identity with respect to ρ over $[a,\theta]$, $a<\theta\leq b$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(u(\rho))}{u(\rho)}v(\rho)w(\rho)d\rho\\
\leq&\frac{1}{\Gamma(\eta)}\int_a^\theta \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1}\frac{\Psi(v(\rho))}{v(\rho)}v(\rho)w(\rho)d\rho,
\end{aligned}$$

which in view of (6) can be written as

$${}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right)\leq{}_a\mathcal{T}^{\eta,\xi}\left(\Psi(v(\theta))w(\theta)\right). \tag{25}$$

Hence, from (25) and (23), we obtain the required result. □

Remark 6. *Setting $\xi=0$ in Theorem 5 will lead to Theorem 3.5 presented by Dahmani [65].*

Remark 7. *Setting $\eta=1$, $\xi=0$ and $x=b$ in Theorem 5 will lead to Theorem 2.*

Theorem 6. *Let the positive functions u, w and v be continuous on $[a,b]$ such that $u(\theta)\leq v(\theta)$ for all $\theta\in[a,b]$. If $\frac{u}{v}$ is decreasing and u and w are increasing on $[a,b]$. Then for any convex function Ψ with the property that $\Psi(0)=0$. The following inequality holds for the tempered integral (6)*

$$\frac{{}_a\mathcal{T}^{\eta,\xi}\left[u(\theta)\right]\,{}_a\mathcal{T}^{\lambda,\xi}\left[\Psi(v(\theta))w(\theta)\right]+{}_a\mathcal{T}^{\lambda,\xi}\left[u(\theta)\right]\,{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(v(\theta))w(\theta)\right]}{{}_a\mathcal{T}^{\eta,\xi}\left[v(\theta)\right]\,{}_a\mathcal{T}^{\lambda,\xi}\left[\Psi(u(\theta))w(\theta)\right]+{}_a\mathcal{T}^{\lambda,\xi}\left[v(\theta)\right]\,{}_a\mathcal{T}^{\eta,\xi}\left[\Psi(u(\theta))w(\theta)\right]}\geq1, \tag{26}$$

where $\eta,\lambda\in\mathbb{C}$, $\Re(\eta)>0$ and $\Re(\lambda)>0$.

Proof. Multiplying both sides of (22) by $\frac{1}{\Gamma(\lambda)}\exp[-\xi(\theta-\vartheta)](\theta-\vartheta)^{\lambda-1}$ and integrating the resultant with respect to ϑ over $[a,\theta]$, $a<\theta\leq b$, we get

$$\begin{aligned}
&{}_a\mathcal{T}^{\lambda,\xi}(u(\theta))\,{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right)+{}_a\mathcal{T}^{\lambda,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}v(\theta)w(\theta)\right){}_a\mathcal{T}^{\eta,\xi}\left(u(\theta)\right)\\
\geq&{}_a\mathcal{T}^{\eta,\xi}\left(v(\theta)\right)\,{}_a\mathcal{T}^{\lambda,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}u(\theta)w(\theta)\right)+{}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)}u(\theta)w(\theta)\right){}_a\mathcal{T}^{\lambda,\xi}\left(v(\theta)\right).
\end{aligned} \tag{27}$$

Since $u(\theta) \leq v(\theta)$ for all $\theta \in [a,b]$ and $\frac{\Psi(\theta)}{\theta}$ is increasing function, therefore for $\rho, \vartheta \in [a,\theta], a < \theta \leq b$, we have

$$\frac{\Psi(u(\rho))}{u(\rho)} \leq \frac{\Psi(v(\rho))}{v(\rho)}, \tag{28}$$

multiplying both sides of (28) by $\frac{1}{\Gamma(\eta)} \exp[-\xi(\theta-\rho)](\theta-\rho)^{\eta-1} v(\rho) w(\rho)$, $\rho \in [a,x]$, $a < \theta \leq b$ and integrating the resultant identity with respect to ρ over $[a,\theta]$, $a < \theta \leq b$, we get

$${}_a\mathcal{T}^{\eta,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)} v(\theta) w(\theta)\right) \leq {}_a\mathcal{T}^{\eta,\xi}\left(\Psi(v(\theta)) w(\theta)\right)\right). \tag{29}$$

By following a similar procedure, one can obtain

$${}_a\mathcal{T}^{\rho,\xi}\left(\frac{\Psi(u(\theta))}{u(\theta)} v(\theta) w(\theta)\right) \leq {}_a\mathcal{T}^{\rho,\xi}\left(\Psi(v(\theta)) w(\theta)\right)\right). \tag{30}$$

Hence, from (27), (29) and (30), we obtain the required inequality (26). □

Remark 8. *Setting $\eta = \lambda$ in Theorem 6 will lead to Theorem 5.*

Remark 9. *Setting $\xi = 0$ in Theorem 6 will lead to Theorem 3.7 presented by Dahmani [65].*

4. Particular Cases

In [62], Li et al. gave the following connection of tempered fractional integral with the Riemann–Liouville fractional integral by

$${}_a\mathcal{I}^{\eta,\xi} u(\theta) = e^{-\xi\theta}\, {}_a\mathcal{I}^{\eta}\left[e^{\xi\theta} u(\theta)\right]. \tag{31}$$

By employing this connection (31) to Theorems 3 and 5, we get the following new results in term of Riemann–Liouville fractional integrals.

Theorem 7. *Let the two positive functions u and v be continuous on $[a,b]$ such that $u(\theta) \leq v(\theta)$ for all $\theta \in [a,b]$. If $\frac{u}{v}$ is decreasing and u is increasing on $[a,b]$. Then for any convex function Ψ with $\Psi(0) = 0$. The following inequality holds*

$$\frac{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} u(\theta)\right]}{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} v(\theta)\right]} \geq \frac{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} \Psi(u(\theta))\right]}{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} \Psi(v(\theta))\right]},$$

where $\eta \in \mathbb{C}$ and $\Re(\eta) > 0$.

Theorem 8. *Let the positive functions u, w and v be continuous on $[a,b]$ such that $u(\theta) \leq v(\theta)$ for all $\theta \in [a,b]$. If $\frac{u}{v}$ is decreasing and u and w are increasing on $[a,b]$. Then for convex function Ψ with $\Psi(0) = 0$. The following inequality holds*

$$\frac{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} u(\theta)\right]}{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} v(\theta)\right]} \geq \frac{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} \Psi(u(\theta)) w(\theta)\right]}{{}_a\mathcal{T}^{\eta}\left[e^{\xi\theta} \Psi(v(\theta)) w(\theta)\right]},$$

where $\eta \in \mathbb{C}$ and $\Re(\eta) > 0$.

Similarly, we can get particular cases of Theorems 4 and 6.
The following Theorems are the particular results of Theorems 3 and 4 which can be obtained by setting $\eta = 1$ and $\theta = b$ in Theorems 7 and 8 respectively.

Theorem 9. *Let the two positive functions u and v be continuous on $[a, b]$ such that $u(\theta) \le v(\theta)$ for all $\theta \in [a, b]$. If $\frac{u}{v}$ is decreasing and u is increasing on $[a, b]$. Then for any convex function Ψ with $\Psi(0) = 0$. The following inequality holds*

$$\frac{\int_a^b e^{\xi\theta} u(\theta) d\theta}{\int_a^b e^{\xi\theta} v(\theta) d\theta} \ge \frac{\int_a^b e^{\xi\theta} \Psi(u(\theta)) d\theta}{\int_a^b e^{\xi\theta} \Psi(v(\theta)) d\theta}.$$

Theorem 10. *Let the positive functions u, w and v be continuous on $[a, b]$ such that $u(\theta) \le v(\theta)$ for all $\theta \in [a, b]$. If $\frac{u}{v}$ is decreasing and u and w are increasing on $[a, b]$. Then for convex function Ψ with $\Psi(0) = 0$. The following inequality holds*

$$\frac{\int_a^b e^{\xi\theta} u(\theta) d\theta}{\int_a^b e^{\xi\theta} v(\theta) d\theta} \ge \frac{\int_a^b e^{\xi\theta} \Psi(u(\theta)) w(\theta) d\theta}{\int_a^b e^{\xi\theta} \Psi(v(\theta)) w(\theta) d\theta}.$$

5. Conclusions

In this paper, we established certain inequalities for tempered fractional integrals via convex functions. We also established certain new particular results by employing the connections of tempered fractional integral with the Riemann–Liouville integral. The obtained results will reduce to the results given by Dahmani [65] by taking the parameter $\xi = 0$. Furthermore, by taking $\eta = 1$ and $\xi = 0$ the obtained inequalities will reduce to the results of Liu et al. ([39], Theorem 9 and 10).

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